The Knot That Rings

Bar-Natan and van der Veen's Θ read through the Vacuum.Net Theory, with new computations

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Abstract

The new knot invariant Θ of Bar-Natan and van der Veen behaves as the Vacuum.Net Theory's addressing rule requires: in every knot computed here the mirror image flips its sign, the knot equal to its own mirror gets zero, and the left and right trefoil are told apart. These three facts were checked by an independent implementation of the paper's formulas, which reproduces the paper's trefoil exactly.

Eight knots were computed. Both of the authors' symmetry conjectures held on all six that were tested, and the genus bound was sharp on all. A new regularity appeared in the torus knots T(2, 2k+1): θ has 6k(k+1) terms and absolute coefficient sum 2k(k+1)(2k+1). Both formulas were stated on three knots and then confirmed on two more that had not yet been computed.

Three further features of Θ (three variables that close, a hexagon from sl_3, traffic along one strand) mirror the theory's trinity, single strand and throughflow, but only as analogies. Two hoped-for links failed: θ does not detect three-colourability, and it has no place on the ring ladder. The paper's remark that its pictures resemble Chladni figures, the patterns of vibrating plates, is turned into a concrete test. Five predictions close the essay, each with a refutation condition.

A same-day follow-up (sections 8–11) tested the predictions. Both held: T(2,13) gave exactly the predicted counts, and all seven amphichiral knots tested have θ = 0. The torus regularity turned out to be an exact closed formula of concentric hexagonal rings. On a random 59-crossing knot the Chladni resemblance is real at an intermediate scale (standing modes 11–30, p = 0.005) but not in the lowest modes. The essay ends with Θ as a fingerprint for knotted proteins and DNA and a reference package for the MAZE.

1. What Θ is

Θ is a pair of polynomials attached to every knot: the old Alexander polynomial Δ in one variable T, and a new polynomial θ in two variables T_1 and T_2. Dror Bar-Natan and Roland van der Veen introduced it in 2025; the edition read here is version 4 of 6 May 2026.

Three properties make it unusual.

- **Fast.** It is computed by inverting one matrix whose size grows with the number of crossings, so its cost grows as a power of that number, not exponentially. The authors compute it fully on knots with more than 300 crossings.
- **Strong.** Among the 313,230 prime knots with up to 15 crossings, Θ leaves 6,758 pairs undistinguished; the hyperbolic volume, HOMFLY-PT polynomial and Khovanov homology together leave about 41,434. It also sometimes separates mutant knots, which none of the others do.
- **Meaningful.** Twice the genus of a knot (the number of handles of the smallest surface it bounds) is at least the degree of θ in T_1. For the 48-crossing Gompf–Scharlemann–Thompson knot this gives a genus of at least 10, where Δ gives only 8.

The construction has three features that matter for what follows.

1. **Three variables, one relation.** Besides T_1 and T_2 the formula uses $T_3 = T_1T_2$. Every quantity is evaluated three times, once per variable, and θ is multiplied by $\Delta(T_1)\cdot\Delta(T_2)\cdot\Delta(T_3)$.
2. **Traffic on the strand.** The inverse matrix G can be read as car traffic along the knot: a car on an over-strand continues with probability T^s or drops to the lower strand with probability $1 - T^s$. The three terms of θ are then Feynman diagrams of interacting cars of three kinds.
3. **A hexagonal picture.** Plotting the coefficients of θ on a sheared grid gives a "hexagonal QR code". The authors conjecture (Conjecture 20) that it is always symmetric under the reflections of a hexagon, and they compare the patterns of large knots to Chladni figures, the shapes sand takes on a vibrating plate. They conjecture that θ comes from the Lie algebra sl_3 (Conjecture 28), whose root system is itself a hexagon.

2. Computed here

Θ was implemented independently for this essay from the paper's formulas (3)–(6), in Python with SymPy, and it reproduces the paper's printed value for the trefoil exactly. The trefoil was also computed from two different diagrams, the paper's own and a closed braid, with the same result, as invariance requires.

Knots were entered as closed braids cut open into long knots: each closing arc adds a rotation number of −1 to its edge, and each braid crossing is upright by construction.

Knot	Braid	Δ	Det.	Genus	deg θ in T_1	Terms	Σ|coeff.|	Hexagonal symmetry	θ(mirror) = −θ
3_1 trefoil, T(2,3)	σ_1^3	$T - 1 + T^{-1}$	3	1	2	12	12	yes	yes
4_1 figure-eight	$(\sigma_1\sigma_2^{-1})^2$	$-T + 3 - T^{-1}$	5	1	— (θ = 0)	0	0	yes	yes
5_1 T(2,5)	σ_1^5	$T^2 - T + 1 - T^{-1} + T^{-2}$	5	2	4	36	60	yes	yes
5_2	$\sigma_1^3\sigma_2\sigma_1^{-1}\sigma_2$	$2T - 3 + 2T^{-1}$	7	1	2	19	180	yes	yes
7_1 T(2,7)	σ_1^7	degree 3, all coefficients ±1	7	3	6	72	168	yes	yes
8_{19} T(3,4)	$(\sigma_1\sigma_2)^4$	$T^3 - T^2 + 1 - T^{-2} + T^{-3}$	3	3	6	90	240	yes	yes
9_1 T(2,9)	σ_1^9	degree 4	9	4	—	120	360	not checked	not checked

Knot	Braid	Δ	Det.	Genus	deg θ in T_1	Terms	Σ|coeff.|	Hexagonal symmetry	θ(mirror) = −θ
T(2,11)	σ_1^{11}	degree 5	11	5	—	180	660	not checked	not checked

Three checks came out as the paper predicts.

- **Hexagonal symmetry (Conjecture 20)** holds for all six knots checked: θ is unchanged under $(T_1, T_2) \to (T_1, 1/(T_1T_2))$ and under $(T_1, T_2) \to (T_1T_2, 1/T_2)$.
- **Mirror antisymmetry (Conjecture 21)** holds for all six: the mirror image has θ of opposite sign. The figure-eight knot is its own mirror image, so its θ must be zero, and it is.
- **Genus bound (Theorem 18).** For every knot checked, the degree in T_1 equals twice the genus, so the bound is sharp on these.

In the six knots checked the coefficients of θ sum to zero, θ(1, 1) = 0. Computing times ran from 0.3 s (trefoil) to 56 s (T(2,11)). The code is available on request.

3. A regularity in the torus knots T(2, 2k+1)

For the torus knots T(2, 2k+1), the number of terms in θ is 6k(k+1) and the sum of the absolute values of its coefficients is 2k(k+1)(2k+1). Both formulas were read off k = 1, 2, 3, written down, and then confirmed exactly on k = 4 and k = 5, which had not been computed when they were stated.

k	Knot	Terms predicted	Terms computed	Σ|c| predicted	Σ|c| computed	Role
1	T(2,3)	12	12	12	12	fit
2	T(2,5)	36	36	60	60	fit
3	T(2,7)	72	72	168	168	fit
4	T(2,9)	120	120	360	360	test
5	T(2,11)	180	180	660	660	test

Written differently, both counts are twelve times a classical sequence:

$$\#\text{terms} = 12 \cdot \frac{k(k+1)}{2}, \qquad \sum |c| = 12 \cdot \frac{k(k+1)(2k+1)}{6} = 12\left(1^2 + 2^2 + \ldots + k^2\right)$$

The factor 12 has a plain explanation: the hexagon's symmetry group has 12 elements, and if the hexagonal symmetry holds and no coefficient sits on a point the symmetries fix, every term comes in a set of 12. The triangular numbers k(k+1)/2 and the sums of squares are the new part. Section 7 states the next case as a prediction.

This is a numerical regularity on five knots of one family, not a theorem. It may follow directly from a closed formula for θ of torus knots; no such formula appears in the paper.

4. Where Θ meets the Vacuum.Net Theory

Six features of Θ have a counterpart in the axioms of the theory; the first three agree with the computations above, the last three are structural analogies.

Θ feature	Vacuum.Net counterpart	Status
Mirror image changes the sign: θ(mirror) = −θ	The address of −Q is the mirror of the address of Q (T12); negation is mirroring, 1 ↔ −1	Confirmed on six knots; conjectured in general by the authors
The figure-eight knot, equal to its mirror, has θ = 0	A configuration equal to its own mirror sits at address 0, the address of rest	Confirmed on the one amphichiral knot computed
θ separates the left and right trefoil; Δ cannot	The trefoil is the first knot and exists in a left and a right form (T19, T20)	Agreement: θ separates the two forms of the theory's first knot, as the Jones polynomial also does but the Alexander polynomial, θ's nearest relative, cannot

Θ feature	Vacuum.Net counterpart	Status
Three variables with $T_3 = T_1T_2$, a hexagonal symmetry, a conjectured origin in sl_3	The trinity: three values that close to zero; the window of three; three-colouring of a crossing as the window rule (T4)	Structural analogy. In exponent space the three variables give three directions $e_1 + e_2 - e_3 = 0$, and the hexagon is the root system of sl_3, three pairs of directions that close
G read as traffic on the single strand; θ as interactions of three kinds of traffic	One strand (N1); life as a closure that lays itself again as long as something flows through it	Structural analogy; the authors use traffic as a reading, not as physics
Genus bound: θ sees how much surface a knot needs	Closing is taking boundary inward; a knot of knots has less open boundary (N5)	Analogy; genus measures boundary, the theory's central quantity for persistence

The first three rows are the strongest. A sign change under mirroring, a zero for the self-mirror knot and the separation of left from right are exactly what the theory's addressing rule requires of any faithful invariant. The Alexander polynomial fails this test, because it is identical for a knot and its mirror image; θ passes it in every case computed.

One further fact fits the theory's bookkeeping: θ_0 is additive when two knots are tied one after the other (Fact 23 of the paper). Adding knots adds their θ_0, as the theory adds the addresses of passages along one strand.

5. Where it does not meet

Three tests that could have tied Θ more tightly to the theory came out negative, and one tempting reading is wrong.

- **No link to three-colouring.** A knot can be coloured with three colours by the theory's window rule exactly when its determinant is divisible by 3 (here: the trefoil and T(3,4)). If θ encoded that rule, its value at a cube root of unity ω should separate these two from the rest. It does not: $\theta(\omega, \omega)$ is −9 for the trefoil, +9 for T(2,5) and

T(2,7), −135 for 5_2, 180 for T(3,4), −27 for T(2,9) and 18 for T(2,11), with no split along divisibility by 3.

- **No pattern modulo 3.** The share of coefficients not divisible by 3 runs from 6 of 19 (5_2) to all of them (trefoil, T(2,5)), again unrelated to three-colourability.
- **No place on the ring ladder.** Θ has no frequency or length, so it has no depth on the hydrogen ladder of *Rhythms Out of Tune*. Any link between Θ and the rings would have to go through a physical model of the knot, which neither the paper nor the theory supplies.
- **The 12 is not closure depth 12.** The counts of section 3 are multiples of 12 because the hexagon has 12 symmetries. That 12 is also a closure depth of the theory is a coincidence of numbers, not a correspondence.

One small regularity did appear: at $T_1 = \omega$, $T_2 = \omega^2$ every value computed is divisible by 3 (3, 0, −3, 27, −3, 30, 9, −6), and at $T_1 = T_2 = \omega$ every value is divisible by 9. Eight knots are too few to call this more than an observation, and it may follow from the hexagonal symmetry alone.

6. From knot to vibration: the Chladni question

The authors see in the hexagonal pictures of large knots patterns "perhaps reminiscent of the Chladni figures formed by powders atop vibrating plates", and say they cannot prove it or even formulate it properly. For the Vacuum.Net programme this is the most interesting sentence in the paper, because the theory reads every stable thing as a closed winding and every perception and treatment as a closure of rhythms.

In mathematical terms the question has a sharp form. A Chladni figure shows the nodal lines of a standing wave on a plate: the places where the vibration is zero. θ is a function on a hexagonal grid of exponents. If the resemblance is real, there should be a wave equation on that grid whose standing waves reproduce the sign pattern of θ. Three things in the paper point that way.

1. **Gaussian integrals.** θ is the first correction of a Gaussian integral over a space of six coordinates per edge of the knot (Fact 33). Gaussian integrals with a perturbation are the mathematics of vibrations around a rest state.
2. **A 2-point function.** The matrix G is called the knot's Green function, the same object that describes how a disturbance spreads through a medium.
3. **Square weaves.** For square weave knots of growing size (the paper's Figure 1.2), the pictures settle into a fixed large-scale pattern, as standing waves on a growing plate keep their mode shape.

A concrete test follows. Take the sign pattern of θ for a large knot, and fit it with the lowest standing modes of the hexagon (the eigenfunctions of the Laplace operator on a hexagonal region). If a handful of modes reproduces the pattern far better than for a shuffled copy of the same coefficients, the knot "rings" in a measurable sense. If not, the resemblance is in the eye.

The link to devices is indirect but real: a knot whose θ is dominated by one hexagonal mode would define a spatial pattern, not a frequency. A three-coil applicator at 0°, 120° and 240°, already proposed in *Rhythms Out of Tune*, has the threefold rotational symmetry on which such a hexagonal pattern is built; with alternating polarity it has the sixfold one.

7. Proposals and predictions

Each item is fixed here before it is tested.

	Prediction or proposal	How to test	Refuted if
K1	The torus knot T(2,13) has θ with 252 terms and Σ|c| = 1,092 (k = 6 in section 3)	Compute θ of σ_1^{13} with the paper's programs or this essay's code	Either count differs
K2	Every amphichiral knot (equal to its mirror image) has θ = 0, as the theory's addressing rule requires and the authors' Conjecture 21 implies	Compute θ for the amphichiral knots up to 10 crossings (4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{17}, 8_{18}, …)	Any amphichiral knot with θ ≠ 0

	Prediction or proposal	How to test	Refuted if
K3	The hexagonal pictures of large knots are dominated by a few low standing modes of the hexagon	Fit the sign pattern of θ with the lowest Laplace eigenmodes of a hexagon; compare with shuffled coefficients	The fit is no better than for shuffled data
K4	Θ can serve as the address of a knot in the MAZE: it is deterministic, computable in polynomial time and separates all but 6,758 of 313,230 knots up to 15 crossings	Give every knot in the MAZE's standard reference its Θ code; check that knot pairs the theory treats as different get different codes	Two knots the theory separates share a Θ code
K5	If θ is the sl_3 invariant (Conjecture 28), the trinity of the theory has an algebraic home: θ is then the near-Alexander invariant of sl_3, the Lie algebra whose root system is the hexagon of three closing directions	Follow the proof or refutation of Conjecture 28	Conjecture 28 is refuted

The direct next step is to test K1 and K2, which take minutes; section 8 reports the outcome.

8. Follow-up results

Two of the five predictions of section 7 were tested the same day and both held, and the torus regularity turned out to be the surface of an exact closed formula.

K1 held. T(2,13) has θ with 252 terms and Σ|c| = 1,092, exactly as predicted. The two counting formulas now stand on three knots computed after they were stated.

K2 held. Braid words and Alexander polynomials were taken from the Knot Atlas, and every Alexander polynomial was reproduced, so the knots are the intended ones. All amphichiral knots tested give θ = 0, and a chiral control does not.

Knot	Symmetry (Knot Atlas)	Δ reproduced	θ
4_1	fully amphichiral	yes	0
6_3	fully amphichiral	yes	0
8_3	fully amphichiral	yes	0
8_9	fully amphichiral	yes	0
8_{12}	fully amphichiral	yes	0
8_{17}	negative amphichiral	yes	0
8_{18}	fully amphichiral	yes	0
6_2 (control)	reversible, chiral	yes	42 terms, not 0

A closed formula for the torus knots T(2, 2k+1). The coefficients of θ lie on concentric hexagonal rings. Ring j sits at hexagonal distance 2j from the centre and carries the value ±j at every one of its 12j lattice points; the odd rings are empty. The sign is + exactly where both exponents are even:

$$\theta\Big(T(2,2k+1)\Big) = \sum_{j=1}^{k} j \sum_{\max\left(|a|,|b|,|a-b|\right)=2j} \varepsilon(a,b)\, T_1^a T_2^b, \qquad \varepsilon(a,b) = \begin{cases} +1 & a,b \text{ both even} \\ -1 & \text{otherwise} \end{cases}$$

The formula was read off the printed grids of k = 1, 2, 3 and then matched coefficient for coefficient on k = 4, 5, 6, whose grids had been computed but not looked at. It contains both counting formulas of section 3: ring j has 12j points with absolute value j, which gives 6k(k+1) terms and $\Sigma|c| = 12(1^2 + \ldots + k^2)$.

In the language of the theory the picture is a set of nested closed rings, one more for each extra pair of crossings, each ring carrying its own depth j as its weight. It is verified here for $k \le 6$, not proved; a proof for all k would follow from the traffic formula applied to the braid σ_1^(2k+1), and is left to the authors or to a later version.

One more knot was computed for section 10: the stevedore knot 6_1 has $\Delta = -2T + 5 - 2T^{-1}$ and θ with 19 terms, $\Sigma|c| = 84$, and centre coefficient −24.

9. The Chladni test

On one random 59-crossing knot, the large-scale pattern of θ is captured by hexagonal standing modes 11 to 30 far better than by chance ($p = 0.005$), but not by the lowest 10 modes, and not at all when the sign of each coefficient is taken on its own. The resemblance to Chladni figures is real at an intermediate scale, on this one knot.

The knot and its Θ. A random braid of 59 crossings on 4 strands (seed fixed in advance) was used, the kind of generic large knot the paper's pictures show. θ was computed numerically by evaluating it at 244 × 244 points on a circle and recovering the coefficients with a two-dimensional Fourier transform; the method had first reproduced the exact θ of four small knots to better than 10^{-12}. For the large knot every coefficient came out within 10^{-4} of an integer. θ has 1,141 non-zero coefficients filling a hexagon of radius 19, the largest of absolute value 513,204, and it obeys the hexagonal symmetry of Conjecture 20 exactly: a further confirmation, on a far larger knot than section 2.

The test, fixed before the data were seen. Standing waves on a hexagonal plate were computed as the eigenmodes of the lattice Laplacian on the hexagon (Dirichlet boundary). Two signals were tested: the sign of each coefficient, and that sign averaged over each point and its six neighbours (the coarse colour a reader sees in the paper's pictures). For each, the share of the pattern explained by the lowest 10 and lowest 30 modes (R^2) was compared with 2,000 shuffles that keep the hexagonal symmetry but move values between symmetry orbits of the same size.

Signal	Modes	R^2 observed	R^2 shuffled, mean	Shuffled 95th percentile	p
sign	10	0.000	0.010	0.037	1.00

Signal	Modes	R^2 observed	R^2 shuffled, mean	Shuffled 95th percentile	p
sign	30	0.011	0.036	0.084	0.88
averaged sign	10	0.000	0.011	0.037	0.99
averaged sign	30	0.130	0.038	0.088	0.005

Reading. Four tests were fixed; with that many, a result needs $p < 0.0125$ to count, and one passes. The coefficient-by-coefficient sign alternates too fast for any smooth mode, which is why the paper's pictures look purple at close range. Once that fine alternation is averaged out, a structure appears that the first 30 modes describe three times better than chance, but the very lowest modes do not carry it. In plate terms: the knot does not ring in its fundamental, it rings in its overtones.

Limits. This is one knot. The next step is the same test on the paper's square weave knots and on a dozen random knots, with the mode band 11–30 fixed in advance as the prediction.

10. Knots in living matter

Knots are not only mathematics: about 0.8 % of the protein chains in the Protein Data Bank are knotted, and cells cut and rejoin their DNA with enzymes that change its knot type. In both places Θ could become a measuring instrument for closure in living matter.

Proteins. The KnotProt survey of 144,554 protein chains found about 1,150 with a knot or slipknot. The knot types are the trefoil 3_1 (715 structures), the figure-eight 4_1, 5_2 and the stevedore knot 6_1 (Jamroz et al. 2014). Knotting patterns are conserved within protein families, and in some enzymes the active site lies near the knot core, so the knots are thought not to be accidental. A search of AlphaFold predictions added the first 7_1 knot and composite knots of two trefoils (*Protein Science*, 2022; Physics World).

Every one of these knot types has been computed in this essay. Their Θ values are:

Protein knot	Chiral?	θ (this essay)
3_1 trefoil	yes	12 terms, $\Sigma\|c\| = 12$; sign gives the handedness

Protein knot	Chiral?	θ (this essay)
4_1 figure-eight	no	0
5_2	yes	19 terms, Σ|c| = 180
6_1 stevedore	yes	19 terms, Σ|c| = 84
7_1	yes	72 terms, Σ|c| = 168; closed formula of section 8

DNA. Tightly packed DNA becomes intertwined and knotted, and must be unknotted before it can be copied or read. Topoisomerases do this: type II enzymes, such as gyrase in *E. coli*, cut both strands, pass one through and reconnect, which flips a crossing. Knot theory has been used for decades to infer how such enzymes act from the knots they leave behind.

What Θ adds, read through the theory. The theory says that identity changes only by a jump of class, and that a charge changes only by cutting the strand (T22). A type II topoisomerase is exactly such a cut: it changes the knot class, and with it Θ, in one step. Three uses follow, each testable.

1. **Handedness as a fingerprint.** θ changes sign with the mirror image, so its sign records the handedness of a knotted protein, which Δ cannot. Proteins are built from one-handed amino acids; whether one handedness dominates per knot type and per family is a question KnotProt data can answer directly.
2. **A measure for enzyme steps.** For DNA knots produced in a reaction, the change in Θ per enzyme step gives a finer record of what the enzyme did than the crossing number alone.
3. **A link to the bacteria line.** Gyrase is a bacterial enzyme and a target of antibiotics. Bacterial DNA topology is therefore a place where the theory's rule of class change by cutting can be compared with measured knot distributions.

These are proposals. The protein and DNA facts above are cited; the reading through the theory is a correspondence, not a result.

11. Θ as an address

The address package. The implementation used here is packaged as a reference for the MAZE: exact code, fast numerical code, and test vectors for nine knots (3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3, 7_1, 8_{19}) that any other implementation must reproduce coefficient for coefficient. The address of a knot is the pair (Δ, θ): mirror images share Δ and have opposite θ, and self-mirror knots sit at θ = 0. Two limits are stated with it: Θ leaves 6,758 of 313,230 knots up to

15 crossings undistinguished, so the hyperbolic volume should be added where uniqueness matters, and the exact code is practical up to about 15 crossings, the numerical code to about 60.

12. Status and references

Statement	Status
Θ is invariant, polynomial-time, strong, and bounds the genus	Proved or measured by Bar-Natan and van der Veen
Independent implementation reproduces the paper's trefoil	Computed
Hexagonal symmetry and mirror antisymmetry on small knots; θ = 0 for the figure-eight	Computed on six knots; the general statements remain the authors' conjectures
Hexagonal symmetry on a random 59-crossing knot	Computed, exact
T(2, 2k+1): 6k(k+1) terms and Σ|c| = 2k(k+1)(2k+1)	Fitted on k = 1–3, confirmed on k = 4–6 (K1 held); not proved
Closed ring formula for θ of T(2, 2k+1)	Read off k = 1–3, exact on k = 4–6; not proved
θ = 0 for amphichiral knots (K2)	Held on all seven tested; chiral control non-zero
Divisibility of θ at cube roots of unity by 3 and 9	Observed on eight knots
Mirror sign, self-mirror zero and handedness match the theory's addressing rule	Correspondence, supported by computation
Three variables, hexagon, traffic and genus as counterparts of trinity, strand and boundary	Structural analogy
Link between θ and three-colourability or the ring ladder	Tested, not found
Chladni resemblance (K3)	One knot: modes 11–30 explain the averaged pattern three times better than chance (p = 0.005); lowest 10 modes and raw signs do not
Θ as a fingerprint of protein and DNA knots	Proposal; biological facts cited

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