# Barriers and Back-Steps

*How mathematics overcame its obstructions, how they match the windings of the Vacuum.Net theory, and a prediction of the next barrier*

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## Abstract

Mathematics does not grow in a straight line. It moves forward until an operation cannot be undone. Then it steps back to the point where the inverse fails. It adds a new kind of object and moves on. The new object always costs a property that was taken for granted. This essay lists the main barriers from negative numbers to the 2023 disproof of Ravenel's telescope conjecture. For each barrier it states how it was solved and which barrier came next. It then sets the list beside the windings of the Vacuum.Net theory. The match is a reading, not a proof. Some links are backed by computations made for this essay. The essay ends with three predictions about the next barrier. Each can be tested.

## 1. The question

Every student meets the pattern. Addition works on counting numbers. Subtraction does not: 3 − 5 has no answer there. Multiplication works on whole numbers. Division does not: 1 ÷ 3 has no answer there. Squaring works on fractions. Taking roots does not: the square root of 2 is not a fraction. Each time, mathematics added a new number and the inverse worked again.

This essay asks whether the pattern goes on to the present day. It also asks where it will go next. A barrier is defined here in one sentence. A barrier is an operation whose inverse fails inside the current domain. A resolution is a larger domain in which the inverse works. The price is a property of the old domain that the new one loses.

The essay compares this history with one theory: the Vacuum.Net theory. That theory claims that mathematics can be grown from a single movement. Section 2 explains it. Section 3 gives the history. Section 4 sets the two side by side. Section 5 reports what was computed. Section 6 gives the predictions.

## 2. The Vacuum.Net theory in brief

The theory starts from movement, not from sets or numbers.[1] There is one strand. It moves and returns. Every distinction is a loop of this strand. Five axioms fix the rules. N1 says there is exactly one strand. N2 says a loop persists only if it closes. What does not close passes on as tension. This rule is called closure. N3 says the first passage through a point fixes a direction. N4 says the response of the medium depends on its state. N5 says that coarse blocks of the strand obey the same rules again.

Each step of the strand is a trit. It is R (forward, +1), O (standing still, 0) or L (backward, −1). An infinite stream of trits has two readings. As a balanced ternary address it names a place between −1/2 and +1/2. As a signed binary calculation it gives a value between −1 and +1.[2] Two different streams can name the same place. This is the edge of the representation.

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[1] [1] Konstapel, J. (2026). Vacuum.Net Theory, Ninth Edition: Foundations 9.2 and Physics 9.0. Constable Research, Leiden.

[2] [3] Konstapel, J. (2026, 20 November). From Movement to Higher Windings. constable.blog.

A book derives mathematics from these rules in 22 rungs.[3] Two rungs matter most here. Rung 10 puts the trits in the plane. It uses the Eisenstein integers $a + b\omega$, with $\omega$ a cube root of one. Its base is $1 - \omega$, the prime above 3. Since $3 = -\omega^2(1 - \omega)^2$, the number 3 is almost a square in this ring. Rung 11 is the node. Its group is the modular group PSL(2,Z), built from a turn of order 2 and a turn of order 3.

The theory sorts loops by how they wind. A research plan names six levels.[4] W0 is the circle: loops wind a whole number of times. W1 is the sphere. W2 is the Hopf construction, where a three-dimensional sphere winds around a two-dimensional one. W3 is the first twist, where winding twice equals not winding at all. W4 is the next such twist. W5 is order and chaos: the regular windings of Bott periodicity against the irregular homotopy groups of spheres. The plan calls W5 its last level.

The book has a stopping point at the edge. To turn every calculation into an address one needs LLPO. LLPO is a principle of choice. It says: of two numbers, at least one is zero, even if you cannot tell which. Constructive mathematics does not accept it.[5] So the theory itself already contains a barrier.

## 3. The chain of barriers

The table gives the main barriers in order. Each row names the inverse that failed, the new object, and the property that was lost. Dates are those of the decisive step, not of the first hint. The list is a selection. It follows the main line of algebra, number, space and logic, and then the homotopy theory of spheres, where the windings live.

| Barrier (inverse that fails) | Resolution | Property lost | Winding / rung |
|---|---|---|---|
| Subtraction on counting numbers | Zero and negative numbers (Brahmagupta, 628) | Every number counts something | W0: winding back (L) |
| Division on whole numbers | Fractions | Each number has a successor | Two readings of a stream |
| Square root of 2 | Real numbers (Eudoxus; Dedekind, 1872) | Every number has a finite name | The edge; LLPO |
| Square root of −1 | Complex numbers (Cardano 1545; Bombelli 1572) | Order (< no longer works) | Rung 10: the plane |
| Roots of the quintic | Groups (Abel 1826; Galois 1832) | Formulas for every solution | Rung 11: the node |
| Division in three dimensions | Quaternions (Hamilton, 1843) | Commutativity | 24-cell; W2 (Hopf) |
| Unique factorization (Fermat) | Ideals (Kummer 1847; Dedekind 1871) | Prime numbers as elements | Rung 10: prime $1 - \omega$ |
| The parallel postulate | Curved space (Lobachevsky, Bolyai, Riemann) | One geometry for all space | The node at n = 6 |
| Naive sets (Russell, 1901) | Axioms (Zermelo, 1908) | Every property defines a set | Closure (N2) |
| Completeness and decision | Incompleteness and computability (Gödel 1931; Turing 1936) | Every truth is provable | The edge: decision |
| The excluded middle | Constructive and homotopy type theory (Bishop 1967; HoTT 2013) | Proof by contradiction | Book proofs in Agda |
| Winding between spheres | Homotopy groups (Hopf 1931; Serre 1951) | Winding as one integer | W2–W4 |

[3] [2] Konstapel, J. (2026). Mathematical Theory Generated with the Vacuum.Net Theory, version 7.0 (reference edition, 19 November 2026). Constable Research, Leiden.

[4] [3] Konstapel, J. (2026, 20 November). From Movement to Higher Windings. constable.blog.

[5] [14] Bishop, E. (1967). Foundations of Constructive Analysis. McGraw-Hill. Bridges, D. and Richman, F. (1987). Varieties of Constructive Mathematics. Cambridge University Press.

| Barrier (inverse that fails) | Resolution | Property lost | Winding / rung |
|---|---|---|---|
| Regular vs irregular stems | Image of J; chromatic theory (Adams; Ravenel 1984) | A global answer | W5 |
| Telescope = chromatic | Disproved (Burklund, Hahn, Levy, Schlank, 2023) | Two ways to isolate periodicity agree | Beyond W5 |

*Table 1. Barriers, resolutions, the price paid, and the matching winding or rung. The last column is a reading (Section 4).*

## 3.1 Numbers

Counting numbers can be added. They cannot always be subtracted. Indian mathematics gave rules for zero and for negative quantities, written down by Brahmagupta in 628.[6] The price was the idea that every number counts things. A debt is not a heap.

Whole numbers can be multiplied but not always divided. Fractions solved this long before. They cost the idea that every number has a next one. Between two fractions there is always another.

Fractions can be squared. Their roots are often not fractions. The diagonal of a unit square is not a ratio of whole numbers. Greek mathematics handled such lengths with Eudoxus' theory of proportion and Euclid's Book X.[7] A full number system came only in 1872, with Dedekind's cuts.[8] The price was that most real numbers have no finite name. A real number is an infinite process. This is where LLPO later appears.

Real numbers can be squared. A negative number has no real root. Cardano met this in 1545 when solving cubic equations. Bombelli in 1572 calculated with such roots and got real answers.[9] The complex numbers followed. They cost order: there is no way to say that i is bigger or smaller than 1.

## 3.2 Structure

Equations up to degree four have formulas in roots. The quintic does not. Abel proved this in 1826. Galois showed why: the symmetries of the roots form a group, and some groups cannot be broken into simple steps.[10] The barrier moved from numbers to symmetries. The price was the hope of a formula for every solution.

Complex numbers divide in two dimensions. Hamilton looked for the same in three dimensions and failed. In 1843 he found it in four, with the quaternions. They cost commutativity: ij = −ji.[11] Frobenius proved in 1878 that the real numbers, the complex numbers and the quaternions are the only associative real division algebras. The stepping up stops after four dimensions unless more properties are given up.

Whole numbers factor into primes in one way only. In 1847 Lamé claimed a proof of Fermat's Last Theorem. It assumed the same for rings with roots of unity. That fails. Kummer repaired it with ideal

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[6] [4] Plofker, K. (2009). Mathematics in India. Princeton University Press.

[7] [5] Heath, T. L. (1908). The Thirteen Books of Euclid's Elements. Cambridge University Press.

[8] [6] Dedekind, R. (1872). Stetigkeit und irrationale Zahlen. Vieweg, Braunschweig.

[9] [7] Stillwell, J. (2010). Mathematics and Its History, 3rd ed. Springer.

[10] [8] Abel, N. H. (1826). Beweis der Unmöglichkeit, algebraische Gleichungen von höheren Graden als dem vierten allgemein aufzulösen. Journal für die reine und angewandte Mathematik 1, 65–84. Galois, É. (1846). Œuvres mathématiques, ed. J. Liouville. Journal de mathématiques pures et appliquées 11, 381–444.

[11] [9] Hamilton, W. R. (1844). On quaternions. Philosophical Magazine 25, 10–13. Frobenius, F. G. (1878). Über lineare Substitutionen und bilineare Formen. Journal für die reine und angewandte Mathematik 84, 1–63.

numbers; Dedekind made them ideals.[12] The price was that primes became sets rather than numbers. These cyclotomic rings, made by adding roots of unity, return at the end of the chain.

## 3.3 Space and logic

Euclid's fifth postulate could not be derived from the others. Lobachevsky and Bolyai built geometries where it fails. Riemann in 1854 made curvature a property of space itself.[13] The price was a single geometry for all space. Whether a space is curved one way, the other way or flat became a question of measurement.

Cantor showed that infinite sets come in sizes. Russell found in 1901 that the set of all sets that do not contain themselves cannot exist. Zermelo answered with axioms in 1908.[14] Gödel then showed in 1931 that any consistent system rich enough for arithmetic has true statements it cannot prove. Turing showed in 1936 that no procedure decides every question. Cohen showed in 1963 that the size of the continuum is not fixed by the usual axioms.[15] The price was the belief that every truth can be reached.

Brouwer and later Bishop built mathematics without the law of the excluded middle.[16] In this view a proof must construct its object. Principles such as LLPO become choices, not facts. Homotopy type theory, from 2013, joins this view with geometry: types are spaces and proofs are paths.[17] In 2024 it was shown that four extra axioms on top of this theory prove LLPO.[18] So the edge can be crossed by adding an axiom. It cannot be crossed by computing.

## 3.4 Windings of spheres

A loop on a circle winds a whole number of times. One expects the same for maps between spheres. Hopf showed in 1931 that the three-dimensional sphere winds around the two-dimensional one in an essential way.[19] Serre showed in 1951 that almost all such winding groups are finite.[20] They hold twists of order 2, 3, 24 and more. Toda tabulated them.[21] No simple rule produces the tables.

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[12][10] Edwards, H. M. (1977). Fermat's Last Theorem: A Genetic Introduction to Algebraic Number Theory. Springer.

[13][11] Gray, J. (1989). Ideas of Space: Euclidean, Non-Euclidean and Relativistic, 2nd ed. Oxford University Press.

[14][12] van Heijenoort, J., ed. (1967). From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931. Harvard University Press.

[15][13] Turing, A. M. (1936). On computable numbers, with an application to the Entscheidungsproblem. Proceedings of the London Mathematical Society 42, 230–265. Cohen, P. J. (1963). The independence of the continuum hypothesis. PNAS 50, 1143–1148.

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[17][15] The Univalent Foundations Program (2013). Homotopy Type Theory: Univalent Foundations of Mathematics. Institute for Advanced Study.

[18][16] Cherubini, F., Coquand, T., Geerligs, F. and Moeneclaey, H. (2024). A Foundation for Synthetic Stone Duality. arXiv:2412.03203.

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[20][18] Serre, J.-P. (1951). Homologie singulière des espaces fibrés. Annals of Mathematics 54, 425–505.

[21][19] Toda, H. (1962). Composition Methods in Homotopy Groups of Spheres. Princeton University Press.

Part of the tables is regular. Adams found it: the image of J. Its orders are denominators of Bernoulli numbers.[22] The number 24 divides each of them. The rest is irregular. Ravenel organised it by height, prime by prime, in his chromatic programme.[23] Devinatz, Hopkins and Smith proved most of his conjectures.[24] Hill, Hopkins and Ravenel used the method to solve the Kervaire invariant problem.[25] Explicit computation now reaches dimension 90.[26]

One conjecture stayed open: the telescope conjecture. It says that two ways to isolate the periodic part of a space agree. In 2023 Burklund, Hahn, Levy and Schlank proved that they differ at every prime and every height of at least 2.[27] This is the most recent barrier on the chain. Its tools are cyclotomic. Ravenel's newest papers try to understand it. They build an intermediate category between the two localizations. Their classical model is the field of p-adic numbers with a p-th root of unity added. Its uniformiser is ω − 1, and a power of it is a unit times p.[28][29]

## 4. Barriers and windings side by side

The last column of Table 1 is a reading. It says where each barrier would sit in the net. It does not claim that the net caused the history. The book itself found no link between the order of its rungs and the dates of the results it uses.[30] The claim is weaker. The same barriers appear in both orders.

W0 is the circle. Winding back is the trit L. So the first barrier, subtraction, is solved at the start: the net has a way back built in. The two readings of a stream and the edge contain the barrier of the reals. The step from calculation to address needs LLPO. This is the same barrier that logic met in Section 3.3.

Rung 10 is the complex plane in its Eisenstein form. It holds three barriers at once. It is a plane of complex numbers. It is a ring where factorization into primes works, by Kummer's methods. And its prime 1 − ω is a ramified prime: its square is 3 up to a unit. Ramified means that a prime of the old ring becomes a square in the new one.

Rung 11, the node, is where groups enter. Its finite images are the symmetry groups of Galois' barrier. Modulo 3 the node becomes SL(2,F3). This group has 24 elements. It is the group of units of the Hurwitz

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[22][20] Adams, J. F. (1960). On the non-existence of elements of Hopf invariant one. Annals of Mathematics 72, 20–104. Adams, J. F. (1966). On the groups J(X) IV. Topology 5, 21–71. Quillen, D. (1971). The Adams conjecture. Topology 10, 67–80.

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[25][23] Hill, M. A., Hopkins, M. J. and Ravenel, D. C. (2016). On the nonexistence of elements of Kervaire invariant one. Annals of Mathematics 184, 1–262.

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[27][28] Burklund, R., Hahn, J., Levy, I. and Schlank, T. M. (2023). K-theoretic counterexamples to Ravenel's telescope conjecture. arXiv:2310.17459.

[28][29] Ravenel, D. C. (2026). What are cyclotomic spectra and why do we need them? arXiv:2606.08109.

[29][30] Ravenel, D. C. (2026, 18 May). Cyclotomic extensions in stable homotopy theory. Preprint, University of Rochester.

[30][2] Konstapel, J. (2026). Mathematical Theory Generated with the Vacuum.Net Theory, version 7.0 (reference edition, 19 November 2026). Constable Research, Leiden.

quaternions, and its elements are the corners of the 24-cell.[31] Here Hamilton's barrier and Hopf's winding (W2) meet in one object.

The node also decides the barrier of Euclid. If one forces the turn T of the node to have order n, one gets the triangle group (2,3,n). Its curvature is measured by $\chi = 1/2 + 1/3 + 1/n - 1 = (6 - n)/(6n)$.[32] For n below 6 it is positive and the group is finite: a sphere. For n = 6 it is zero: the flat plane of Rung 10. For n above 6 it is negative: hyperbolic space. The hexagon is the line between Euclid and Lobachevsky. The same number gives 24: for SL(2,Z) the Euler characteristic is $\zeta(-1) = -1/12$.[33]

W3 to W5 are the windings of Section 3.4. W5 is the split between the regular image of J and the irregular rest. The irregular part seen by topological modular forms lives only at the primes 2 and 3.[34] These are the orders of the two turns of the node. At the prime 3 the irregular part is built from a finite group G24 of order 24.[35] This group sits in a quaternion algebra in which 3 has a square root.[36]

## 5. What was computed for this essay

Six results come from the research behind this essay.[37] None is new mathematics. All are either computations or readings of known results. They are listed with their status.

- Computed. The 24-cell, seen as the group SL(2,F3), maps onto four points with fibres that are hexagons. A simplicial version has Hopf invariant 1. The fibration itself is known; the trit reading is not in the literature checked.
- Known, read anew. The commutator subgroup of the modular group is free on two generators. Its quotient is a punctured torus of hexagonal type, the curve with j = 0.[38] Filling the puncture gives the Eisenstein torus of Rung 10. Dividing by $1 - \omega$ there is the inverse of a map of degree 3. Its kernel is the three trits.
- Computed. Inside the node, over Z/3 and Z/9, the turn U of order 6 is never conjugate to its inverse. Only a mirror (determinant −1) swaps them. This answers the book's Open 7: inside the node the choice between ω and its conjugate cannot be derived.
- Computed. In the ramified quaternion order at the prime 3, an inner element t of order 4 swaps ω and its conjugate. Applying it twice gives −1, not 1. The swap has monodromy −1, as the book suggested by analogy.

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[31] [32] Coxeter, H. S. M. (1973). Regular Polytopes, 3rd ed. Dover.

[32] [32] Coxeter, H. S. M. (1973). Regular Polytopes, 3rd ed. Dover.

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[36] [27] Henn, H.-W. (2017). A mini-course on Morava stabilizer groups and their cohomology. arXiv:1702.05033.

[37] [31] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 0.1–1.9, with Python scripts and cubical Agda files. Constable Research, Leiden (unpublished).

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- Computed. Inside that same order, ω satisfies $3 = -\omega^2(1-\omega)^2$, the identity of Rung 10. The group of 24 that builds W5 at the prime 3 has a normal subgroup C3 generated by this ω.[39] The node over the 3-adic trits has a different group of 24, with Q8 normal. Same parts, glued the other way.
- Computed. The book asked whether the share of Eisenstein integers that never reach 0 under division by $1-\omega$ settles (IX.3). Over 14.5 million points in discs, it does not. It oscillates between about 29.6 % and 32.0 %, with period one digit (norm × 3). The pattern repeats to four decimals. No source for this count was found.

Together these results point to one object. It is the Eisenstein plane with its ramified prime $1-\omega$, seen from the node. The latest barrier of mathematics uses the same object as its classical model.[40]

## 6. A prediction of the next barrier

The pattern of Section 3 has three parts. An inverse fails. Mathematics steps back to the point of failure and adds an object that squares or inverts to the obstruction. The new domain loses a property. The telescope barrier fits. The two localizations cannot be turned into each other. The repair now under way adds roots of unity in a ramified way. The question is which property will be lost and where the next inverse will fail.

The net gives three predictions. They are readings. Each states how it could be shown wrong.

- Prediction 1 (place). The hardest cases of the telescopic theory will remain at the primes 2 and 3 and at height 2. At the prime 3 the decisive computation will run through the ramified field obtained by adding ω to the 3-adic numbers, and through a group of order 24 with a normal C3. The prediction fails if the prime 3 at height 2 turns out to behave like the primes 5 and above.
- Prediction 2 (shape). Quantities that separate the two localizations will not settle to limits. They will oscillate, with a period of one power of the uniformiser, as the share in IX.3 does. The prediction fails if such quantities are shown to converge.
- Prediction 3 (the next barrier). Ravenel asks whether a telescopic analogue of the Morava stabilizer group exists. He notes that none is known or even expected.[41] The net predicts that it will not exist as a group. Its place will be taken by a relation of 'same place', like the edge of the trits. Then the next barrier is not computational but logical. To name the telescopic part one will need a decision of the type of LLPO. The repair will come from type theory with added axioms, as in synthetic Stone duality.[42] The prediction fails if a telescopic stabilizer group is constructed by computation alone.

If the third prediction holds, the chain closes a loop. It began with numbers, passed through space and logic, and ended in the windings of spheres. The next barrier would bring it back to logic: to the question of what may be decided without being computed. In the language of the net, the strand returns to the edge it started from.

## 7. Conclusion

Mathematics has met at least fourteen major barriers. Each was an inverse that failed. Each was solved by a step back and a new object. Each solution cost a property. The windings of the Vacuum.Net theory pass through the same barriers in their own order. The match is a reading, supported at several points

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[39][26] Goerss, P., Henn, H.-W., Mahowald, M. and Rezk, C. (2005). A resolution of the K(2)-local sphere at the prime 3. Annals of Mathematics 162, 777–822. See also the related text arXiv:0706.2175.

[40][30] Ravenel, D. C. (2026, 18 May). Cyclotomic extensions in stable homotopy theory. Preprint, University of Rochester.

[41][30] Ravenel, D. C. (2026, 18 May). Cyclotomic extensions in stable homotopy theory. Preprint, University of Rochester.

[42][16] Cherubini, F., Coquand, T., Geerligs, F. and Moeneclaey, H. (2024). A Foundation for Synthetic Stone Duality. arXiv:2412.03203.

by computation. Its most concrete point is the ramified prime 1 − ω of Rung 10. It is the classical model of the tool now used on the newest barrier. The predictions of Section 6 can be checked as the telescopic theory develops.

## Annotated references


**[1]** Konstapel, J. (2026). Vacuum.Net Theory, Ninth Edition: Foundations 9.2 and Physics 9.0. Constable Research, Leiden.

*The source of the axioms N1–N5 used in Section 2. Read it for the full statement of strand, closure, memory, context and scale.*

**[2]** Konstapel, J. (2026). Mathematical Theory Generated with the Vacuum.Net Theory, version 7.0 (reference edition, 19 November 2026). Constable Research, Leiden.

*The book of 22 rungs. It gives Rung 10 (the Eisenstein plane), Rung 11 (the node), the edge at LLPO and the open points Open 7 and IX.3. Each statement carries a status label.*

**[3]** Konstapel, J. (2026, 20 November). From Movement to Higher Windings. constable.blog.

*Defines the winding levels W0–W5 as a research plan. Useful to see which levels are classical facts and which links to the net are still open.*

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*Standard history of Indian mathematics. Covers Brahmagupta (628) and his rules for zero and negative quantities.*

**[5]** Heath, T. L. (1908). The Thirteen Books of Euclid's Elements. Cambridge University Press.

*Translation with commentary. Book V gives Eudoxus' theory of proportion; Book X treats incommensurable magnitudes.*

**[6]** Dedekind, R. (1872). Stetigkeit und irrationale Zahlen. Vieweg, Braunschweig.

*Defines real numbers by cuts. The first complete answer to the barrier of the square root of two.*

**[7]** Stillwell, J. (2010). Mathematics and Its History, 3rd ed. Springer.

*A single-volume history that follows problems rather than people. It covers cubic equations, complex numbers, the quintic, quaternions, ideals and non-Euclidean geometry. Best starting point for Section 3.*

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*Abel proves the quintic has no general solution by radicals. Galois explains why, using groups.*

**[9]** Hamilton, W. R. (1844). On quaternions. Philosophical Magazine 25, 10–13. Frobenius, F. G. (1878). Über lineare Substitutionen und bilineare Formen. Journal für die reine und angewandte Mathematik 84, 1–63.

*Hamilton gives up commutativity to divide in more dimensions. Frobenius proves that the real numbers, the complex numbers and the quaternions are the only associative real division algebras.*

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*Tells how Lamé's 1847 proof failed on unique factorization and how Kummer's ideal numbers repaired it. Central for the link to cyclotomic rings such as Z[ω].*

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*History of the parallel postulate, Lobachevsky, Bolyai and Riemann's 1854 lecture on curved space.*

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*Contains Russell's 1902 letter to Frege, Zermelo's 1908 axioms and Gödel's 1931 incompleteness paper, with introductions.*

**[13]** Turing, A. M. (1936). On computable numbers, with an application to the Entscheidungsproblem. Proceedings of the London Mathematical Society 42, 230–265. Cohen, P. J. (1963). The independence of the continuum hypothesis. PNAS 50, 1143–1148.

*Turing shows that some questions cannot be decided by any procedure. Cohen shows the continuum hypothesis is independent of the usual axioms.*

**[14]** Bishop, E. (1967). Foundations of Constructive Analysis. McGraw-Hill. Bridges, D. and Richman, F. (1987). Varieties of Constructive Mathematics. Cambridge University Press.

*Bishop builds analysis without the law of the excluded middle. Bridges and Richman define the omniscience principles, including LLPO.*

**[15]** The Univalent Foundations Program (2013). Homotopy Type Theory: Univalent Foundations of Mathematics. Institute for Advanced Study.

*The book of homotopy type theory. Types are spaces and equalities are paths. This is the language of the Agda proofs of the project.*

**[16]** Cherubini, F., Coquand, T., Geerligs, F. and Moeneclaey, H. (2024). A Foundation for Synthetic Stone Duality. arXiv:2412.03203.

*Homotopy type theory plus four axioms. The axioms prove LLPO and Markov's principle and allow cohomology. Shows that the LLPO edge can be crossed by adding axioms rather than by computing.*

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