

Closure on a Ring

The Bethe Ansatz, the Work of J.-S. Caux, and the Closure Rule of the Vacuum.Net

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1. Introduction

In 1931 Hans Bethe solved a chain of magnetic atoms exactly. His method rests on one condition. A wave that runs around a ring must return in phase with itself. If it does, the state exists. If it does not, there is no state.

Jean-Sébastien Caux, professor at the Institute for Theoretical Physics of the University of Amsterdam, has spent two decades turning that condition into numbers. His list at the university holds 77 refereed publications from 2008 to 2025. Nearly all of them concern one subject: chains in one dimension that can be solved exactly. His results have been checked against neutron scattering, cold atoms and chains built atom by atom.

The Vacuum.Net theory starts from the same condition. What returns onto itself in phase persists. What does not close, scatters.

This essay lays the two side by side. It states the rule of the net. It states Bethe's condition. It shows by counting that they are one rule in two notations. It then reads the main groups of results in the publication list in the terms of the net.

2. The net in brief

The picture is a fishnet in water. One strand runs everywhere and crosses itself. The strands are under tension. Where strands wind around each other, knots form. Between the strands lie meshes.

Four words carry the theory.

- **Strand:** the net itself.
- **Winding:** a strand closed on itself.
- **Knot:** windings that hold each other in three directions.
- **Mesh:** the opening between strands.

At every crossing the strand has one of three states: +1, 0 or -1. This unit is the trit. Three crossings in a row form a window. A window is **closed** when its three values sum to zero, counted in threes. An open window carries tension. Only what closes, stays.

Three states on three places give $3 \times 3 \times 3 = 27$ windows. Nine of them close.

- Three equal values: (+,+,+), (0,0,0), (-,-,-). That is 3 windows.
- Three different values: the six orders of (-, 0, +). That is 6 windows.

The other 18 are open.

A ring of three different values has no starting point. Six orders divided by three rotations gives two rings.

- **Left ring:** $- \rightarrow 0 \rightarrow +$
- **Right ring:** $+ \rightarrow 0 \rightarrow -$

They are mirror images. Their content is the same. Only the direction differs.

3. Bethe's condition

Take N particles on a ring of circumference L . In the model of Lieb and Liniger (1963) the particles repel each other on contact with strength c . Particle j has wave number k_j . Bethe's condition reads:

$$k_j \cdot L + \sum 2 \cdot \arctan((k_j - k_l) / c) = 2\pi \cdot I_j$$

The sum runs over all other particles l . The number I_j is an integer, or a half-integer when N is even.

The equation has three parts.

- $k_j \cdot L$ is the phase the particle collects along the ring.
- **Each term in the sum** is the phase shift from one crossing with one other particle.
- $2\pi \cdot I_j$ says that the total must be a whole number of turns.

In the terms of the net: a strand runs once around. On the way it crosses every other strand. Each crossing adds its phase. The strand must arrive in phase with itself. That is closure.

The numbers I_j are the address of the state. A set of N different numbers fixes one state, and every state has one set. The address is computed, never assigned.

Caux built a program on this. It is called ABACUS (Caux, 2009). It walks through the addresses one by one. For each address it solves the condition and computes what that state contributes to a measurement. The method is counting, address by address. The standard text of the field, Gaudin's *La fonction d'onde de Bethe*, was translated into English by Caux in 2014.

4. From 2π to three

Bethe's phase is continuous. The trit has three states. The bridge is one line.

Give each trit a phase of 120° per unit. The value $+1$ is 120° . The value 0 is 0° . The value -1 is -120° . A window of three trits a, b, c then has total phase $(a + b + c) \times 120^\circ$.

- Sum 0: total phase 0° .
- Sum +3: total phase 360° .
- Sum -3: total phase -360° .
- Sum ± 1 or ± 2 : total phase $\pm 120^\circ$ or $\pm 240^\circ$.

The first three are whole turns. The last are not. So "sum zero, counted in threes" is the same statement as "total phase is a whole number of turns". The closed window is Bethe's condition for a strand that has three phases.

5. Three strands

Why can these chains be solved exactly? The answer is again a window of three.

Take three particles on a line. They can meet in two orders. First 1 with 2, then 1 with 3, then 2 with 3. Or first 2 with 3, then 1 with 3, then 1 with 2. A chain is exactly solvable when both orders give the same result:

$$S_{12} \cdot S_{13} \cdot S_{23} = S_{23} \cdot S_{13} \cdot S_{12}$$

This is the Yang-Baxter equation (Yang, 1967). When it holds, every meeting of many particles falls apart into meetings of two. Nothing is lost and nothing is added.

Draw the particles as strands and the meetings as crossings. The equation then says that one strand may slide across the crossing of two others. Knot theory knows this move as the third Reidemeister move (Kauffman, 1991). The condition for exact solvability is the condition under which a braid of three strands keeps its form.

Caux and Mossel (2011) examined what "integrable" means for quantum systems and found that the usual definitions do not hold. Read with the net, the working definition is short. A chain is integrable when every three-strand window can be crossed in either order.

6. The trit chain

One model in the publication list is a chain of trits in the literal sense. Vlijm and Caux (2014) compute the correlations of the spin-1 chain of Babujian and Takhtajan. Every site of this chain has three states: +1, 0 and -1.

Take three sites. Count.

Step 1. There are $3^3 = 27$ states.

Step 2. States with sum zero: (0,0,0) and the six orders of (-1, 0, +1). That is 7.

Step 3. States with sum +3 or -3: (+,+,+) and (-,-,-). That is 2.

Step 4. Together $7 + 2 = 9$. These are exactly the nine closed windows of Section 2.

Physics sorts the same 27 states by total spin. Three spins of size 1 combine to total spin 3, 2, 1 or 0. The numbers of states are:

$$27 = 7 + 5 + 5 + 3 + 3 + 3 + 1$$

There is one block of spin 3, two of spin 2, three of spin 1 and one of spin 0. Each block contains exactly one state with sum zero. Check: $1 + 2 + 3 + 1 = 7$. That is the 7 of Step 2.

The last block is the one that matters. It holds a single state with total spin zero. A state of spin zero does not change under any rotation. It adds nothing to the outside. It is complete.

Which state is it? It must change sign when two sites are swapped. Three sites with three states allow exactly one such combination:

$$|\text{spin } 0\rangle = (-,0,+) + (0,+,-) + (+,-,0) - (+,0,-) - (0,-,+) - (-,+,0)$$

Read the terms. The first three are the left ring in its three rotations. The last three are the right ring in its three rotations. So:

lspin 0 = left ring – right ring

The six windows that close with three different values form, with these signs, the one state of three trits that is complete in itself. The two directions enter with opposite sign.

The three windows with equal values have their own place. The states (+,+,+) and (−,−,−) are the reference states of Bethe's method. Every other state is built by counting steps away from them.

7. The seam

The best-known result in the list concerns the spinon.

Take a chain of spins that alternate: $\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow$. Flip the fourth spin:

$\uparrow \downarrow \uparrow \uparrow \uparrow \downarrow \uparrow \downarrow$

Two places now show equal neighbours: between sites 3 and 4, and between sites 4 and 5. Each is a seam in the pattern. Let the neighbours of a seam swap, and the seam moves two sites:

$\uparrow \downarrow \uparrow \uparrow \downarrow \uparrow \uparrow \downarrow$

The two seams drift apart. Each travels alone. The flip changed the total spin by 1. Each seam carries half of it. This is the spinon: a carrier of spin $\frac{1}{2}$ that exists only as one of a pair (Faddeev and Takhtajan, 1981).

Now take the net. Put a left ring against a right ring:

− , 0 , + , + , 0 , −

Window	Sum	State
(−, 0, +)	0	closed
(0, +, +)	2	open
(+, +, 0)	2	open
(+, 0, −)	0	closed

One seam gives two open windows. They come as a pair, as the spinons do.

The numbers are known. Pairs of two spinons carry 72.89% of the total signal of the chain (Karbach and colleagues, 1997). Caux and Hagemans (2006) added the groups of four. Together they reach about 98%. Neutron measurements on copper sulphate confirmed the count (Mourigal and colleagues, 2013). A second experiment in the list followed the spinons at finite temperature (Lake and colleagues, 2013).

The net adds a reading of the half. In the net an electron is a winding that must turn twice before it meets itself. One turn is half a closure. A seam is where one such half is exposed.

In a chain the properties of one electron can also travel apart. Schlappa and colleagues (2012), with Caux, measured in Sr_2CuO_3 that the spin and the orbital state of one excited electron move as two

separate carriers. In the net, spin, charge and magnetic moment are three views of one knot. On a single strand the views separate.

8. What a crossing leaves behind

A soliton is a wave that keeps its form. Two solitons that meet pass through each other. Afterwards each has the same form and speed as before. One thing has changed: each is displaced by a fixed distance.

Doyon, Yoshimura and Caux (2018) showed that a gas of such solitons obeys the same equations as the quantum chains above. The theory is called generalized hydrodynamics. Its core is one calculation.

Let a tracer move with speed $v = 1$ through a gas of resting solitons. The gas has density $n = 0.1$ per unit length. Every crossing shifts the tracer forward by $\Delta = 2$.

- In one unit of time the tracer covers a distance v_{eff} .
- On that distance it meets $n \cdot v_{\text{eff}}$ solitons.
- Each meeting adds Δ . So $v_{\text{eff}} = v + \Delta \cdot n \cdot v_{\text{eff}}$.
- Solve: $v_{\text{eff}} = v / (1 - n \cdot \Delta) = 1 / (1 - 0.2) = 1.25$.

The tracer moves 25% faster than its own speed. Double the density and the result is $1 / (1 - 0.4) = 1.67$.

The speed of a carrier is therefore a function of the state of its surroundings. The fourth axiom of the Vacuum.Net theory says the same about every constant: it is a property of the local state of the net. In these chains that statement is exact and measured.

Two further papers move the state on purpose. Bastianello, Alba and Caux (2019) let the coupling vary in place and time. Koch, Bastianello and Caux (2021) turn the coupling slowly from repelling to attracting. Free particles then bind into groups, called strings, that travel as one. Windings come to hold each other when the state of the medium is changed and held.

9. A chain that does not forget

Pour hot water into cold and the mixture forgets where it came from. An exactly solvable chain does not.

In 2006 Kinoshita, Wenger and Weiss set two clouds of cold atoms swinging against each other in a line. After thousands of collisions the clouds were still swinging. The experiment is called the quantum Newton's cradle. Caux and colleagues computed it twice (van den Berg and colleagues, 2016; Caux and colleagues, 2019).

The reason lies in the address. Every closed mode keeps its number. A chain of N particles has N such numbers. None of them is lost in a collision. Caux and Konik (2012) showed how to build the final state from these numbers. Ilievski and colleagues (2015) showed that the list is complete only when one family of numbers is kept for every length of string.

The net states this as its third axiom. The net remembers how it was laid. A knot forgets how hard it was pulled, never that it was tied.

A related result concerns selection. Change the coupling of a chain in one stroke. Which state survives? Caux and Essler (2013) showed that one state does, fixed by how well each address fits

the starting pattern. For the alternating pattern $\uparrow \downarrow \uparrow \downarrow$ the fit is strict. Only states whose wave numbers come in pairs $+k$ and $-k$ fit at all (Brockmann and colleagues, 2014). Everything else has weight zero. The survivor is selected by fitting, and what fits is the mirror pair.

10. Drive, disturbance and direction

Three smaller groups in the list complete the picture.

Drive. Claeys and colleagues (2018) switch a system of spins back and forth with a fixed period. At most periods little happens. At periods that fit the level spacing of the system, the spins line up. Only a drive that returns in phase leaves a mark.

Disturbance. Brandino, Caux and Konik (2015) disturb an exactly solvable gas slightly. Most of its conserved numbers survive in adjusted form. Closure is not fragile. It bends before it opens.

Direction. Veenstra and colleagues (2025), with Caux, build a mechanical chain in which the push from left to right differs from the push from right to left. The chain carries solitons that pulse with their own period and run one way. This is a winding with a clock and a fixed direction, on a laboratory table.

11. Place on the ladder

The net has a ladder: strand, winding, knot, mesh. The chains in this essay are one strand with crossings. They sit on the rung of the winding.

That fixes what the comparison covers. It covers the closure rule, the address, the memory of the net and the constant as a state function. Knots, which close in three directions, and meshes do not occur in a single chain. They belong to the next rungs.

12. Where the count continues

Three places lie open.

The chain of three phases. Fateev and Zamolodchikov (1982) found an exactly solvable chain in which every site has three phases of 120° . In that chain the rule of Section 4 and the method of Section 3 are the same object. The publication list does not contain it. The tools in the list fit it.

Rings of three and six. In Delft, chains of magnetic atoms are laid one atom at a time (Toskovic and colleagues, 2016, with Caux). The net counts that a ring of three different values cannot grow by one site. It grows by three, to six, and a ring of six divides into two rings of three. A ring of three and a ring of six spin-1 atoms would put that count on a surface.

The one-way chain. The mechanical chain of Section 10 has a built-in direction. The net counts that a strand of three different values cannot reverse direction without an open window. A chain that is one-way by construction is the place to see a seam.

13. Status

- **From sources.** Bethe's condition. The Yang-Baxter equation. The spinon and its share of 72.89%. The soliton gas. The Newton's cradle. The pair rule for the alternating pattern.

- **Counted here.** The 27 states and the 9 closed windows as sum 0 and sum ± 3 . The split $27 = 7 + 5 + 5 + 3 + 3 + 3 + 1$. The spin-zero state as left ring minus right ring. The effective speed of 1.25.
- **Read.** Bethe's condition as closure. Yang-Baxter as the three-strand window. The spinon as the seam. The state-dependent speed as the fourth axiom. The conserved numbers as the memory of the net.
- **Open.** The count for the chain of three phases. The rings of three and six on a surface.

Annotated references

The net

Konstapel, J. (2026). *The Vacuum.Net Theory: Foundational Paper*, fifth edition. Constable Research, Leiden, 6 August 2026. *Why read it?* The one assumption, the selection rule and the axioms used in Sections 8 and 9. *Reading advice:* start with the statement of closure, then the axioms on memory and on state.

Konstapel, J. (2026). *Life and Consciousness from One Rule: Closure in the Net*. Constable Research, Leiden, 29 September 2026. *Why read it?* The source of the window, the 27 and the 9, the two kinds of closure and the step $3 \rightarrow 6 \rightarrow 3 + 3$. It also gives the mapping of trits to phases of 120° used in Section 4. *Reading advice:* the tables per depth.

Konstapel, J. (2026). *One Direction or None: Mirror Molecules, the 2026 Nobel Prize in Chemistry and the Closure Rule of the Vacuum.Net*. Constable Research, Leiden, 10 October 2026. *Why read it?* The left ring, the right ring and the seam table repeated in Section 7. *Reading advice:* Section 4 on the open seam.

Konstapel, J. (2026). *The Vacuum.Net Theory, complete derivation*. Constable Research, Leiden, 28 September 2026. *Why read it?* The ladder strand, winding, knot, mesh, on which Section 11 places the chains.

The condition

Bethe, H. (1931). "Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette." *Zeitschrift für Physik* 71, 205–226. *Why read it?* The origin. One chain, one condition, solved by counting. *Reading advice:* the passage where the integers first appear; they are the addresses.

Lieb, E. H., and Liniger, W. (1963). "Exact analysis of an interacting Bose gas. I." *Physical Review* 130, 1605–1616. *Why read it?* The form of the condition printed in Section 3, for particles on a ring. Clear and short.

Gaudin, M., translated by Caux, J.-S. (2014). *The Bethe Wavefunction*. Cambridge University Press. *Why read it?* The full method in one book. *Reading advice:* chapters 1 and 4; the translator's notes show where the field has moved since 1983.

Yang, C. N. (1967). "Some exact results for the many-body problem in one dimension with repulsive delta-function interaction." *Physical Review Letters* 19, 1312–1315. *Why read it?* The three-particle condition of Section 5, in four pages.

Kauffman, L. H. (1991). *Knots and Physics*. Singapore: World Scientific. *Why read it?* Shows that the Yang-Baxter equation and the sliding of a strand over a crossing are one statement. *Reading advice:* Part I, sections 8 to 11.

Caux, J.-S., and Mossel, J. (2011). "Remarks on the notion of quantum integrability." *Journal of Statistical Mechanics* P02023. *Why read it?* An honest account of why the word "integrable" has no settled meaning in quantum physics. The background to the short definition in Section 5.

The count

Caux, J.-S. (2009). "Correlation functions of integrable models: a description of the ABACUS algorithm." *Journal of Mathematical Physics* 50, 095214. *Why read it?* How to walk through the addresses in order of weight. The working method behind most of the list. *Reading advice:* the section on scanning.

Vlijm, R., and Caux, J.-S. (2014). "Computation of dynamical correlation functions of the spin-1 Babujian–Takhtajan chain." *Journal of Statistical Mechanics* P05009. *Why read it?* The chain with three states per site, computed in full. The model of Section 6.

Takhtajan, L. A. (1982). *Physics Letters A* 87, 479–482. **Babujian, H. M. (1982).** *Physics Letters A* 90, 479–482. *Why read them?* The two papers that found the solvable chain of three states. A few pages each.

The seam

Faddeev, L. D., and Takhtajan, L. A. (1981). "What is the spin of a spin wave?" *Physics Letters A* 85, 375–377. *Why read it?* Three pages that established the carrier of spin $\frac{1}{2}$ and that it comes in pairs.

Karbach, M., Müller, G., Bougourzi, A. H., Fledderjohann, A., and Mütter, K.-H. (1997). "Two-spinon dynamical structure factor of the one-dimensional $s = 1/2$ Heisenberg antiferromagnet." *Physical Review B* 55, 12510. *Why read it?* The source of the 72.89%.

Caux, J.-S., and Hagemans, R. (2006). "The four-spinon dynamical structure factor of the Heisenberg chain." *Journal of Statistical Mechanics* P12013. *Why read it?* The groups of four, which bring the count to about 98%.

Mourigal, M., Enderle, M., Klöpperpieper, A., Caux, J.-S., Stunault, A., and Rønnow, H. M. (2013). "Fractional spinon excitations in the quantum Heisenberg antiferromagnetic chain." *Nature Physics* 9, 435–441. *Why read it?* The measurement. Neutrons on copper sulphate against the count, in absolute units. *Reading advice:* the figure that sets measured and computed intensity side by side.

Lake, B., Tennant, D. A., Caux, J.-S., and colleagues (2013). "Multispinon continua at zero and finite temperature in a near-ideal Heisenberg chain." *Physical Review Letters* 111, 137205. *Why read it?* The same carriers followed as the chain warms.

Schlappa, J., and colleagues (2012). "Spin–orbital separation in the quasi-one-dimensional Mott insulator Sr_2CuO_3 ." *Nature* 485, 82–85. *Why read it?* The properties of one electron seen travelling apart.

State and memory

Doyon, B., Yoshimura, T., and Caux, J.-S. (2018). "Soliton gases and generalized hydrodynamics." *Physical Review Letters* **120**, 045301. *Why read it?* The calculation of Section 8 in its general form: every crossing is a shift, and the speed follows from the density. *Reading advice:* the first two pages.

Bastianello, A., Alba, V., and Caux, J.-S. (2019). "Generalized hydrodynamics with space-time inhomogeneous interactions." *Physical Review Letters* **123**, 130602. *Why read it?* The coupling itself made a function of place and time.

Koch, R., Bastianello, A., and Caux, J.-S. (2021). "Adiabatic formation of bound states in the one-dimensional Bose gas." *Physical Review B* **103**, 165121. *Why read it?* Free particles bound into strings by changing the state slowly.

Kinoshita, T., Wenger, T., and Weiss, D. S. (2006). "A quantum Newton's cradle." *Nature* **440**, 900–903. *Why read it?* The experiment: a gas that keeps swinging.

Caux, J.-S., and Konik, R. M. (2012). "Constructing the generalized Gibbs ensemble after a quantum quench." *Physical Review Letters* **109**, 175301. *Why read it?* How the conserved numbers fix the final state.

Ilievski, E., De Nardis, J., Wouters, B., Caux, J.-S., Essler, F. H. L., and Prosen, T. (2015). "Complete generalized Gibbs ensembles in an interacting theory." *Physical Review Letters* **115**, 157201. *Why read it?* The list of conserved numbers made complete: one family per string length.

Caux, J.-S., and Essler, F. H. L. (2013). "Time evolution of local observables after quenching to an integrable model." *Physical Review Letters* **110**, 257203. *Why read it?* Selection of one surviving state by fit.

Brockmann, M., De Nardis, J., Wouters, B., and Caux, J.-S. (2014). "A Gaudin-like determinant for overlaps of Néel and XXZ Bethe states." *Journal of Physics A* **47**, 145003. *Why read it?* The pair rule: only wave numbers in pairs $+k, -k$ fit the alternating pattern.

Drive, disturbance, direction

Claeys, P. W., De Baerdemacker, S., El Araby, O., and Caux, J.-S. (2018). "Spin polarization through Floquet resonances in a driven central spin model." *Physical Review Letters* **121**, 080401. *Why read it?* A drive that works only at fitting periods.

Brandino, G. P., Caux, J.-S., and Konik, R. M. (2015). "Glimmers of a quantum KAM theorem." *Physical Review X* **5**, 041043. *Why read it?* What survives when exact solvability is slightly broken.

Veenstra, J., Gamayun, O., Brandenbourger, M., van Gorp, F., Terwisscha-Dekker, H., Caux, J.-S., and Coulais, C. (2025). "Nonreciprocal breathing solitons." *Physical Review X* **15**, 031045. *Why read it?* A one-way chain with pulsing solitons, built and filmed. *Reading advice:* watch the supplementary video first.

Where the count continues

Fateev, V. A., and Zamolodchikov, A. B. (1982). "Self-dual solutions of the star-triangle relations in Z_N -models." *Physics Letters A* **92**, 37–39. *Why read it?* The exactly solvable chain with three phases per site. Three pages.

Toskovic, R., van den Berg, R., Spinelli, A., Eliens, I. S., van den Toorn, B., Bryant, B., Caux, J.-S., and Otte, A. F. (2016). "Atomic spin-chain realization of a model for quantum criticality." *Nature Physics* 12, 656–660. *Why read it?* Chains laid atom by atom and measured site by site. *Reading advice:* the first figure shows the chains as built.

University of Amsterdam. Profile page of Prof. dr. J.-S. Caux, publication list. <https://www.uva.nl/profiel/c/a/j.s.caux/j.s.caux.html> *Why read it?* The full list of 77 refereed publications on which this essay draws, most with open downloads.