Closure Without Alignment: Exact Results for Balanced-Ternary Closure Dynamics and Their Relation to the Physics of Swarming

J. Konstapel, Leiden, 24 September 2026

Abstract

Swarm physics treats collective order as alignment: neighbours become alike. Vacuum.Net proposes a second kind of order, closure: neighbours complete each other. This article gives exact results for the simplest closure dynamics on a ring of balanced-ternary states $T_i \in \{-1, 0, +1\}$.

Four results are proven:

1. For a random ring updated asynchronously, the closed fraction after one step is $C_1(p) = 1/3 + (2/3)\cdot p(1-p)^2$. Its maximum lies exactly at $p = 1/3$.
2. Synchronous updating ($p = 1$) returns closure to chance level, $C = 1/3$.
3. Closure rises while alignment stays at chance level. Closure and alignment are therefore independent order parameters.
4. Only under the mod-3 closure rule do the open windows form a closed ternary system of their own. They are trits that split and annihilate, with their sum conserved mod 3. The trit register closes over levels.

The exact stationary state and a control test sharpen two claims. The stationary optimum lies near $p \approx 0.2$–0.25, not at $1/3$. A plain integer zero-sum rule also produces heterogeneous closure. What is unique to the mod-3 rule is the self-closing register, not the closure level. In one dimension that register yields no bulk invariant. The decisive test therefore moves to two-dimensional diagrams with real crossings.

1. Introduction

The Vicsek model shows that self-driven particles that copy the heading of their neighbours pass from disorder to collective motion. The Kuramoto model shows the same for coupled phases. In both cases order means sameness.

Starling data point further. Each bird interacts with a fixed number of neighbours, about six or seven, regardless of distance (Ballerini et al., 2008). Velocity fluctuations correlate across the whole flock (Cavagna et al., 2010). The flock behaves as one connected net, not as a crowd of copiers.

Vacuum.Net starts from that net. Picture a fishing net. The knots hold because each knot is tied from strands that differ, not from strands that are the same. A knot of one strand folded on itself does not hold. In Vacuum.Net each crossing carries a relation with three states. A crossing holds when its three relations close.

An earlier version of this work tested closure inside a phase model with an operator designed for the purpose. There, closure feedback doubled the lifetime of closed configurations near coupling $K \approx 1$. That version named its own limits. The operator was provisional, and part of the order could come from ordinary synchronisation. It proposed one next step: replace the operator with the exact trit algebra and look for closure without synchronisation.

This article takes that step. It uses no phase coupling and no feedback. The rule is the trit algebra itself.

2. Definitions

State. A ring of N positions. Each position carries a trit $T_i \in \{-1, 0, +1\}$. Indices run modulo N.

Window. $W_i = (T_{i-1}, T_i, T_{i+1})$, the local crossing around position i.

Window charge. $D_i = \mathrm{bal}(T_{i-1} + T_i + T_{i+1})$. Here bal(x) maps x mod 3 to its representative in $\{-1, 0, +1\}$.

Closure. Window i is closed when $D_i = 0$. This is Fox's tricolouring condition for three arcs at a crossing, written in balanced ternary.

Closed triples. Of the 27 triples, 9 are closed:

- the 6 permutations of $(-1, 0, +1)$, called *heterogeneous*;
- the 3 constant triples (a, a, a), since $3a \equiv 0$, called *homogeneous*.

Observables.

- R: the fraction of adjacent pairs with $T_{i-1} = T_i$ (alignment).
- C: the fraction of closed windows.

- C_{het}: the fraction of heterogeneously closed windows.

For a uniformly random ring: $R = 1/3$, $C = 9/27 = 1/3$, $C_{het} = 6/27 = 2/9 \approx 0.222$.

Dynamics. At each step every position is selected independently with probability p. A selected position takes its new value from the old state:

- *closure rule:* $T_i \leftarrow \mathrm{bal}(-(T_{i-1} + T_{i+1}))$;
- *copy rule:* $T_i \leftarrow T_{i-1}$ or T_{i+1}, each with probability 1/2.

After the update, each position is replaced with probability η by a uniformly random trit (noise).

3. Exact result: the first step and the value p = 1/3

Theorem 1. Start from a uniformly random ring, with $\eta = 0$ and the closure rule. After one step:

$$C_1(p) = 1/3 + (2/3)\cdot p(1-p)^2$$

$$C_{het,1}(p) = (2/3)\cdot C_1(p)$$

Proof. Fix window i. Only positions i−1, i and i+1 can change its sum. There are three cases.

Case A. Only the centre is updated (probability $p(1-p)^2$). The new centre is $-(T_{i-1} + T_{i+1})$. The window sum is then 0. The window is closed with probability 1.

Case B. A neighbour is updated, say i−1. Its new value is $-(T_{i-2} + T_i)$. Position i−2 is uniform and appears nowhere else in the window. It shifts the window sum by a uniform amount mod 3. The window is closed with probability 1/3. The same holds for i+1 through position i+2.

Case C. Nothing is updated. The window is still uniform. It is closed with probability 1/3.

Adding the cases:

$$C_1 = p(1-p)^2 \cdot 1 + (1 - p(1-p)^2) \cdot 1/3 = 1/3 + (2/3)\cdot p(1-p)^2.$$

Heterogeneous part. In case A the window is $(a, -(a+b), b)$ with a and b uniform. It is heterogeneous exactly when $a \neq b$, which has probability 2/3. In cases B and C the window is uniform over the 27 triples, so heterogeneous closure has probability $6/27 = (2/3)\cdot(1/3)$. In every case, heterogeneous closure is 2/3 of total closure. ∎

The maximum.

- The derivative of $p(1-p)^2$ is $(1-p)(1-3p)$.
- It vanishes at $p = 1/3$.
- There $C_1 = 1/3 + (2/3)\cdot(1/3)\cdot(4/9) = 35/81 \approx 0.432$.
- At $p = 1$, $C_1 = 1/3$.

The value 1/3 has a simple reading: the centre must move while both neighbours stand still. A full enumeration of all $3^5 \times 2^3$ cases gives the same polynomial.

4. Exact result: synchronous updating stays at chance level

Theorem 2. At $p = 1$ and $\eta = 0$, one closure step on a uniform ring gives $C = 1/3$ exactly.

Proof. All positions update at once, so $T'_j = -(T_{j-1} + T_{j+1})$. The new window sum is:

$$D'_i = -(T_{i-2} + T_{i-1} + 2T_i + T_{i+1} + T_{i+2})$$

Position $i-2$ enters with coefficient -1 and is uniform and independent of the rest. D'_i is therefore uniform on $\{-1, 0, +1\}$, and it equals 0 with probability 1/3. ∎

The exact stationary state at $p = 1$ with $\eta = 0.05$ also gives $C = 0.3333$ (Section 6). Synchronous closure sits at chance level, now and in the long run. Every position repairs its own window and breaks both neighbouring ones in the same step.

5. Closure is independent of alignment

Simulation settings. $N = 1000$; 5000 steps; $p = 1/3$; $\eta = 0.05$; five independent seeds. Values are the mean ± one standard deviation over the seeds.

	R	C	C_{het}
Random ring	0.333	0.333	0.222
Closure rule	0.323 ± 0.019	0.484 ± 0.020	0.333 ± 0.025
Copy rule	0.681 ± 0.021	0.512 ± 0.023	0.038 ± 0.007

- *Closure rule.* C rises by 45% above chance and C_{het} by 50%. R stays at chance level.
- *Copy rule.* R doubles. Total closure also rises, but through homogeneous runs such as (+1, +1, +1). Heterogeneous closure nearly vanishes.

Total closure C therefore does not separate the two rules. C_{het} does: 0.333 against 0.038.

This answers the open question of the earlier version. That version asked whether there is a regime with closure but without synchronisation. There is: closure above chance with alignment at chance.

6. The stationary optimum

Theorem 1 concerns one step. For the long run, the complete Markov chain on a ring of $N = 7$ was solved exactly: $3^7 = 2187$ states, $\eta = 0.05$. The stationary distribution is the eigenvector of the transition matrix for eigenvalue 1.

p	0.1	0.2	0.25	0.3	1/3	0.4	0.5	0.7	1.0
C	0.462	0.479	**0.480**	0.477	0.474	0.466	0.447	0.394	0.333
C_{het}	0.308	**0.319**	0.319	0.317	0.314	0.308	0.295	0.259	0.222

The simulation at $N = 1000$ (2000 steps, five seeds) shows the same shape:

p	0.05	0.1	0.2	1/3	0.5	0.7	0.9	1.0
C	0.432	0.459	0.481	0.481	0.443	0.394	0.350	0.339

The stationary state has a broad plateau between $p \approx 0.2$ and $p \approx 0.35$, with its top near 0.22–0.25. So $p = 1/3$ is the exact optimum of the first step. In the stationary state it lies on the plateau, slightly past its top.

7. Control test: what is specific to mod 3

A closure rule fixes a local constraint. Any rule that fixes a local constraint may raise the fraction of windows that satisfy it. Two controls separate the mod-3 structure from constraint-fixing in general.

- *Integer rule.* $T_i \leftarrow -(T_{i-1} + T_{i+1})$ when that value lies in $\{-1, 0, +1\}$. Otherwise T_i is drawn at random. The sum is zero in ordinary integers, without mod 3.
- *Shifted rule.* $T_i \leftarrow \mathrm{bal}(1 - (T_{i-1} + T_{i+1}))$. The window sum closes on +1 instead of 0.

Settings: $N = 1000$, 3000 steps, $p = 1/3$, $\eta = 0.05$, five seeds.

Rule	R	C_{het}	Own constraint met
Mod 3 ([illegible] 0)	[illegible]	[illegible]	[illegible]

Rule	R	C_{het}	Own constraint met
Integer (sum = 0)	0.238	0.366	0.463
Shifted (sum ≡ +1)	0.337	—	0.471

Every rule meets its own constraint in about 47% of windows. The integer rule produces even more heterogeneous closure than the mod-3 rule.

The closure level alone therefore does not mark the trit algebra. It marks local constraint-fixing under asynchronous updating. Section 8 gives the property that belongs to mod 3 alone.

8. The self-closing register

Theorem 3. Under the mod-3 closure rule, the window charges D_i follow a closed dynamics of their own. An update at position i with charge $q = D_i$ gives:

- $D_i \to 0$;
- $D_{i-1} \to \mathrm{bal}(D_{i-1} - q)$;
- $D_{i+1} \to \mathrm{bal}(D_{i+1} - q)$;
- all other windows unchanged.

Proof. The update changes T_i by $\mathrm{bal}(-(T_{i-1} + T_{i+1}) - T_i) = -q$. Position i belongs to windows $i-1$, i and $i+1$. Each of these sums therefore changes by $-q$. Window i goes from q to 0. ∎

What this means in trits.

- *Every open window is a trit charge.* Charge +1 or −1 means open; 0 means closed.
- *A charge q splits into two charges −q on its neighbours.* In balanced ternary, $(-1) + (-1) = -2 \equiv +1$ and $(+1) + (+1) = +2 \equiv -1$. The total charge is conserved mod 3.
- *Charges q and −q that meet annihilate.* Closure is annihilation of charge.
- *Noise creates only neutral charge.* A change δ at position j adds δ to three windows: $\delta + \delta + \delta \equiv 0$.

Numerical check. In 20,000 random single-site updates on rings of N = 30, Theorem 3 predicted the new charges exactly in 20,000 cases. For the integer rule the same prediction failed in 3023 cases. There the sum can leave {−1, 0, +1}, and chance must fill in the value. The charges then no longer form a system of their own.

Levels. The charge of a window of charges is again a trit:

$D_{i-1} + D_i + D_{i+1} \equiv T_{i-2} - T_{i-1} - T_{i+1} + T_{i+2} \pmod 3$

Only in balanced ternary with closure at 0 does closure on one level yield a valid state on the next level. The register is self-similar: the same winding at a different depth. This is the property that belongs to the trit algebra and to no other local rule tested here.

9. The limit of one dimension

Theorem 4. On a ring, the total charge of a block of windows a…b depends only on four boundary positions:

$\Sigma_{i=a..b} D_i \equiv T_{a-1} + 2T_a + 2T_b + T_{b+1} \pmod 3$

Proof. Every interior position appears in three windows of the block. Its coefficient is $3 \equiv 0$. The boundary positions appear once or twice. ∎

The block charge is a boundary term. Nothing in the interior of a block is conserved on its own. On the whole ring the total charge is always zero. A one-dimensional strand has no crossings of different strands, so it cannot carry a knot invariant. That fits Fox's definition. Tricolourability is a property of diagrams in which three arcs meet at every crossing.

10. Results and next test

Established:

1. The exact one-step law $C_1(p) = 1/3 + (2/3)\cdot p(1-p)^2$, with its maximum exactly at $p = 1/3$.
2. Synchronous updating gives closure at chance level.
3. Closure and alignment are independent order parameters. C_{het} is the observable that separates them.
4. The stationary optimum is a plateau around $p \approx 0.2$–0.35.
5. Local constraint-fixing alone produces heterogeneous closure. The mod-3 rule is not needed for that.
6. Only the mod-3 rule gives a self-closing trit register: charges that split, annihilate and are conserved mod 3.
7. In one dimension the register yields no bulk invariant.

The next test moves to a two-dimensional knot diagram in which every crossing joins three arcs. There the number of valid tricolourings is a knot invariant. The question is whether asynchronous closure dynamics near $p \approx 1/3$ preserves that invariant under noise, while the copy rule and the integer rule do not.

A positive result would give a property that only the trit algebra carries. A negative result would place the effect in arithmetic, not in topology. Both outcomes can be computed directly.

A second route runs through measured systems. The cultured neural networks of the Pisa group, with their multi-electrode arrays, record local states over many sites. They offer a place to look for split-and-annihilate statistics of open windows in real data.

Appendix A. Methods

The computations were run by Claude (Anthropic) on the author's instruction, in Python 3 with NumPy and SymPy.

- *Theorem 1.* Checked by exact enumeration in SymPy over all $3^5 \times 2^3$ cases.
- *Stationary distributions.* Computed as the eigenvector for eigenvalue 1 of the full 2187 × 2187 transition matrix. Given the old state, positions update independently, so each row is the Kronecker product of seven single-site distributions.
- *Simulations.* Vectorised over the ring. All new values are computed from the old state, then applied.

Core code:

```
import numpy as np
T = np.array([-1, 0, 1])
def bal(v): return ((v + 1) % 3) - 1

def step(s, rule, p, eta, g):
    l, r = np.roll(s, 1), np.roll(s, -1)
    if rule == 'closure':
        new = bal(-(l + r))
    elif rule == 'copy':
        new = np.where(g.random(s.size) < 0.5, l, r)
    elif rule == 'integer':
        t = -(l + r)
        new = np.where(abs(t) <= 1, t, g.choice(T, s.size))
    s = np.where(g.random(s.size) < p, new, s)
```

```python
    return np.where(g.random(s.size) < eta, g.choice(T, s.size), s)

def observables(s):
    l, r = np.roll(s, 1), np.roll(s, -1)
    closed = bal(l + s + r) == 0
    het = closed & (l != s) & (s != r) & (l != r)
    return (l == s).mean(), closed.mean(), het.mean()
```

Annotated references

Fox, R. H. (1970). "Metacyclic invariants of knots and links." *Canadian Journal of Mathematics* 22, 193–201. *Why read?* The source of tricolouring as a knot invariant. It gives the rule that the three arcs at a crossing sum to zero mod 3. That is the closure condition of Section 2. *Reading advice:* Read the definitions first. The metacyclic generalisation can wait.

Przytycki, J. H. (1998). "3-colorings and other elementary invariants of knots." *Banach Center Publications* 42, 275–295. *Why read?* The most accessible account of why the number of tricolourings is invariant under Reidemeister moves. It is the direct basis for the two-dimensional test in Section 10. *Reading advice:* Sections 1–2 are enough to set up the test.

Martin, O., Odlyzko, A. M. & Wolfram, S. (1984). "Algebraic properties of cellular automata." *Communications in Mathematical Physics* 93, 219–258. *Why read?* Treats cellular automata whose rules are linear mod a prime, which is exactly the class of the mod-3 closure rule. It gives the algebraic tools behind Theorems 2 and 3. *Reading advice:* Focus on the polynomial representation of linear rules. The cycle-length results can be skipped.

Schönfisch, B. & de Roos, A. (1999). "Synchronous and asynchronous updating in cellular automata." *BioSystems* 51, 123–143. *Why read?* Documents that asynchronous updating changes the order a rule can build. It places Theorem 2 in known territory. *Reading advice:* Read it as a map of update schemes, and compare with the p-dependence of Section 6.

Fatès, N. (2014). "A guided tour of asynchronous cellular automata." *Journal of Cellular Automata* 9, 387–416. *Why read?* A survey of asynchronous cellular automata, including the fully asynchronous and α-asynchronous (here: p) cases. *Reading advice:* The sections on α-asynchronism correspond directly to the parameter p used here.

Vicsek, T., Czirók, A., Ben-Jacob, E., Cohen, I. & Shochet, O. (1995). "Novel type of phase transition in a system of self-driven particles." *Physical Review Letters* 75, 1226–1229. *Why read?* The founding alignment model. It is the reference point against which closure is defined. *Reading advice:* Four pages. Read it whole.

Kuramoto, Y. (1984). *Chemical Oscillations, Waves, and Turbulence.* Springer. *Why read?* The origin of the phase-coherence order parameter R. The earlier phase-model version of this work built on it. *Reading advice:* Chapter 5 on mutual entrainment is enough.

Ballerini, M. et al. (2008). "Interaction ruling animal collective behavior depends on topological rather than metric distance." *PNAS* 105, 1232–1237. *Why read?* Field data showing that starlings interact with about six or seven neighbours regardless of distance. It supports reading a flock as a net of relations. *Reading advice:* The results section and the figures carry the argument.

Cavagna, A. et al. (2010). "Scale-free correlations in starling flocks." *PNAS* 107, 11865–11870. *Why read?* Shows that velocity fluctuations correlate across the whole flock. It is a candidate data set for looking for closure statistics independent of alignment. *Reading advice:* Focus on how the fluctuations are defined. That is where a closure observable could attach.

Shankar, S., Souslov, A., Bowick, M. J., Marchetti, M. C. & Vitelli, V. (2022). "Topological active matter." *Nature Reviews Physics* 4, 380–398. *Why read?* The current overview of topological defects in active systems. The split-and-annihilate charges of Section 8 are a discrete relative of those defects. *Reading advice:* The sections on defect creation and annihilation are the most relevant.

Escaff, D., Rosas, A., Toral, R. & Lindenberg, K. (2016). "Synchronization of coupled noisy oscillators: Coarse graining from continuous to discrete phases." *Physical Review E* 94, 052219. *Why read?* Shows how continuous phases reduce to a few discrete states. It links the earlier phase-model version to the pure trit dynamics used here. *Reading advice:* The three-state reduction is the part to study.

"From Neurons to Computation: Biological Reservoir Computing for Pattern Recognition." arXiv:2505.03510 (2025). *Why read?* Describes cultured neural networks read out over multi-electrode arrays. It is a possible source of measured data for the second route in Section 10. *Reading advice:* The methods section on stimulation and readout shows which local states can be recorded.