

Curves in the Vacuum.Net

J. Konstapel, Constable Research, Leiden, 24 October 2026

Version 1.0. © 2026 J. Konstapel, Constable Research, Leiden. All rights reserved. Quotation with attribution is permitted; reproduction of the whole requires written permission.

Abstract

The Vacuum.Net is built from one concept: the loop. This essay derives from it what a path in the net is, which connection points paths have, and how each is computed. Every theorem is proven in Appendix A. Every computed result in Appendix B was recomputed by an independent judge.

The main results:

1. **Direction gives three values.** $\mathbb{Z}_3$ is the smallest cyclic group in which a direction differs from its reverse. Its elements $-1, 0, +1$ are the trits.
2. **Closure is a loop of trits.** Three trits close when their sum is 0 modulo 3. Of the 27 windows, 9 close.
3. **Three elementary closures.** Every closure breaks down completely into standings (0), meetings $(x, -x)$ and rings (x, x, x) .
4. **Every number is a path.** Every integer has exactly one balanced-ternary word. Its length is its depth in scales.
5. **Nine paths close everywhere.** They repeat one of the nine closed windows.
6. **The seam rule.** Two closed windows join without tension only when they are equal.
7. **The net never closes completely.** Under the equal-block merging of Theorem 2, meetings and rings never return together: $2^a = 3^b$ has no solution except $a = b = 0$. The net comes closest to closing at depths 1, 2, 5, 12, 41, 53, 306, 665, 15601, ...

1. The Loop

1.1 Definition

The Vacuum.Net theory, version 4 (17 October 2026), defines:

- An ordered index set I : the course of the strand.
- A set P of places.
- The strand: a map $\sigma: I \rightarrow P$.
- A loop (touching): a pair $i < j$ in I with $\sigma(i) = \sigma(j)$. The strand returns to a place it has been.
- A knot: a place visited at least twice.

Five axioms:

- **N1 Strand.** There is exactly one strand. Every distinction is a loop.

- **N2 Closure.** A loop persists if and only if it lies in a closed window. What does not close passes on as tension.
- **N3 Memory.** At every knot the passages are ordered. The first passage fixes the direction. Later passages are taken relative to it.
- **N4 Context.** Every constant is a function of the strand.
- **N5 Scale.** Merged into blocks, the coarse strand obeys N1 to N4 again. What does not close at one scale is the material of the next.

The net is the strand. It is the whole universe: there is nothing outside it (N1).

1.2 Direction gives three values

A loop has a direction (N3). A direction must differ from its reverse. That needs an element x with $x \neq -x$. In $\mathbb{Z}_1$ and $\mathbb{Z}_2$ every element equals its reverse; in $\mathbb{Z}_2$, $1 = -1$. $\mathbb{Z}_3$ is the smallest cyclic group where a direction and its reverse differ: $1 \neq -1 = 2$ (A.1).

So a passage through a knot has one of three values relative to the first passage. These values are the trits:

- **+1:** it leaves the knot the way the first passage left it.
- **-1:** it leaves the knot the way the first passage came in.
- **0:** it leaves another way.

Rule. Let i be a later passage through a knot, not the last passage of the strand, and let f be the first passage through $\sigma(i)$. Then $t(i) = +1$ if $\sigma(i+1) = \sigma(f+1)$; $t(i) = -1$ if $\sigma(i+1) = \sigma(f-1)$; $t(i) = 0$ otherwise. The rule uses only σ . It is mechanical.

1.3 Closure is a loop of trits

Read the trits as steps on $\mathbb{Z}_3$. A run of steps returns to its start if and only if its sum is 0 modulo 3 (A.2). A closed run of trits is therefore itself a loop, one scale up. This is N5.

The shortest return that goes round is $+1, +1, +1$. It takes three steps, because 1 has order 3 in $\mathbb{Z}_3$. This fixes the window at three trits. A window closes when its sum is 0 modulo 3.

Of the 27 windows, 9 close: the three constant windows and the six orders of $(-1, 0, +1)$. The other 18 have two equal trits and one different (A.3).

1.4 Three elementary closures

Theorem 1 (Elementary closures). Every closed trit word breaks down to nothing by removing, one at a time:

- a **standing**: 0 (length 1);
- a **meeting**: $x, -x$ (length 2; it goes and comes back);
- a **ring**: x, x, x (length 3; it goes round once).

No other word does. Proven, A.4; computed for all 265,720 closed words up to length 12.

Lift the nonzero steps to the integers. A closed walk then moves $3w$ places, where w is its winding number. Among closures with nonzero steps, a meeting is the shortest with $w = 0$ and a ring the shortest with $w = \pm 1$ (A.4). Apart from standing still, there are two ways to close: to meet and to go round. Their lengths are 2 and 3.

1.5 Numbers, depth and position

A number counts loops. Every integer has exactly one representation $\sum d_k \cdot 3^k$ with digits $d_k \in \{-1, 0, +1\}$: balanced ternary (A.5). Words of k digits give exactly the integers from $-(3^k - 1)/2$ to $(3^k - 1)/2$. Every number is a trit word, and every trit word is a number.

Digit k stands at scale 3^k (N5). A number $n \geq 1$ has $\lfloor \log_3(2n - 1) \rfloor + 1$ digits (A.6). The position $p = \log_3 q$ therefore measures the depth of q in scales, up to rounding.

Examples, writing T for -1 : $2 = 1T$, $3 = 10$, $8 = 10T$, $-8 = T01$.

1.6 The ladder and the comma

Theorem 2 (Ladder). Let the strand be merged, at each scale, into blocks of equal size, each block one elementary closure of length 2 or 3. Then a passage after a merges of meetings and b merges of rings contains exactly $2^a \cdot 3^b$ passages of the finest scale. Proven, A.7.

Theorem 3 (No complete closure). $2^a = 3^b$ holds only for $a = b = 0$. Meetings and rings never return together. Proven, A.8.

Under the merging of Theorem 2 the net as a whole never closes. By N2 the remainder passes on as tension: there is always a next passage. The strand then has no last passage, and the net is infinite.

It comes closest to closing where $b \cdot \log_2 3$ is nearest a whole number a . These are the convergents of $\log_2 3$:

Depth b	a	Comparison	Remainder
1	2	$3^1 \sim 2^2$	498.045 cents
2	3	$3^2 \sim 2^3$	203.910 cents
5	8	$3^5 \sim 2^8$	90.225 cents
12	19	$3^{12} \sim 2^{19}$	23.460 cents
41	65	$3^{41} \sim 2^{65}$	19.845 cents
53	84	$3^{53} \sim 2^{84}$	3.615 cents
306	485	$3^{306} \sim 2^{485}$	1.770 cents
665	1054	$3^{665} \sim 2^{1054}$	0.0756 cents
15601	24727	$3^{15601} \sim 2^{24727}$	0.0315 cents

The remainder at depth 12 is the Pythagorean comma. Each near-closure is finite: a cycle at depth 665 holds $3^{665} \approx 1.9 \times 10^{317}$ finest passages.

2. What a Path Is

2.1 In the net

Definition P1. A **path** is the strand on an interval J of I . Its **trit word** is the sequence of its trits, by the rule of 1.2.

Definition P2. A path is **closed in window k** when its k -th window closes. It is **closed everywhere** when every window closes. Its **open windows** are its tension (N2).

Every path is a piece of the one strand (N1). Two paths meet only where the strand meets itself: at a knot.

Counts (A.9). A strand of N passages has $N(N + 1)/2$ paths. Each has one fixed trit word; only its later, non-final passages carry a trit. Of the 3^n possible words of length $n \geq 3$, 3^{n-1} close their first window and exactly 9 close everywhere. A knot visited m times holds $m(m - 1)/2$ loops.

2.2 In position space

A phenomenon is described by n magnitudes. Each is a ratio to a reference of its own kind, so each is a pure number q . Its position is $p = \log_3 q$ (1.5). The phenomenon is a point in n -dimensional position space. When it changes, the point moves and traces a path.

Definition P3. Along such a path the **local exponent** between q_i and q_j is $e = dp_j/dp_i$. The **slope trit** is the direction of p_j relative to p_i : rising +1, flat 0, falling -1 (1.2).

A point alone is a coordinate. A path has direction and shape. The point disappears; the path carries the content.

3. Connection Points on One Path

3.1 In the net

Connection point	Definition	What is fixed	Proof
Knot	a place visited $m \geq 2$ times	$m(m - 1)/2$ loops	A.9
Closed window	three trits with sum $0 \bmod 3$	3 constant windows, 6 orders of $(-1, 0, +1)$	A.3
Open window	any other three trits	two equal trits and one different: 18 cases	A.3

Theorem 4 (Propagation). Let s_k be the sum of window k . Then $s_{k+1} \equiv s_k + t_{k+3} - t_k \pmod{3}$. Proven, A.10.

Theorem 5 (Nine paths close everywhere). A trit word of length $n \geq 3$ closes in every window if and only if its first window closes and the word has period 3. For every n there are exactly 9 such words. Proven, A.10; computed for $n = 3$ to 8.

A path without tension is one of nine: a constant run, or one order of $(-1, 0, +1)$ repeated. Every other path carries tension.

3.2 In position space

Connection point	Definition	Trits around it	Closure
Regular point	the slope keeps its direction	+1, +1 or -1, -1	none
Turning point	the slope changes direction	+1, 0, -1 or -1, 0, +1	a meeting (A.11)
Zero	the magnitude changes sign; $p = \log_3 q \rightarrow -\infty$	sign trits $\pm 1, 0, \mp 1$ or $\pm 1, \mp 1$	a meeting (A.11)
Return	the path passes a point it passed before	a knot at this level	P1

Theorem 6 (Closed paths are meetings). On a closed path of one magnitude, rises and falls alternate and are equal in number. The reduced slope word breaks down into meetings only; its winding number is 0. Proven, A.12; computed on 2,000 random periodic paths, rechecked on 3,000.

4. Connection Points Between Paths

4.1 In the net

Shared knot. Two paths are two pieces of the one strand. They meet where the strand meets itself. Section 3.1 applies to that knot.

Seam. A seam joins the end of one closed window A to the start of another closed window B. Two windows straddle the joint.

Theorem 7 (Seam rule). The number of open windows at the seam of A and B is $[A_1 \neq B_1] + [A_3 \neq B_3]$. The seam is closed if and only if $B = A$. Proven, A.13; computed over all 81 ordered pairs: 9 closed seams, 36 with one open window, 36 with two.

A window continued in its own order gives no open window. A seam between two different closed windows always carries tension.

4.2 In position space

Theorem 8 (Two laws). A power law $q_j = C \cdot q_i^e$ is a straight line in the plane of p_i and p_j , with slope e . Two laws between the same two magnitudes are identical, parallel, or meet in exactly one point. Lines in a space of three or more magnitudes can also be skew. Proven, A.14.

Relation	Exponents	Constants	Connection
identical	equal	equal	the same law
parallel	equal	different	the same exponent, a different constant
intersecting	different	—	one point where both hold

Theorem 9 (Two cycles). Two periodic paths with frequencies f_1 and f_2 return together if and only if f_1/f_2 is rational. A chain of 12 fifths never returns onto a whole number of octaves: $3^{12} = 531,441$ and $2^{19} = 524,288$. Their ratio is the Pythagorean comma, 23.460 cents. Proven, A.15. This is Theorem 3 seen between two paths.

5. Computing a Connection Point

Four fixed steps. No step involves a choice.

1. **Strand.** Take the strand: a word sequence, a measured series, or a set of magnitudes.
2. **Trits.** For a strand, by the rule of 1.2. For magnitudes, by the slope trit (P3). For an exact number, by its balanced-ternary word (1.5).
3. **Windows.** Mark every window as closed or open.
4. **Connection points.** Read knots, seams, turning points, zeros and returns by Sections 3 and 4. Break every closure down into standings, meetings and rings (Theorem 1).

5.1 Worked cases

Case	Strand	Trits	Connection points	Status
A word sequence	"the net is a strand and the strand is the net": 11 passages	later passages: the 0, strand 0, is 0, the +1	knots: the (3 visits), net, is, strand (2 each): 6 loops; window (0, 0, 0) closed, (0, 0, +1) open	computed
Light	wavelength against frequency	slope $-1 = T$	a straight line (Theorem 8); $\lambda \cdot f$ constant	proven
Kepler	orbital period against distance, eight planets	measured slope 1.4978 ± 0.0007 ; law $3/2$; $3 = 10$, $2 = 1T$	a straight line (Theorem 8)	computed from input
π junction	critical current on cooling (Ryazanov et al. 2001)	sign +1, 0, -1	a zero: a meeting (A.11)	input and proof
Little–Parks	resistance of a ring against flux (Little and Parks 1962)	periodic in flux	a closed path: meetings only (Theorem 6)	input and proof

Inputs for the Kepler case. NASA Planetary Fact Sheet: distance from the Sun 57.9, 108.2, 149.6, 228.0, 778.5, 1,432.0, 2,867.0, 4,515.0 (10^6 km); orbital period 88.0, 224.7, 365.2, 687.0, 4,331, 10,747, 30,589, 59,800 (days). Least squares on base-3 positions gives slope 1.49781, standard error 0.00074.

Appendix A: Proofs

Trits are read modulo 3. The values -1 , 0 and $+1$ are distinct modulo 3, so two trits are congruent only when they are equal.

A.1 The smallest group with direction

In $\mathbb{Z}_1$ the only element is $0 = -0$. In $\mathbb{Z}_2$, $1 = -1$. In $\mathbb{Z}_3$, $-1 = 2 \neq 1$. ■

A.2 Walks on $\mathbb{Z}_3$

After steps $t_1 \dots t_m$ the position is $t_1 + \dots + t_m$ modulo 3. The walk returns if and only if this sum is 0 modulo 3. The step +1 returns first after 3 steps, because 1 and 2 are not 0 modulo 3 and 3 is. ■

A.3 The 27 windows

There are $3^3 = 27$ windows. The sum map from $(\mathbb{Z}_3)^3$ to $\mathbb{Z}_3$ is a surjective homomorphism. Its kernel, the closed windows, has $27/3 = 9$ elements.

- All equal: $3a \equiv 0$. Three windows.
- All different: $-1 + 0 + 1 = 0$. Six orders.
- Two equal, one different: $2a + b \equiv 0$ gives $b \equiv -2a \equiv a$, a contradiction. None closes.

$3 + 6 = 9$ close. The other 18 have two equal trits and one different: 3 values for the pair $\times 2$ for the other $\times 3$ positions. ■

A.4 Elementary closures

Each removal takes away a piece with sum 0 modulo 3, so it keeps the sum of the word modulo 3. Only closed words can therefore reduce to nothing.

Conversely, let w be closed. Remove every 0. If the remaining word has two neighbours $x, -x$, remove them. If it has none, all its entries are equal, because two different nonzero trits are opposite. Its length L then satisfies $L \cdot x \equiv 0$, so $L \equiv 0$ modulo 3, and it splits into rings. Each step shortens the word and keeps it closed. By induction it reduces to nothing.

Lift the nonzero steps ± 1 to the integers. A walk returns on $\mathbb{Z}_3$ if and only if its total displacement is $3w$ for a whole number w . The displacement is at most the length. So $|w| = 1$ needs at least 3 steps, and x, x, x reaches it. A nonempty walk of nonzero steps with $w = 0$ needs an even length, at least 2, and $x, -x$ reaches it. With 0 allowed, the standing has length 1. ■

A.5 Balanced ternary

For every integer n exactly one $d_0 \in \{-1, 0, +1\}$ has $n \equiv d_0$ modulo 3. For $|n| \geq 2$, $|(n - d_0)/3| < |n|$. Repeating gives the digits, and each digit is forced, so the word is unique. The largest value with k digits is $1 + 3 + \dots + 3^{k-1} = (3^k - 1)/2$, and the set is symmetric. The 3^k words give 3^k distinct integers in a range of exactly 3^k integers, so they give each once. ■

A.6 Depth

By A.5, $n \geq 1$ has d digits if and only if $(3^{d-1} - 1)/2 < n \leq (3^d - 1)/2$. Multiply by 2: $3^{d-1} < 2n + 1$ and $2n - 1 < 3^d$. Since 3^{d-1} and $2n + 1$ are odd, the first gives $3^{d-1} \leq 2n - 1$. So $3^{d-1} \leq 2n - 1 < 3^d$, and $d - 1 = \lfloor \log_3(2n - 1) \rfloor$. Computed in whole numbers, without rounding. ■

A.7 The ladder

At each scale every coarse passage is one block, and every block at that scale has the same size: 2 for a merge of meetings, 3 for a merge of rings. A coarse passage therefore holds the product of the block

sizes below it: $2^a \cdot 3^b$. The equal size is needed: a ring merged into one passage and then joined to a single passage in a meeting gives 4 finest passages, not 6. ■

A.8 No complete closure

For $a > 0$ the number 2^a is even and 3^b is odd. For $a = 0$, $3^b = 1$ forces $b = 0$. So $2^a = 3^b$ only for $a = b = 0$, and $\log_2 3$ is irrational. The continued fraction of $\log_2 3$ is $[1; 1, 1, 2, 2, 3, 1, 5, 2, 23, \dots]$. Its convergents a/b give the table of 1.6. The remainder is $1200 \cdot |b \cdot \log_2 3 - a|$ cents. ■

A.9 Counts

A path is fixed by its first and last index, with $\text{first} \leq \text{last}$: $N(N+1)/2$ choices. A word of length $n \geq 3$ has 3^n choices. Its first window closes in 9 of 27 cases, the other $n-3$ trits are free: $9 \cdot 3^{n-3} = 3^{n-1}$. A knot visited m times gives one loop per pair of visits: $m(m-1)/2$. ■

A.10 Propagation and the nine everywhere-closed words

Let $s_k = t_k + t_{k+1} + t_{k+2}$. Then $s_{k+1} = t_{k+1} + t_{k+2} + t_{k+3} = s_k - t_k + t_{k+3}$. This proves Theorem 4.

If every window closes, $s_{k+1} \equiv s_k \equiv 0$, so $t_{k+3} \equiv t_k$, so $t_{k+3} = t_k$: period 3. Conversely, if the first window closes and $t_{k+3} = t_k$, every window has the same sum 0. A word of period 3 is fixed by its first three trits, and these form a closed window. By A.3 there are 9. ■

A.11 Turning points and zeros are meetings

At a turning point the slope trits before, at and after are $+1, 0, -1$ (or $-1, 0, +1$). At a zero the sign trits are $\pm 1, 0, \mp 1$, or $\pm 1, \mp 1$ when the zero itself is not sampled. Remove the 0: what remains is $x, -x$, a meeting. At a zero $|q| \rightarrow 0$, so $p = \log_3 |q| \rightarrow -\infty$. ■

A.12 Closed paths are meetings

Let g be continuous, periodic, non-constant and piecewise monotone. Split one period into maximal stretches on which g is weakly monotone, flat parts absorbed. Neighbouring stretches then differ in direction. (A constant g has no stretches and winding 0.) On a closed path the last stretch is followed by the first. A cyclic sequence of two symbols in which neighbours differ has even length and equal numbers of each. The reduced slope word is $+1, -1, +1, -1, \dots$: it breaks down into meetings only, and its displacement and winding are 0. ■

A.13 The seam rule

Let A and B be closed windows. The joined word is $A_1 A_2 A_3 B_1 B_2 B_3$. The windows across the joint are $W_1 = (A_2, A_3, B_1)$ and $W_2 = (A_3, B_1, B_2)$.

- A closes, so $A_2 + A_3 \equiv -A_1$. The sum of W_1 is $\equiv B_1 - A_1$. W_1 closes if and only if $B_1 = A_1$.
- B closes, so $B_1 + B_2 \equiv -B_3$. The sum of W_2 is $\equiv A_3 - B_3$. W_2 closes if and only if $B_3 = A_3$.

So the number of open windows is $[A_1 \neq B_1] + [A_3 \neq B_3]$. A closed window is fixed by its first and last trit, because its middle is $\equiv -(\text{first} + \text{last})$. So the seam is closed if and only if $B = A$. For a fixed A , of the 9 windows B : 1 gives no open window, 4 give one, 4 give two. Over 9 windows A : 9, 36, 36. ■

A.14 Two laws

A power law $q_i = C \cdot q_i^e$ gives $p_j = e \cdot p_i + \log_3 C$: a straight line with slope e . Write two lines as $a + s \cdot u$ and $b + t \cdot v$. If u and v are parallel, the lines are identical or disjoint. If not, they meet in one point when $b - a$ lies in the plane of u and v , and are skew otherwise. In two dimensions non-parallel lines always meet. ■

A.15 Two cycles

Let two periodic paths have least periods $1/f_1$ and $1/f_2$. They return together at a time $t > 0$ if and only if $t = m/f_1 = n/f_2$ for whole numbers $m, n > 0$. That holds if and only if $f_1/f_2 = m/n$ is rational. For 12 fifths and 7 octaves: $(3/2)^{12}/2^7 = 3^{12}/2^{19} = 531,441/524,288 = 1.0136433$, which is $1200 \cdot \log_2(1.0136433) = 23.460$ cents. ■

Appendix B: Computations and Independent Check

Every number was computed twice. A respondent script produced it. A separate judge wrote its own code from scratch, without reading the first, and recomputed it.

Claim	Respondent	Judge	Verdict
$\mathbb{Z}_3$ is the smallest cyclic group with $x \neq -x$	$\mathbb{Z}_1$ to $\mathbb{Z}_5$	$\mathbb{Z}_1$ to $\mathbb{Z}_5$	verified
Balanced ternary is unique; k digits give $-(3^k - 1)/2 \dots (3^k - 1)/2$ bijectively	$k \leq 8$; round trip on $\pm 10^5$	$k \leq 8$; round trip on $\pm 10^5$	verified
Digits of $n = \lfloor \log_3(2n - 1) \rfloor + 1$	$n < 2 \cdot 10^5$	$n \leq 10^6$, whole-number arithmetic	verified
27 windows: 9 close (3 constant, 6 orders); 18 open, all with one pair	enumerated	enumerated	verified
Every closed word reduces by standings, meetings and rings; no open word does	all 265,720 closed words, $n \leq 12$	exhaustive both ways, $n \leq 9$	verified
Shortest closure with nonzero steps: 2 for $w = 0$, 3 for $ w = 1$	enumerated	enumerated	verified
Ladder $2^a \cdot 3^b$ needs equal block size (counterexample $4 \neq 6$)	proof A.7	constructed	verified
$2^a = 3^b$ only for $a = b = 0$; convergents and remainders of 1.6	exact	exact	verified
First window closed in 3^{n-1} of 3^n words; 9 everywhere-closed words	$n = 3$ to 8	$n = 3$ to 9	verified
Propagation $s_{k+1} \equiv s_k + t_{k+3} - t_k$	$n = 3$ to 9	$n = 3$ to 10	verified
Seam rule; 9 / 36 / 36 over 81 pairs	all 81 pairs	all 81 pairs	verified
Closed paths have equal numbers of rises and falls	2,000 random periodic paths	3,000 random periodic paths	verified

Claim	Respondent	Judge	Verdict
$3^{12} = 531,441$; $2^{19} = 524,288$; 1.0136433; 23.460 cents	exact	exact	verified
Word sequence of 5.1: 11 passages, 4 knots, 6 loops; trits 0, 0, 0, +1; one window closed, one open	computed	computed	verified
Continued fraction of $\log_2 3$; $3^{665} \approx 1.9 \times 10^{317}$	exact	exact	verified
$N(N + 1)/2$ paths; $m(m - 1)/2$ loops per knot	proof A.9	proof A.9	verified
Kepler slope on NASA data	1.49781 ± 0.00074	1.4978081 ± 0.0007389	verified

Annotated References

The theory

1. Konstapel, J. (2026). [The Vacuum.Net Theory: Definition, Theorems and Tests, version 4](#). Constable Research, Leiden, 17 October 2026. Why read? The definition and the five axioms of Section 1.1.
2. Konstapel, J. (2026). ["The Way Before the Net."](#) constable.blog, 20 October 2026. Why read? Zero as the meeting of an act with its mirror. The two roads to closure: meeting and going round. Used in Section 1.4 and A.11.
3. Konstapel, J. (2026). ["The Mathematics of Closure."](#) constable.blog, 15 October 2026. Why read? The convergents of $\log_2 3$ behind the closure depths.
4. Konstapel, J. (2026). "The Possibilities and Impossibilities of the Net." constable.blog, 19 October 2026. Why read? What the net can do in its own quantities: read the state of a place, hold two distant structures in one state, select matter by address, keep a configuration in circulation. A map of 10 quantities by 5 approaches.
5. Konstapel, J. (2026). [The Respondent–Judge Protocol](#). Constable Research, Leiden, 17 October 2026. Why read? The check of Appendix B: a separate judge with its own code.

Numbers and the register

6. Hardy, G. H., and Wright, E. M. *An Introduction to the Theory of Numbers*. Oxford University Press, 6th ed. 2008. Why read? Unique factorisation, irrationality and continued fractions. Used in A.8.
7. Knuth, D. E. *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms*. Addison-Wesley, Reading, Mass., 3rd ed. 1998. Why read? Balanced ternary: the register of trits. Section 4.1.
8. Spencer-Brown, G. (1969). *Laws of Form*. Allen & Unwin, London. Why read? A calculus built on one act of distinction. Background to N1.
9. Kauffman, L. H. (2005). "EigenForm." *Kybernetes* 34(1/2), 129–150. doi:10.1108/03684920510575780. Why read? Forms that return on themselves: closure as a fixed point.

10. Varela, F. J. (1975). "A calculus for self-reference." *International Journal of General Systems* 2, 5–24. Why read? A third value beside true and false, close to the trit 0.

Paths, returns and knots

11. Pólya, G. (1921). "Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz." *Mathematische Annalen* 84, 149–160. [doi:10.1007/BF01458701](https://doi.org/10.1007/BF01458701). Why read? A random path returns with certainty in one and two dimensions, not in three. The basis of the free path model of the strand.
12. Watson, G. N. (1939). "Three triple integrals." *Quarterly Journal of Mathematics* 10, 266–276. [doi:10.1093/qmath/os-10.1.266](https://doi.org/10.1093/qmath/os-10.1.266). Why read? The integral from which the return probability in three dimensions, 0.3405, follows (McCrea and Whipple, 1940).
13. Montroll, E. W. (1956). "Random walks in multidimensional spaces, especially on periodic lattices." *Journal of the SIAM* 4(4), 241–260. [doi:10.1137/0104014](https://doi.org/10.1137/0104014). Why read? Return probabilities in four and five dimensions.
14. Rolfsen, D. (1976). *Knots and Links*. Publish or Perish, Berkeley. Why read? The standard theory of closed curves in space and how they wind.
15. Adams, C. C. (1994). *The Knot Book*. W. H. Freeman, New York. Why read? A readable entry to knots, crossings and invariants.
16. Milnor, J. (1965). *Topology from the Differentiable Viewpoint*. University Press of Virginia. Why read? Degree and winding number: how a closed curve counts its turns. Background to A.4.

Curves, cycles and their special points

17. Strogatz, S. H. *Nonlinear Dynamics and Chaos*. Westview Press, 2nd ed. 2015. Why read? Fixed points, limit cycles, bifurcations and quasi-periodic motion on a torus. Background to Sections 3.2 and 4.2.
18. Lissajous, J. A. (1857). "Mémoire sur l'étude optique des mouvements vibratoires." *Annales de chimie et de physique*, 3rd series, 51, 147–231. Why read? The figures of two combined vibrations, which close only at rational ratios. Theorem 9.
19. Thom, R. (1972). *Stabilité structurelle et morphogénèse*. Benjamin, Reading. Why read? Folds and cusps: the stable kinds of sudden change in a curve.
20. Arnold, V. I. (1992). *Catastrophe Theory*. Springer, 3rd ed. Why read? A short, exact account of the special points of smooth maps.
21. Feigenbaum, M. J. (1978). "Quantitative universality for a class of nonlinear transformations." *Journal of Statistical Physics* 19, 25–52. [doi:10.1007/BF01020332](https://doi.org/10.1007/BF01020332). Why read? Period doubling: a cascade of branchings by factor two.
22. Mandelbrot, B. B. (1982). *The Fractal Geometry of Nature*. W. H. Freeman, San Francisco. Why read? Curves that look alike at every scale. Background to N5.
23. Verhulst, P.-F. (1838). "Notice sur la loi que la population suit dans son accroissement." *Correspondance mathématique et physique* 10, 113–121. Why read? The S-curve of growth with a ceiling.

Scaling, laws and dimensions

24. Buckingham, E. (1914). "On physically similar systems; illustrations of the use of dimensional equations." *Physical Review* 4, 345–376. [doi:10.1103/PhysRev.4.345](https://doi.org/10.1103/PhysRev.4.345). Why read? The Π theorem: a law can be written in its dimensionless invariants. The basis of the pure-number positions of Section 2.2.
25. Bridgman, P. W. (1922). *Dimensional Analysis*. Yale University Press. Why read? Why dimension is multiplicative, as used in A.9.
26. Barenblatt, G. I. (1996). *Scaling, Self-Similarity, and Intermediate Asymptotics*. Cambridge University Press. Why read? Power laws as straight lines in logarithmic space, and when exponents are exact.
27. Clauset, A., Shalizi, C. R., and Newman, M. E. J. (2009). "Power-law distributions in empirical data." *SIAM Review* 51(4), 661–703. [doi:10.1137/070710111](https://doi.org/10.1137/070710111). Why read? How to fit an exponent honestly. Needed before any exponent is called exact.
28. Stanley, H. E. (1971). *Introduction to Phase Transitions and Critical Phenomena*. Oxford University Press. Why read? Critical exponents: straight stretches next to a threshold.
29. Larson, D. B. (1959). *The Structure of the Physical Universe*. North Pacific Publishers, Portland. Why read? Motion as the one component of the universe, with space and time as its two reciprocal aspects.
30. Hestenes, D. (1966). *Space-Time Algebra*. Gordon and Breach, New York. Why read? Rotation as two reflections; the rotor that returns only after two turns.
31. Grassmann, H. (1844). *Die lineale Ausdehnungslehre*. Wigand, Leipzig. Why read? The extended form as the trace of a moving element: the strand as a map.

Music and the comma

32. Barbour, J. M. (1951). *Tuning and Temperament: A Historical Survey*. Michigan State College Press. Why read? The Pythagorean comma and the cycles of 12, 41 and 53 fifths.
33. Benson, D. J. (2006). *Music: A Mathematical Offering*. Cambridge University Press. Why read? The arithmetic of intervals, cents and temperaments.
34. Kepler, J. (1619). *Harmonices Mundi*. Linz. Why read? The third law, published in a book on musical ratios in the heavens.

Measurements used

35. Williams, D. R. NASA, NSSDCA (last updated 18 March 2025). [Planetary Fact Sheet](#). Why read? The distances and periods behind the Kepler case.
36. Ryazanov, V. V., Oboznov, V. A., Rusanov, A. Yu., Veretennikov, A. V., Golubov, A. A., and Aarts, J. (2001). "Coupling of two superconductors through a ferromagnet: evidence for a π junction." *Physical Review Letters* 86(11), 2427–2430. [doi:10.1103/PhysRevLett.86.2427](https://doi.org/10.1103/PhysRevLett.86.2427). Why read? A critical current that passes through zero and changes sign: a zero in the sense of Section 3.2.

37. Little, W. A., and Parks, R. D. (1962). "Observation of quantum periodicity in the transition temperature of a superconducting cylinder." *Physical Review Letters* 9, 9–12. [doi:10.1103/PhysRevLett.9.9](https://doi.org/10.1103/PhysRevLett.9.9). Why read? A closed curve in flux: the periodic case of Section 5.1.
38. Hertlein, C., Helden, L., Gambassi, A., Dietrich, S., and Bechinger, C. (2008). "Direct measurement of critical Casimir forces." *Nature* 451(7175), 172–175. [doi:10.1038/nature06443](https://doi.org/10.1038/nature06443). Why read? The first direct measurement of critical Casimir forces near a critical point. The fit takes the exponent $\nu = 0.63$ of the 3D Ising class as given.

Method

39. Tarski, A., Mostowski, A., and Robinson, R. M. (1953). *Undecidable Theories*. North-Holland, Amsterdam. Why read? What it means to translate one theory into another.
40. Lakatos, I. (1970). "Falsification and the Methodology of Scientific Research Programmes." In *Criticism and the Growth of Knowledge*, Cambridge University Press, 91–196. Why read? Why claims must be fixed before the data are read.

Topology of closed curves

41. Gauss, C. F. (1833). Note of 22 January 1833 on the linking of two closed curves. *Werke*, Vol. 5, p. 605. Göttingen, 1867. Why read? The linking number: an integer that counts how often two closed curves wind round each other. The first invariant of two curves. Section 4.
42. Alexander, J. W. (1928). "Topological invariants of knots and links." *Transactions of the AMS* 30(2), 275–306. [doi:10.1090/S0002-9947-1928-1501429-1](https://doi.org/10.1090/S0002-9947-1928-1501429-1). Why read? The first polynomial that tells knots apart.
43. Jones, V. F. R. (1985). "A polynomial invariant for knots via von Neumann algebras." *Bulletin of the AMS* 12(1), 103–111. [doi:10.1090/S0273-0979-1985-15304-2](https://doi.org/10.1090/S0273-0979-1985-15304-2). Why read? A sharper knot invariant. It links knots to physics (see 61).
44. White, J. H. (1969). "Self-linking and the Gauss integral in higher dimensions." *American Journal of Mathematics* 91(3), 693–728. [doi:10.2307/2373348](https://doi.org/10.2307/2373348). Why read? Linking = twist + writhe. A closed ribbon splits its winding into two parts. Used for DNA.

Cycles, rotation and locking

45. Poincaré, H. (1885). "Sur les courbes définies par les équations différentielles (3e partie)." *Journal de Mathématiques Pures et Appliquées*, 4e série, 1, 167–244. Why read? The rotation number of a circle map. A curve on a torus closes if and only if that number is rational. The exact form of Theorem 9.
46. Kronecker, L. (1884). "Näherungsweise ganzzahlige Auflösung linearer Gleichungen." *Sitzungsberichte der Akademie der Wissenschaften zu Berlin*, 1179–1193 and 1271–1299. Why read? At an irrational ratio the curve never closes. It comes arbitrarily close to every point. The other half of Theorem 9.
47. Weyl, H. (1916). "Über die Gleichverteilung von Zahlen mod. Eins." *Mathematische Annalen* 77(3), 313–352. [doi:10.1007/BF01475864](https://doi.org/10.1007/BF01475864). Why read? The multiples of an irrational number spread evenly. Background to the remainders of Section 1.6 (table).

48. Arnold, V. I. (1961). "Small denominators. I. Mappings of the circumference onto itself." *Izvestiya Akad. Nauk SSSR, Ser. Mat.* 25(1), 21–86. English: *AMS Translations*, Series 2, 46 (1965), 213–284. Why read? Arnold tongues. Coupled cycles lock onto a rational ratio over a finite range. Closure is then stable, not exceptional.
49. Kuramoto, Y. (1975). "Self-entrainment of a population of coupled non-linear oscillators." In H. Araki (ed.), *International Symposium on Mathematical Problems in Theoretical Physics*, Lecture Notes in Physics 39, Springer, 420–422. [doi:10.1007/BFb0013365](https://doi.org/10.1007/BFb0013365). Why read? Many cycles fall into one common rhythm above a coupling threshold.
50. Bennett, M., Schatz, M. F., Rockwood, H., and Wiesenfeld, K. (2002). "Huygens's clocks." *Proceedings of the Royal Society A* 458(2019), 563–579. [doi:10.1098/rspa.2001.0888](https://doi.org/10.1098/rspa.2001.0888). Why read? Huygens' 1665 observation repeated: two clocks on one beam lock in anti-phase. A measured case of Theorem 9 with coupling.
51. Noether, E. (1918). "Invariante Variationsprobleme." *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Klasse*, 235–257. Why read? Every continuous symmetry gives a conserved quantity. The physical source of the constants in Theorem 8.

Physics: closed paths and whole numbers

52. Thomson, W. (Lord Kelvin) (1869). "On vortex atoms." *Proceedings of the Royal Society of Edinburgh* 6, 94–105 (read 18 February 1867). [doi:10.1017/S0370164600045430](https://doi.org/10.1017/S0370164600045430). Why read? The first theory of atoms as knotted closed curves in a medium. The historical forerunner of the knot in the net.
53. Aharonov, Y., and Bohm, D. (1959). "Significance of electromagnetic potentials in the quantum theory." *Physical Review* 115(3), 485–491. [doi:10.1103/PhysRev.115.485](https://doi.org/10.1103/PhysRev.115.485). Why read? A particle on a closed path picks up a phase from a field it never touches. Only the closed path counts.
54. Deaver, B. S., and Fairbank, W. M. (1961). "Experimental evidence for quantized flux in superconducting cylinders." *Physical Review Letters* 7(2), 43–46. [doi:10.1103/PhysRevLett.7.43](https://doi.org/10.1103/PhysRevLett.7.43). Why read? Flux through a superconducting ring comes in whole units of $h/2e$. A closed curve that admits only integers.
55. Doll, R., and Näbauer, M. (1961). "Experimental proof of magnetic flux quantization in a superconducting ring." *Physical Review Letters* 7(2), 51–52. [doi:10.1103/PhysRevLett.7.51](https://doi.org/10.1103/PhysRevLett.7.51). Why read? The same result, measured independently in the same month.
56. Josephson, B. D. (1962). "Possible new effects in superconductive tunnelling." *Physics Letters* 1(7), 251–253. [doi:10.1016/0031-9163\(62\)91369-0](https://doi.org/10.1016/0031-9163(62)91369-0). Why read? Current across a junction depends on the phase difference. The basis of every junction case in Section 5.
57. Bulaevskii, L. N., Kuzii, V. V., and Sobyanin, A. A. (1977). *JETP Letters* 25, 290. Why read? The theoretical prediction of the π junction, confirmed in 36. A sign change predicted 24 years before it was measured.
58. Berry, M. V. (1984). "Quantal phase factors accompanying adiabatic changes." *Proceedings of the Royal Society A* 392(1802), 45–57. [doi:10.1098/rspa.1984.0023](https://doi.org/10.1098/rspa.1984.0023). Why read? A system led round a closed loop returns with an extra phase set by the loop's geometry.

59. von Klitzing, K., Dorda, G., and Pepper, M. (1980). "New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance." *Physical Review Letters* 45(6), 494–497. [doi:10.1103/PhysRevLett.45.494](https://doi.org/10.1103/PhysRevLett.45.494). Why read? A resistance that takes exact values $h/(ie^2)$, i an integer. A measured whole-number closure.
60. Thouless, D. J., Kohmoto, M., Nightingale, M. P., and den Nijs, M. (1982). "Quantized Hall conductance in a two-dimensional periodic potential." *Physical Review Letters* 49(6), 405–408. [doi:10.1103/PhysRevLett.49.405](https://doi.org/10.1103/PhysRevLett.49.405). Why read? The integer of 59 is a winding number. Topology explains the exact value.
61. Witten, E. (1989). "Quantum field theory and the Jones polynomial." *Communications in Mathematical Physics* 121(3), 351–399. [doi:10.1007/BF01217730](https://doi.org/10.1007/BF01217730). Why read? Knot invariants come out of a quantum field theory. Knots and physics in one formalism.
62. Faddeev, L., and Niemi, A. J. (1997). "Stable knot-like structures in classical field theory." *Nature* 387(6628), 58–61. [doi:10.1038/387058a0](https://doi.org/10.1038/387058a0). Why read? A field theory whose stable particles are knots.
63. Kleckner, D., and Irvine, W. T. M. (2013). "Creation and dynamics of knotted vortices." *Nature Physics* 9(4), 253–258. [doi:10.1038/nphys2560](https://doi.org/10.1038/nphys2560). Why read? Knotted vortices made in water and filmed. They untie by reconnection. Kelvin's idea (52) tested.
64. Rowlands, P. (2007). *Zero to Infinity: The Foundations of Physics*. Series on Knots and Everything, Vol. 41. World Scientific. Why read? Physics built from a total of zero. The nilpotent condition is a closure condition.

Chemistry: chirality, order and molecular knots

65. Pasteur, L. (1848). "Mémoire sur la relation qui peut exister entre la forme cristalline et la composition chimique, et sur la cause de la polarisation rotatoire." *Comptes rendus de l'Académie des sciences* 26, 535–538. Why read? Left- and right-handed crystals of one compound. The chemical counterpart of the direction of a curve.
66. Mendelejeff, D. (1869). *Zeitschrift für Chemie* 12, 405–406. Why read? The periodic system: properties of elements return in periods. In today's table the periods have lengths 2, 8, 8, 18, 18, 32, 32. A closure pattern in chemistry.
67. Frisch, H. L., and Wasserman, E. (1961). "Chemical topology." *Journal of the American Chemical Society* 83(18), 3789–3795. [doi:10.1021/ja01479a015](https://doi.org/10.1021/ja01479a015). Why read? The programme for molecules that are knots and links. Isomers that differ only in topology.
68. Dietrich-Buchecker, C. O., Sauvage, J.-P., and Kintzinger, J.-P. (1983). "Une nouvelle famille de molécules : les métallo-caténanes." *Tetrahedron Letters* 24(46), 5095–5098. [doi:10.1016/S0040-4039\(00\)94050-4](https://doi.org/10.1016/S0040-4039(00)94050-4). Why read? The first efficient synthesis of two interlocked molecular rings. Two closed curves with linking number 1 (41).
69. Dietrich-Buchecker, C. O., and Sauvage, J.-P. (1989). "A synthetic molecular trefoil knot." *Angewandte Chemie International Edition in English* 28(2), 189–192. [doi:10.1002/anie.198901891](https://doi.org/10.1002/anie.198901891). Why read? The first molecule tied as a trefoil knot.

70. Anelli, P. L., Spencer, N., and Stoddart, J. F. (1991). "A molecular shuttle." *Journal of the American Chemical Society* 113(13), 5131–5133. [doi:10.1021/ja00013a096](https://doi.org/10.1021/ja00013a096). Why read? A ring that moves between two stations on a thread. A mechanical bond used as a switch.
71. Koumura, N., Zijlstra, R. W. J., van Delden, R. A., Harada, N., and Feringa, B. L. (1999). "Light-driven monodirectional molecular rotor." *Nature* 401(6749), 152–155. [doi:10.1038/43646](https://doi.org/10.1038/43646). Why read? A molecule that turns one way only. Chirality sets the direction of a closed cycle.
72. Taylor, W. R. (2000). "A deeply knotted protein structure and how it might fold." *Nature* 406(6798), 916–919. [doi:10.1038/35022623](https://doi.org/10.1038/35022623). Why read? A natural protein whose chain forms a knot. Knotted curves occur in living matter.

All 72 entries were checked one by one against Crossref, publisher records or the source itself by a separate judge. For entry 57 the authors, journal, volume and first page are confirmed.