The History of Music and the Vacuum.Net Theory

From the San to the present, read through the windings of the net — with a practical part for making music

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Edition v3.1 — 21 November 2026Part I: the history (Chapters 1–7). Part II: composing with the net (Chapters 8–15). Appendices: register, reference code.

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Introduction

Why this book

Music has a history of rules. Each rule works for a while. Then someone tries to go further and the rule breaks. Musicians do not give up at that point. They make something new that repairs the break. The repair always costs something. A property that the old music took for granted is lost.

This book follows that pattern from the oldest music we know to the music of today. It names the composers and traditions where each step happened. For each step it asks three questions. Where did the old theory get stuck? Which obstacle started the next development? And where does that step sit in the windings of the Vacuum.Net theory?

The book is self-contained. The theory, the mathematics and the music are explained in the text itself. The footnotes point to the sources. The annotated reference list at the end tells the reader where to go deeper.

The barrier rule

The pattern comes from a recent essay on the history of mathematics.[1] It can be stated in four steps.

- A rule works.
- Someone tries to undo it, or to carry it one step further.
- That attempt fails. The failure is a barrier.
- A new object is made that passes the barrier. It costs a property the old system had.

Mathematics shows this many times. Subtraction could not be undone within the counting numbers. Negative numbers repaired that. They cost the idea that every number counts something. Music shows the same rule. An early example is the Pythagorean comma. Twelve pure fifths do not land on seven octaves. The repair was temperament. It closed the circle of fifths. It cost the purity of the intervals. Chapter 3 tells that story in full.

The Vacuum.Net theory in brief

The Vacuum.Net theory starts with movement. In the beginning there is a pulsation, a going back and forth: <-.->. Nothing happens without movement. What stands at a crossing of the net is therefore not a value. It is a direction: forward, backward, or standing still. Three directions make a *trit*. Rows of trits are numbers in balanced ternary.

The net itself is made of such rows. Each row, called a stream, is joined by a path to its tail. The tail is the same row with its first trit removed. A stream that is its own tail, such as a row of only standing still, closes on itself. Its step is a loop.[2]

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[1] Konstapel, J. (2026). The Next Mathematics. constable.blog, 20 November 2026. Source of the barrier rule.

[2] Konstapel, J. (2026). Upward in the Windings: What the net makes by itself, and which coupling makes which level, v2.0. Constable Research, Leiden.

A *winding* is a movement that comes back to where it started. One trip around a circle is a winding of the first level. A loop of loops that sweeps a surface is a winding of the second level. Higher levels exist. The theory asks which windings the net makes by itself, and which ones appear only when nets are coupled or when cells are glued on by hand. The answer is a table with one row per level.[3] Each cell of that table carries a status. It is proved by machine (in the proof language Agda), a known theorem, an argument on paper, a reading, or open.

Four kinds of construction lift the net from level to level. The bare net makes parts, turns and order. Gluing a cell by hand makes a twist: a turn that vanishes after a fixed number of rounds. *Suspension* hangs the net between two poles. It lifts a choice to a turn and a turn to a surface. The *join* connects every point of one net to every point of another. It makes interlocked cycles: the winding of the third level that mathematicians call the Hopf map.[4]

The bridge: the geometry of chords

Music can be put into this language because music theory already has a geometry. Dmitri Tymoczko showed in 2006 that chords are points in a space.[5] A pitch class, a note without its octave, is a point on a circle. A chord of n notes in a fixed order is a point on an n-dimensional torus. A chord without order is a point of the torus divided by the swapping of voices. That quotient is an *orbifold*. Voice leading, the way one chord moves to the next, is a path in that space. Callender, Quinn and Tymoczko then showed that five equivalences shape all of music theory. They are octave, permutation, transposition, inversion and cardinality: “OPTIC”.[6]

Three results from mathematics turn this geometry into windings. They are the backbone of Chapters 4 and 5.

- Chords with at most three distinct pitch classes, doublings ignored, form the three-sphere S^3. Transposition turns this sphere around in circles. Only two chords are special: the tritone and the augmented triad. They are the two exceptional circles of the motion.[7] [Known]
- Dividing out transposition gives a map from S^3 to a two-sphere. That map is a winding of the third level. Its Hopf invariant is -6, which is 2×3. The circle of a single note and the circle of a major triad wind six times around each other.[8] [Computed; follows from known theorems]

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[3]Konstapel, J. (2026). Upward in the Windings: What the net makes by itself, and which coupling makes which level, v2.0. Constable Research, Leiden.

[4]Konstapel, J. (2026). Upward in the Windings: What the net makes by itself, and which coupling makes which level, v2.0. Constable Research, Leiden.

[5]Tymoczko, D. (2006). The Geometry of Musical Chords. Science 313, 72–74.

[6]Callender, C., Quinn, I. & Tymoczko, D. (2008). Generalized Voice-Leading Spaces. Science 320, 346–348.

[7]Tuffley, C. (2002). Finite subset spaces of S^1. Algebraic & Geometric Topology 2, 1119–1145. See the proof of his Theorem 3.

[8]Computed for this book: Gauss linking integral of two regular orbits of the (2,–3) circle action on S^3. Result −6.000. Script available on request.

- Only at one, and at three, pitch classes is this space a closed manifold.[9] With four or five pitch classes it is not. In this sense the triad is topologically unique. [Known theorem; the musical reading is new as far as we found]

The third-level winding also links music to recent work by Bludov, Fomin and Musin. They prove that any essential map from S^3 to S^2 needs at least nine tetrahedra over each triangle of the target.[10] The map that transposition makes is such an essential map.

The windings as the plan of the book

The table below sets the levels of the net next to the music. It is the plan of the book. Each row is a reading until a chapter shows otherwise.

Level	What appears in the net	What appears in music	Chapter
0 · parts	which streams belong together	the pulse; voices that only make sense together (hocket)	1
1 · turns	a cycle that returns ($\mathbb{Z}$)	the timeline cycle; later the octave as a circle	1, 2
1 · order	steps whose order matters	asymmetric timelines; modes as rotations of one scale	1, 2
1 · twist	a turn that vanishes after n rounds; needs a glued cell	the tempered circle of twelve fifths	3
3 · interlocked	the Hopf winding (join)	chords of at most three pitch classes: S^3; transposition gives winding −6	4
above 3	spheres of higher dimension	four and five pitch classes: no longer a manifold; chromaticism, atonality	5
new objects	glued cells, new couplings	series, spectra, process, electronics, algorithms	6

Method and labels

The work follows a fixed order. First collect. Then follow every question that arises. Draw short conclusions. Collect again and see where the lines converge. Every claim in the book carries one of three labels.

- [Known]: found in the literature, with a footnote.
- [Computed]: calculated for this book. The calculation is described in a footnote.
- [Reading]: an interpretation in the language of the net. It is not a proof.

The scope is worldwide. No book can name every composer who ever lived. This book is complete in a stated sense. Every historical chapter covers all seven regions of the world: Africa; the Middle East and North Africa; South Asia; East Asia; Southeast Asia and Oceania; Europe; and the Americas. Women composers and popular music are included

[9] Kallel, S. & Sinha, D. (2009). Remarks on Finite Subset Spaces. arXiv:0902.3773, Lemma 7.4 (a result of Wagner).

[10] Bludov, M., Fomin, S.V. & Musin, O.R. (2025). On essential simplicial maps $S^3 \to S^2$. arXiv:2512.08584.

wherever they shaped a barrier. Appendix A lists every name with its region and shows the coverage in a table without empty cells. The choice follows the barriers, not fame.

Part I — The History

Part I follows the barriers of music from the oldest living traditions to the present. Each chapter names the people and traditions at one barrier, shows where the old theory got stuck, and places the step in the windings of the net.

Chapter 1 — The Pulse and the First Turn

c. 45,000 – 1400 BCE · San and Pygmy music, the first flutes, the first written song

1.1 Before the instrument: the pulse

The theory begins with a pulsation. Music begins in the same place. Before there is a melody there is a beat. Steps, clapping and breathing all go back and forth. Nothing in music happens without that movement. [Reading]

The pulse is level 0 of the net. It makes parts. It does not yet make a cycle. A row of claps can go on forever. Only when the row returns to its start does a turn appear. That return is the first real step of this chapter.

1.2 The oldest living music: the San and the Pygmy peoples

Two groups of hunter-gatherers in Africa keep a musical style that many scholars see as very old. They are the San of the Kalahari and the Pygmy peoples of the Central African forest. Their populations are among the oldest human lineages known from genetics.[11]

The Aka of the Central African Republic sing a contrapuntal polyphony in four voices. Every member of the community masters it. Each singer varies his or her part, so the music seems to change all the time. UNESCO listed this tradition as heritage of humanity in 2008.[12] Simha Arom recorded and analysed this music from the 1960s on. His book of 1991 made its structure visible to the outside world.[13]

The Ju|'hoansi San sing during the healing dance, the centre of their social life. Emmanuelle Olivier isolated the voices by re-recording them one by one. She found three vocal lines at different pitches, each with its own melody. Half of the repertoire is sung by healers. They say they receive their songs in sleep or trance, from the spirits of the dead.[14] Olivier also stresses that no two San singers vary a song in the same way. A song is made step by step in the interaction between singers. Old songs fall out of use and new ones replace them.[15]

Victor Grauer argued that the Pygmy and San styles share one root. He pointed to the same features in both: polyphony, yodel, singing without words, and above all *hocket*. In hocket each voice sings only some of the notes. The melody exists only when all voices sound together.[16] His thesis is contested. Ian Cross argued that the two styles differ more

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[11]Grauer, V.A. (2006). Echoes of Our Forgotten Ancestors. The World of Music 48(2). Expanded in: Sounding the Depths (2011).

[12]UNESCO (2008). Polyphonic singing of the Aka Pygmies of Central Africa. Representative List of the Intangible Cultural Heritage of Humanity.

[13]Arom, S. (1991). African Polyphony and Polyrhythm: Musical Structure and Methodology. Cambridge University Press.

[14]Olivier, E. (1998). Bushman vocal music: the illusion of polyphony. In: Language, identity and conceptualization among the Khoisan. Köln, 359–370.

[15]Olivier, E. The "Success" of San Music. Cultural Survival Quarterly.

[16]Grauer, V.A. (2006). Echoes of Our Forgotten Ancestors. The World of Music 48(2). Expanded in: Sounding the Depths (2011).

than Grauer allows, and that styles can change fast.[17] This book does not need the common root. It needs only the structure. [Known: the structure. Contested: the shared root.]

Hocket fits level 0 of the net exactly. The bottom row of the table asks which streams belong together. In hocket no single voice is the song. The parts belong together only in the whole. [Reading]

1.3 The timeline: the first turn

Most of this music runs on a *timeline*: a short rhythm, often struck on a bell or clapped, that repeats without end. Adrian Poole compared the timelines of Pygmy and San music. They have cycles of 8, 9, 12, 16, 20, 24 or 32 pulses. Almost all of them are maximally or nearly maximally even: the strokes are spread as evenly as the cycle allows. Many contain a cell of three against two.[18] Poole is careful about ancestry. His tree of timelines has no clear root.

A timeline is a winding of level 1. It is a movement that returns to its start. The cycle can be travelled once, twice or any whole number of times. That is the integer loop of the net: the row "1 · turns" of the table. [Reading]

Many timelines are asymmetric. Arom called this "rhythmic oddity": the cycle cannot be split into two equal halves that match.[19] Where you start in such a cycle matters. Starting at another stroke gives a different feel. In the net this is the row "1 · order": steps whose order matters. [Reading]

1.4 Euclid in the drum: maximal evenness

In 2005 Godfried Toussaint found that the Euclidean algorithm generates these rhythms. The algorithm is Euclid's method for the greatest common divisor of two numbers. Put k strokes into n pulses as evenly as possible, and the steps of Euclid's algorithm give the answer. More than forty timelines from world music come out of it.[20] Demaine and co-authors proved that these Euclidean rhythms are exactly the rhythms with maximal evenness.[21]

The most common timeline of sub-Saharan Africa is the 12-pulse "standard pattern". It has seven strokes. Kofi Agawu traced how widely it is used and how scholars have read

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[17]Cross, I. (2006). Clues from our present peers? A response to Victor Grauer. The World of Music 48(2).

[18]Poole, A. Study of timeline rhythms in Pygmy and Bushmen musical cultures. Empirical Musicology Review (after 2016).

[19]Arom, S. (1991). African Polyphony and Polyrhythm: Musical Structure and Methodology. Cambridge University Press.

[20]Toussaint, G. (2005). The Euclidean Algorithm Generates Traditional Musical Rhythms. Proceedings of BRIDGES, 47–56.

[21]Demaine, E.D. et al. (2009). The distance geometry of music. Computational Geometry 42, 429–454.

it.[22] Its strokes fall on pulses 0, 2, 4, 5, 7, 9 and 11. The gaps between strokes are 2-2-1-2-2-2-1. These are exactly the steps of the major scale. [Computed][23]

This is a striking fact. The same rule spreads seven strokes over twelve pulses in time and seven notes over twelve semitones in pitch. John Clough and Jack Douthett called such sets *maximally even*. Their list for twelve contains the diatonic scale, the whole-tone scale, the diminished-seventh chord, the augmented triad and the bare tritone.[24] The last two are the two exceptional circles of the three-sphere in Chapter 4. Maximal evenness is the thread that runs from the bell in the forest to the geometry of the triad. [Known: the facts. Reading: the thread.]

1.5 The first instruments: pitch fixed in bone

The oldest certain instruments are flutes from the caves of the Swabian Alb in southern Germany. At Hohle Fels a flute was found made from the wing bone of a griffon vulture. It has five finger holes and a V-shaped mouthpiece.[25] New radiocarbon dates gave the flutes from nearby Geissenklösterle an age of about 42,000 to 43,000 years.[26] Older claims, such as the Divje Babe bone, are disputed and not used here.[27]

At Jiahu in Henan, China, six complete flutes of red-crowned crane bone were found, with fragments of some thirty more. They are about 9,000 years old and have five to eight holes. The best preserved one could still be played and its tones were analysed.[28]

A flute does something a voice does not. It fixes pitch in matter. The holes are steps that stay where they are. A singer can bend a note. A hole cannot. The flute is the first store of the musical step. [Reading]

1.6 The first written song

Around 1400 BCE, in the palace of Ugarit on the Syrian coast, scribes wrote songs on clay tablets in cuneiform. One of them is almost complete. It is the hymn to the goddess Nikkal, known as h.6. It is the oldest substantially complete notated music in the

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[22]Agawu, K. (2006). Structural analysis or cultural analysis? Competing perspectives on the “standard pattern” of West African rhythm. Journal of the American Musicological Society 59(1), 1–46.

[23]Computed for this book: the strokes of the 12-pulse standard pattern lie on pulses 0, 2, 4, 5, 7, 9, 11. The gaps are 2-2-1-2-2-2-1. These are the steps of the major scale. The Euclidean rhythm E(7,12) gives a rotation of the same cycle.

[24]Clough, J. & Douthett, J. (1991). Maximally Even Sets. Journal of Music Theory 35(1/2), 93–173.

[25]Conard, N.J., Malina, M. & Münzel, S.C. (2009). New flutes document the earliest musical tradition in southwestern Germany. Nature 460, 737–740.

[26]Higham, T. et al. (2012). Testing models for the beginnings of the Aurignacian and the advent of figurative art and music. Journal of Human Evolution 62, 664–676.

[27]The Divje Babe bone (Slovenia, 1995) has been proposed as a Neanderthal flute. Many specialists read its holes as carnivore bites. It is not used as evidence here.

[28]Zhang, J., Harbottle, G., Wang, C. & Kong, Z. (1999). Oldest playable musical instruments found at Jiahu early Neolithic site in China. Nature 401, 366–368.

world.[29] Its notation names intervals on a nine-stringed lyre. It refers to a diatonic tuning system known from Babylonian tablets.[30] The colophon names two people: Urhiya, from whose collection the song came, and Ammurapi, who copied it.[31] They are the first named people in this book. The composer of h.6 is unknown.

1.7 The barrier: a living song cannot be carried back

The music of the San and the Aka lives by variation. Every performance is new. No two singers vary a part alike. Songs are dropped and replaced. That is its strength.

But it sets a barrier. A song cannot be carried unchanged across a great distance or many generations. Try to undo the performance, to bring back exactly what was sung, and it fails. The song has already changed. [Reading, in the sense of the barrier rule]

The repair came in two steps. First the instrument fixed the steps in bone. Then the tablet fixed the strings and their intervals in signs. Ugarit shows the result: a named tuning, a named song, a named copyist. The new object is the fixed and written step. The property that was lost is the living variation. Music became something that can be copied. It also became something that can freeze.

This repair opens the next chapter. Once strings are named and tuned, they can be compared. Once they are compared, the octave appears as a circle. And on that circle the next barrier waits: the fifths that do not close.

1.8 Where the reading stops

Three limits must be stated plainly.

- The links between hocket and level 0, and between timelines and level 1, are readings. No proof in Agda or on paper connects them to the net yet.
- The common root of San and Pygmy music is contested. Nothing in this chapter depends on it.
- The match between the standard pattern and the major scale is computed and exact. Whether it reflects one cause, or two separate uses of the same arithmetic, is open.

1.9 Further figures at this barrier

Other living traditions show the same first levels in other places.

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[29] Laroche, E. (1968). Ugaritica V; Kilmer, A.D. (1974). The cult song with music from ancient Ugarit. Revue d'Assyriologie 68; Dumbrill, R. (2005). The Archaeomusicology of the Ancient Near East.

[30] Kilmer, A.D. & Crocker, R.L. (1976). Sounds from Silence. Bit Enki Records. Reconstruction of the Hurrian hymn and the Babylonian tuning texts.

[31] World's Oldest Musical Notation Deciphered on Cuneiform Tablet. Biblical Archaeology Review (BAS Library). Gives the colophon naming Urhiya and Ammurapi.

- Among the Inuit, two women face each other and exchange breath sounds in the vocal game katajjaq. The game ends when one runs out of breath or laughs.[32] It is hocket and pulse at once: level 0 as play.
- In India the Vedic chant has been passed on orally for more than three thousand years with great precision. It is the opposite answer to the San: exact repetition instead of variation. UNESCO lists the tradition of Vedic chanting as intangible heritage. [Known]
- Aboriginal Australian song traditions carry knowledge of land and law in song cycles, the songlines, over a drone such as the didjeridu. A route through the land is sung as a sequence: a path with places on it. [Known: the practice. Reading: the path.]

[32]Katajjaq: two Inuit women face each other and exchange breath sounds until one runs out of breath or laughs. The Canadian Encyclopedia, "Inuit Vocal Games"; D'Souza, F. (2021). From Vocal Game to Protest Song: The Complexities of Katajjaq. SOAS Undergraduate Research Journal 2(1), 53–61.

Chapter 2 — The Octave as a Circle

c. 1800 BCE – 1300 CE · Babylon, China, India, Greece, the Islamic world, medieval Europe

2.1 Counting forward and back: the Babylonian strings

The first theory of pitch we can read comes from Mesopotamia. Cuneiform tablets name the nine strings of a lyre or harp. The strings are numbered first, second, third, fourth, fifth, and then "fourth behind", "third behind", "second behind", "behind". In numbers: 1-2-3-4-5-4-3-2-1.[33]

That numbering counts forward to a middle and then back. It is the first written trace of a movement that goes out and returns. In the language of the net it has two directions around a centre: -> and <-. [Reading]

An Old Babylonian text from Ur, UET VII 74, gives instructions to retune the instrument string by string.[34] Each step moves the whole tuning to the next one. Seven tunings follow each other and the last leads back to the first. Because each step uses fourths and fifths, the result is Pythagorean, whatever the starting point.[35] The Hurrian hymn of Chapter 1 was written in this system. [Known]

Here pitch makes its first full turn. Not a rhythm returns, as in Chapter 1, but a whole system of tunings. This is level 1 of the net in the domain of pitch. [Reading]

2.2 China: the twelve pitch-pipes

In 433 BCE Marquis Yi of Zeng was buried with his orchestra. The tomb held a set of some sixty-five bronze bells on three levels. Each bell gives two tones, depending on where it is struck. In the middle range the set plays all twelve semitones.[36] The inscriptions on the bells and chime stones are the earliest known Chinese texts on music theory.[37]

Chinese theory built the twelve *lü*, the standard pitch-pipes, by alternately shortening and lengthening a pipe by one third. That is a chain of fifths and fourths. Around 122 BCE the Huainanzi recorded that twelve such steps do not close exactly. The court official Jing Fang (78–37 BCE) went on to sixty steps. He found that after fifty-three

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[33]Music tablet from Sippar (BM 65217 + 66616). Iraq (Cambridge). The article reviews CBS 10996 and UET VII 126, where the nine strings are numbered first to fifth and then "fourth behind" back to "behind": 1-2-3-4-5-4-3-2-1.

[34]Gurney, O.R. (1968). An Old Babylonian Treatise on the Tuning of the Harp. Iraq 30, 229–233; Gurney, O.R. (1994). Babylonian Music Again. Iraq 56. The text UET VII 74 gives step-by-step retuning of the instrument.

[35]The Tuning of the Babylonian Harp. Iraq (Cambridge). The author notes that a tuning by fourths and fifths ends Pythagorean, whatever the starting intonation.

[36]Bianzhong of Marquis Yi of Zeng (433 BCE): 64–65 bells in eight groups on three levels; each bell gives two tones; the middle range gives all twelve semitones. Hubei Provincial Museum; summary in Smithsonian Magazine.

[37]Bagley, R. (2005). The Prehistory of Chinese Music Theory. Proceedings of the British Academy 131, 41–90. The inscriptions on the Zeng instruments are the earliest Chinese texts on music theory.

fifths the chain comes very close to its start: 31 octaves.[38] [39] The remaining gap is only 3.6 cents. [Known; the cents value computed][40]

2.3 India: twenty-two shrutis and the rotation of scales

The Natyashastra, attributed to Bharata and dated between 400 BCE and 200 CE, divides the octave into twenty-two *shrutis*.[41] It knows two parent scales, the sa-grama and the ma-grama. Each scale can start on each of its seven degrees. These seven rotations are the *mūrchanās*.[42]

A mūrchanā is a scale read from another starting point. The steps are the same but their order around the circle starts elsewhere. That is the row "1 · order" of the net: steps whose order matters. The Greek modes and the later church modes follow the same idea. [Reading]

2.4 Greece: numbers against the ear

The Pythagoreans measured intervals as ratios of string lengths: 2:1 for the octave, 3:2 for the fifth, 4:3 for the fourth. Two whole tones subtracted from a fourth leave a small remainder. They called it the *leimma*. Aristoxenus of Tarentum (4th century BCE) took the other side. He let the ear judge, built scales from tetrachords, and accepted that some divisions cannot be expressed as ratios.[43] [Known]

The quarrel is a first version of the barrier of this chapter. Ratios close only on themselves. The ear wants a circle that closes. The two do not agree.

2.5 The Islamic world: theory and practice

Al-Farabi (872–950/951) wrote the Great Book of Music, the Kitab al-Musiqa al-Kabir. He split music into a theoretical and a practical part, and described the intervals and instruments of his time.[44] Three centuries later Safi al-Din al-Urmawi (d. 1294) fixed a system of seventeen tones per octave, built from Pythagorean steps. His Kitab al-Adwar

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[38] Hegesh, N. (2021). Mind the Gap: Acoustical Answers to Cosmological Concerns in First-Century BCE China. Isis. On Jing Fang (78–37 BCE) and his sixty tuning standards.

[39] Jing Fang noticed that 53 fifths come very close to 31 octaves. The Pythagorean comma between 12 fifths and 7 octaves had been published c. 122 BCE in the Huainanzi. See McClain & Ming (1979), Chinese Cyclic Tunings in Late Antiquity, Ethnomusicology 23.

[40] Computed for this book: Pythagorean comma $3^{12}/2^{19}$ = 23.46 cents; syntonic comma 81/80 = 21.51 cents; 53 fifths against 31 octaves: 3.62 cents; equal-tempered fifth 1.96 cents narrow; equal-tempered major third 13.69 cents wide of 5/4.

[41] Bel, B. (1988, rev. 2020). A Mathematical Discussion of the Ancient Theory of Scales according to Natyashastra. He dates the text between 400 BCE and 200 CE and finds two cycles of fifths a major third apart.

[42] Bharata, Natyashastra, ch. 28: twenty-two shrutis in the octave; two gramas; mūrchanās as the scales starting on each degree. English translation by M. Ghosh (1951–61).

[43] Aristoxenus, Elementa harmonica (4th c. BCE). See Barker, A. (2007), The Science of Harmonics in Classical Greece, Cambridge University Press. Aristoxenus accepted irrationality as a fact of musical life.

[44] Al-Farabi (872–950/951), Kitab al-Musiqa al-Kabir (The Great Book of Music), divided into theory and practice. French translation in R. d'Erlanger, La Musique arabe, vol. 1–2 (1930–35).

became the most copied Arabic treatise on music.[45] Seventeen is another attempt to bring the chain of fifths near its start.

2.6 Europe: the staff and the hand

Guido of Arezzo wrote his Micrologus around 1025–26. He set staff lines a third apart and gave singers the syllables ut, re, mi, fa, sol, la from the hymn Ut queant laxis.[46] With the staff, pitch became a place on a page. Around 1200 the singers of Notre-Dame, Léonin and Pérotin, built long polyphonic pieces on chant. Hildegard of Bingen (1098–1179) left a large body of monophonic song. All of this ran on Pythagorean intervals. [Known]

2.7 The barrier: the circle of fifths does not close

Every tradition in this chapter met the same obstacle. Twelve pure fifths overshoot seven octaves by 23.46 cents: the Pythagorean comma.[47] Go around the circle of fifths and you do not come back where you started. The undo fails. The chain is a spiral, not a circle. [Computed]

China tried 60 steps, Islam 17, Europe kept 12 with one bad fifth. None closed. A second gap waited as soon as thirds became important. The Pythagorean major third is 81/64. The pure third is 5/4. They differ by the syntonic comma, 21.5 cents.[48]

In the net this is exactly the row "1 · twist". A turn that should vanish after twelve rounds does not vanish by itself. It needs a cell glued on by hand.[49] That cell is temperament: the subject of the next chapter. [Reading]

2.8 Where the reading stops

- The forward-and-back numbering of the Babylonian strings is a fact. Reading it as the two directions of the net is an interpretation.
- Bharata's shrutis are interpreted in very different ways. Nothing here depends on one reading.

2.9 Further figures at this barrier

Several more figures carried the circle from antiquity to the Middle Ages.

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[45]Safi al-Din al-Urmawi (d. 1294), Kitab al-Adwar. A Pythagorean system of seventeen tones per octave. Wright, O. (1978), The Modal System of Arab and Persian Music AD 1250–1300, Oxford University Press.

[46]Guido of Arezzo, Micrologus (c. 1025–26). Staff lines a third apart and the syllables ut-re-mi-fa-sol-la from the hymn Ut queant laxis.

[47]Computed for this book: Pythagorean comma $3^{12}/2^{19}$ = 23.46 cents; syntonic comma 81/80 = 21.51 cents; 53 fifths against 31 octaves: 3.62 cents; equal-tempered fifth 1.96 cents narrow; equal-tempered major third 13.69 cents wide of 5/4.

[48]Computed for this book: Pythagorean comma $3^{12}/2^{19}$ = 23.46 cents; syntonic comma 81/80 = 21.51 cents; 53 fifths against 31 octaves: 3.62 cents; equal-tempered fifth 1.96 cents narrow; equal-tempered major third 13.69 cents wide of 5/4.

[49]Konstapel, J. (2026). Upward in the Windings: What the net makes by itself, and which coupling makes which level, v2.0. Constable Research, Leiden.

- *Ptolemy* (c. 100–170) wrote the Harmonics. He tested ratios against the ear and proposed tetrachord divisions with pure thirds. *Boethius* (c. 477–524) compiled De institutione musica from Greek sources, mainly Nicomachus and Ptolemy. He split music into three kinds: the music of the cosmos, of the human being and of instruments. Europe read music theory through him for a thousand years.[50]
- *Kassia* (9th century) wrote hymns for the Byzantine church. She is among the first women composers whose music survives.
- *Ziryab* (c. 789–857) came from Baghdad to Córdoba in 822. He is credited with adding a fifth pair of strings to the oud and with one of the first music schools in Córdoba.[51] Through him the theory of the Islamic world met Europe.
- *Ibn Sina* (980–1037) treated music as part of the mathematical sciences in his encyclopaedia, between al-Farabi and al-Urmawi.
- *Saint Yared* (6th century) is honoured as the creator of Ethiopian chant, zema, in three modes. Its notation, melekket, was codified in the 16th century.[52] *The octoechos*, a system of eight modes, came from 6th-century Syria into Byzantine chant; its 9th-century book is one of the first with musical notation.[53] Eight modes in a weekly cycle: a level-1 turn through the year.
- In the Andes the panpipe *siku* is split into two rows, ira and arka. Two players alternate note by note to make one melody: hocket, as in Chapter 1.[54] In northern Vietnam the *Dong Son* culture cast large bronze drums. In Japan the court orchestra *gagaku*, brought from Tang China, still plays.[55]

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[50]Burkholder, J.P., Grout, D.J. & Palisca, C.V. A History of Western Music (Norton), ch. 2: Boethius compiled De institutione musica mainly from Nicomachus and the first book of Ptolemy's Harmonics; his three kinds of music are musica mundana, humana and instrumentalis.

[51]Ziryab (c. 789–857) reached Córdoba in 822 and served Abd al-Rahman II. He is credited with a fifth pair of strings on the oud and with one of the first schools of music in Córdoba. Reynolds, D.F., "Ziryab", UC Santa Barbara.

[52]Britannica, "Ethiopian chant": the three zema modes are said to have been revealed to Saint Yared in the 6th century; the melekket notation was codified in the 16th century; the oldest manuscripts date from the 14th.

[53]The octoechos, eight modes, is thought to come from 6th-century Syria; the Byzantine book Octoechos (9th century) is one of the first hymn books with notation.

[54]The siku is played by two people: the ira and arka rows alternate note by note to make one melody. Ohio State University, Andean Music Teacher Guide; Turino, T. (1993). Moving Away from Silence. University of Chicago Press.

[55]UNESCO (2001/2008). Kun Qu Opera, developed under the Ming dynasty in Kunshan. Gagaku was inscribed in 2009.

Chapter 3 — The Twist: Comma and Temperament

c. 1300 – 1750 · Ars nova, meantone, equal temperament in China and Europe, Bach; answers in India, Java and the Arab world

3.1 Two cycles at once: the Ars nova

Around 1320 Philippe de Vitry and Johannes de Muris described a new art of notation. It allowed duple as well as triple division of note values. Composers built motets on *isorhythm*: a rhythmic pattern, the talea, repeated against a melodic pattern, the color, of a different length.[56] Guillaume de Machaut wrote the first complete Mass setting conceived as one work. The new style also brought a taste for thirds and sixths.[57] [Known]

Isorhythm runs two cycles of different length at the same time. Together they trace a path on a torus, the surface of two independent turns. Only after the least common multiple of the two lengths do both return together. [Reading]

3.2 The barrier of the third

The taste for thirds collided with Pythagorean tuning. Its major third is 21.5 cents wider than the pure third.[58] John Dunstable and the English "contenance angloise" brought full sonorities of thirds and sixths to the Continent. Dufay, Ockeghem, Josquin and later Palestrina wrote in that sound.[59]

The repair was *meantone*. Pietro Aron described quarter-comma meantone in 1523. Gioseffo Zarlino gave exact versions in 1558. He also founded just intonation on the numbers one to six, the *senario*, and was the first to put the triad above the interval as the unit of harmony.[60] Meantone made thirds pure by narrowing every fifth. It left one fifth so bad it was called the wolf. The circle still did not close. [Known]

3.3 Closing the circle: equal temperament

The Ming prince Zhu Zaiyu published an exact calculation of twelve-tone equal temperament in 1584. Simon Stevin in Holland gave a mathematical definition with a less precise calculation in 1585 or later. They worked independently. Both treated it as a

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[56] Philippe de Vitry, Ars nova (c. 1320–22); Johannes de Muris, Ars novae musicae (c. 1320). Isorhythm repeats a rhythmic pattern (talea) against a melodic pattern (color).

[57] Britannica, "Western music: Ars Nova": the new style brought triple and duple divisions and a taste for thirds and sixths; Dunstable's presence on the Continent shaped Dufay.

[58] Computed for this book: Pythagorean comma $3^{12}/2^{19} = 23.46$ cents; syntonic comma $81/80 = 21.51$ cents; 53 fifths against 31 octaves: 3.62 cents; equal-tempered fifth 1.96 cents narrow; equal-tempered major third 13.69 cents wide of 5/4.

[59] Britannica, "Western music: Ars Nova": the new style brought triple and duple divisions and a taste for thirds and sixths; Dunstable's presence on the Continent shaped Dufay.

[60] Pietro Aron, Toscanello in musica (Venice 1523): the first description of quarter-comma meantone. Zarlino, Le istitutioni harmoniche (Venice 1558): 2/7-comma meantone with exactness, just intonation on the senario (1–6), and the primacy of the triad.

computation more than a musical practice.[61] Zhu credited the zither, the qin, with knowledge that the twelve pitch-pipes could not convey.[62]

In Germany Andreas Werckmeister used the word “wohltemperiert” from 1681 and described several well temperaments in 1691. In 1722 Johann Sebastian Bach published Das wohltemperirte Clavier: preludes and fugues in all twenty-four keys.[63] Equal temperament itself became the standard only in the late eighteenth century or later.[64] [Known]

Equal temperament closes the circle by force. Each fifth is made 1.96 cents narrow. Twelve of them now land exactly on seven octaves.[65] The cost is that no interval except the octave is pure. The major third is 13.7 cents too wide.[66] [Computed]

3.4 Other answers to the same barrier

Not every culture closed the circle in the same way.

- India closed it by counting. Around 1650 Venkatamakhin gave sixteen names to the twelve semitones and derived 72 parent scales, the *melas*. He claimed the scheme contained every scale past and future. He could name ragas for only nineteen of them.[67]
- Java closed it locally. A gamelan has two tunings, slendro with five nearly equal steps and pelog with seven unequal ones. There is no standard: every gamelan has its own tuning and is a world of its own.[68]

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[61]Kuttner, F.A. (1975). Prince Chu Tsai-Yü’s Life and Work. Ethnomusicology 19, 163–206; Robinson, K. (1980). A Critical Study of Chu Tsai-yü’s Contribution to the Theory of Equal Temperament. Zhu Zaiyu published in 1584; Stevin’s derivation came in 1585 or later.

[62]Rehding, A. Fine-Tuning a Global History of Music Theory: Divergences, Zhu Zaiyu, and Music-Theoretical Instruments. Zhu credits the qin for knowledge that the twelve pitch-pipes could not convey.

[63]Werckmeister, A. Orgel-Probe (1681) and Musicalische Temperatur (Quedlinburg 1691). Bach, J.S. Das wohltemperirte Clavier (1722): preludes and fugues in all 24 keys. Equal temperament became standard only in the late 18th century or later.

[64]Werckmeister, A. Orgel-Probe (1681) and Musicalische Temperatur (Quedlinburg 1691). Bach, J.S. Das wohltemperirte Clavier (1722): preludes and fugues in all 24 keys. Equal temperament became standard only in the late 18th century or later.

[65]Computed for this book: Pythagorean comma $3^{12}/2^{19} = 23.46$ cents; syntonic comma $81/80 = 21.51$ cents; 53 fifths against 31 octaves: 3.62 cents; equal-tempered fifth 1.96 cents narrow; equal-tempered major third 13.69 cents wide of 5/4.

[66]Computed for this book: Pythagorean comma $3^{12}/2^{19} = 23.46$ cents; syntonic comma $81/80 = 21.51$ cents; 53 fifths against 31 octaves: 3.62 cents; equal-tempered fifth 1.96 cents narrow; equal-tempered major third 13.69 cents wide of 5/4.

[67]Venkatamakhin, Chaturdandi Prakashika (c. 1650). Sixteen svara names on twelve semitones give 72 melas; he could identify ragas for only 19 of them.

[68]Javanese gamelan: slendro (five roughly equal steps) and pelog (seven unequal steps). There is no standard form; each gamelan has its own tuning. Lindsay, J. (1992), Javanese Gamelan, Oxford University Press.

- The Arab world moved from seventeen tones to twenty-four. Mikhail Mishaqa proposed twenty-four equal quarter tones around 1840. The Cairo Congress of 1932 discussed standardising on it.[69]

3.5 The twist in the net

The row "1 · twist" of the net is a turn that vanishes after n rounds. The bare net cannot make it. It needs a cell glued on by hand. Upward in the Windings proves this in Agda for n = 2; the general case is a classical result.[70]

Equal temperament is that cell for n = 12. Twelve fifths become the identity only because the comma is glued away. The barrier rule is visible in full. The rule was "stack pure fifths". Its undo, going around the circle and back, failed. The new object is the tempered fifth. The lost property is purity. Gamelan shows the other choice: keep the tuning living, give up the shared circle. [Reading]

3.6 The barrier that opens the next chapter

A closed circle of keys invites travel. To travel between keys, music needs a unit larger than the interval. Zarlino had already pointed to the triad. The next chapter follows the triad as it becomes the centre of music, and shows why it is a very special object.

3.7 Further figures at this barrier

The meantone centuries had many more players.

- *Francesco Landini* (c. 1325–1397) led the Italian trecento. *Gilles Binchois* (c. 1400–1460), *Johannes Ockeghem* (c. 1410–1497), *Heinrich Isaac* (c. 1450–1517) and *Jacob Obrecht* (1457/58–1505) built Franco-Flemish polyphony in the new sound of thirds. *Orlando di Lasso* (1532–1594), *William Byrd* (c. 1540–1623) and *Jan Pieterszoon Sweelinck* (1562–1621) brought it to its fullest form.
- *Nicola Vicentino* (1511–c. 1576) built the archicembalo, a keyboard with 31 keys to the octave. Extended quarter-comma meantone comes very close to 31 equal steps.[71] The fifths of the two systems differ by only 0.2 cents.[72] *Christiaan Huygens* (1629–1695) recognised this in 1661.[73] The circle can close with pure thirds, if one accepts 31 steps instead of 12. [Known; difference computed]

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[69]Mikhail Mishaqa (1800–1888), Essay on the Art of Music (c. 1840): 24 equal quarter tones. The Cairo Congress of Arab Music (1932) discussed standardisation on 24 equal steps.

[70]Konstapel, J. (2026). Upward in the Windings: What the net makes by itself, and which coupling makes which level, v2.0. Constable Research, Leiden.

[71]Vicentino, N. (1555). L'antica musica ridotta alla moderna prattica. His archicembalo had 31 keys to the octave. Extended quarter-comma meantone comes very close to 31 equal steps; Huygens recognised this in 1661 and published it in 1691.

[72]Computed for this book: the 31-ET fifth (696.77 cents) differs from the quarter-comma fifth (696.58) by 0.20 cents. The 31-ET major third is 387.10 cents against the pure 386.31.

[73]Vicentino, N. (1555). L'antica musica ridotta alla moderna prattica. His archicembalo had 31 keys to the octave. Extended quarter-comma meantone comes very close to 31 equal steps; Huygens recognised this in 1661 and published it in 1691.

- *Carlo Gesualdo* (1566–1613) pushed chromaticism to the edge of what meantone allows. *Vincenzo Galilei* (c. 1520–1591) argued for equal semitones on the lute. *Marin Mersenne* (1588–1648) described tunings and acoustics in his Harmonie universelle (1636).
- *Purandara Dasa* (1484–1564) is called the grandfather of Carnatic music. He laid out graded exercises and made the scale Mayamalavagowla the first that beginners learn.[74] *Tansen* (c. 1500–1589) was the leading musician at the Mughal court of Akbar. Indian music kept its own closure: through teaching and the raga, not through temperament.
- In West Africa the *griots* of the Mali Empire carried history in song, with kora and balafon, from the 13th century on. In China *Kunqu* opera grew under the Ming dynasty in Kunshan.[75] In Japan *Yatsuhashi Kengyō* (1614–1685) founded modern koto music and its standard tuning, hirajōshi.[76] In the Ottoman world *Dimitrie Cantemir* (1673–1723) wrote down the repertoire in his own alphabetic notation.[77]
- In Peru *Juan Pérez Bocanegra* printed Hanacpachap cussicuinin in 1631: polyphony in Quechua, among the first printed in the Americas.[78]

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[74]Purandara Dasa (1484–1564) is called the Pitamaha, the grandfather, of Carnatic music. He structured graded exercises (svaravalis, alankaras) and made raga Mayamalavagowla the first scale for beginners.

[75]UNESCO (2001/2008). Kun Qu Opera, developed under the Ming dynasty in Kunshan. Gagaku was inscribed in 2009.

[76]Yatsuhashi Kengyō (1614–1685) founded modern koto music; hirajōshi remained the standard tuning. Yamada Kengyō (1751–1817) founded the Yamada school. Britannica, “Japanese music: Koto music”.

[77]Dimitrie Cantemir (1673–1723) wrote an Ottoman music treatise with an alphabetic notation. Hamparsum notation recorded the Ottoman repertoire from the early 19th century. Wright, O. (1992–2000). Demetrius Cantemir: The Collection of Notations. SOAS.

[78]Berliner, P. (1978). The Soul of Mbira. University of California Press. Pérez Bocanegra, J. (1631). Ritual formulario. Lima.

Chapter 4 — The Triad and the Three-Sphere

c. 1600 – 1850 · Monteverdi, Rameau, Euler, Bach to Schubert; the geometry of three-note chords

4.1 Words free the dissonance

In 1600 and 1603 the Bolognese theorist Giovanni Maria Artusi attacked madrigals by an unnamed composer for their “crudities”. The composer was Claudio Monteverdi. One target was Cruda Amarilli. Monteverdi answered in the preface to his fifth book of madrigals in 1605. He called the old way the prima pratica and his own the seconda pratica, in which the words govern the harmony.[79] Dissonance was now allowed for the sake of meaning. The bass and the top voice took over from equal voices. [Known]

4.2 The chord as a unit

In 1722 Jean-Philippe Rameau published his Traité de l’harmonie. Every chord has a root. A chord with its third or fifth in the bass is the same chord, inverted. All chords reduce to two prototypes: the consonant triad and the dissonant seventh chord. A “fundamental bass” of roots carries the music.[80] In 1739 Leonhard Euler drew a lattice of tones joined by fifths and thirds.[81] It became the Tonnetz of later theory. [Known]

4.3 Departure and return

On this ground stood a century and a half of music. Corelli and Vivaldi fixed the cadence. Bach and Handel wove counterpoint over it. Haydn, Mozart and Beethoven built the sonata form. A sonata leaves the home key, travels through others, and returns. That is a winding of level 1 in the space of keys. Schubert loved keys a third apart, a path that does not follow the circle of fifths. [Known: the music. Reading: the winding.]

4.4 The geometry of three notes

Why did the triad rule for so long? Modern mathematics gives an answer that composers could not have known.

Take all chords of at most three distinct pitch classes, ignoring doublings. Christopher Tuffley proved that this space is the three-sphere S^3. Transposition turns the whole sphere in circles. Each circle is the set of all transpositions of one chord. Almost all circles are alike. Two are exceptional. The circle of the tritone repeats after half an octave: multiplicity 2. The circle of the augmented triad repeats after a third of an octave: multiplicity 3.[82] [Known]

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[79] Artusi, G.M. (1600; 1603). L’Artusi, overo Delle imperfettioni della moderna musica. Monteverdi, C. (1605). Il quinto libro de madrigali, preface on the seconda pratica.

[80] Rameau, J.-Ph. (1722). Traité de l’harmonie réduite à ses principes naturels. Paris: Ballard. Root, fundamental bass, inversion; the triad and the seventh chord as the two prototypes.

[81] Euler, L. (1739). Tentamen novae theoriae musicae. St Petersburg. Contains the lattice of fifths and thirds later called the Tonnetz.

[82] Tuffley, C. (2002). Finite subset spaces of S^1. Algebraic & Geometric Topology 2, 1119–1145. See the proof of his Theorem 3.

Dividing out transposition gives a map from S^3 onto a two-sphere with two special points: the tritone and the augmented triad. That map is a winding of the third level. Its Hopf invariant is −6, which is 2 × 3. The circle of single notes and the circle of a major triad wind six times around each other.[83] [Computed; follows from Tuffley and the theory of Seifert fibrations]

Three more facts fit the picture.

- In Tymoczko's model of three-note chords the four augmented triads run up the centre of a twisted prism. The major and minor triads surround them.[84]
- Richard Cohn showed that major and minor triads are the minimal perturbations of the perfectly even augmented triad. They form four "hexatonic" cycles of six triads in which each step moves one voice by a semitone.[85] [86]
- In twelve-tone equal temperament, exactly one two-note chord and one three-note chord are symmetric under transposition: the tritone and the augmented triad.[87] They are the discrete traces of the two exceptional circles. [Computed]

One more theorem makes the triad unique. The space of chords with at most n distinct pitch classes is a closed manifold only for n = 1 and n = 3.[88] With four or five pitch classes the space folds in ways a manifold cannot. [Known theorem; the musical reading is new as far as we found]

4.5 The triad in the net

Upward in the Windings reaches S^3 by the *join*: every point of one net connected to every point of another. That route gives the Hopf winding with invariant 1.[89] The musical route reaches the same S^3 by letting voices merge. It gives invariant −6. Both are level 3. They are different couplings. In the net, merging voices is a cell glued on by hand: {x, x, y} is set equal to {x, y, y}. [Reading]

A reading of the history follows. For two and a half centuries composers worked in the only chord space that closes as a manifold. At its centre sit the two symmetric chords.

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[83] Computed for this book: Gauss linking integral of two regular orbits of the (2,−3) circle action on S^3. Result −6.000. Script available on request.

[84] Review of Tymoczko, A Geometry of Music (Project MUSE): in the three-note chord space the four augmented triads run up the centre of a twisted prism; the twelve major and twelve minor triads surround them.

[85] Cohn, R. (1996). Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions. Music Analysis 15(1), 9–40.

[86] Cohn, R. (2012). Audacious Euphony: Chromatic Harmony and the Triad's Second Nature. Oxford University Press. Consonant triads as minimal perturbations of the perfectly even augmented triad.

[87] Computed for this book: of the 351 transposition classes of non-empty pitch-class sets in 12-tone equal temperament, 16 are symmetric under a transposition. With two notes there is exactly one (the tritone); with three notes exactly one (the augmented triad).

[88] Kallel, S. & Sinha, D. (2009). Remarks on Finite Subset Spaces. arXiv:0902.3773, Lemma 7.4 (a result of Wagner).

[89] Konstapel, J. (2026). Upward in the Windings: What the net makes by itself, and which coupling makes which level, v2.0. Constable Research, Leiden.

The music of departure and return ran on a three-sphere without anyone knowing it. [Reading]

4.6 The barrier: the centre pulls

The smooth cycles that Cohn found lead step by step towards the augmented triad. There the sense of key dissolves: from an augmented triad you cannot tell which key you came from. The undo fails. At the same time the seventh chord, a chord of four notes, gains weight. Four pitch classes add no new level: the space stays a sphere of dimension three in homotopy, with its winding doubled.[90] Music must leave the three-sphere to go further. The next chapter shows what happened when it did.

4.7 Further figures at this barrier

Tonal harmony spread through all of Europe and met another closed system in South India.

- In France *Jean-Baptiste Lully* (1632–1687) shaped opera and orchestra. In England *Henry Purcell* (1659–1695) built arias on repeating bass lines: a level-1 loop under a free melody. *Georg Philipp Telemann* (1681–1767), *Domenico Scarlatti* (1685–1757), *Christoph Willibald Gluck* (1714–1787) and *Carl Philipp Emanuel Bach* (1714–1788) carried tonal practice into the classical era.
- In the nineteenth century *Gioachino Rossini* (1792–1868) and *Giuseppe Verdi* (1813–1901) made tonal opera popular across the world. *Hector Berlioz* (1803–1869) turned the orchestra into colour. *Felix Mendelssohn* (1809–1847) revived Bach. *Frédéric Chopin* (1810–1849) and *Robert Schumann* (1810–1856) drove chromatic harmony on the piano towards the boundary of Chapter 5.
- In South India *Syama Sastri* (1762–1827), *Tyagaraja* (1767–1847) and *Muthuswami Dikshitar* (1775–1835) are honoured as the trinity of Carnatic music. Their songs (kritis) live inside the raga system that Venkatamakhin had closed by counting. Two closed systems, the European triad and the South Indian mela, flowered in the same decades.
- Women composed at the centre of tonal music. *Barbara Strozzi* (1619–1677) published more secular vocal music than almost anyone of her time. *Élisabeth Jacquet de La Guerre* (1665–1729) was the first French woman to have an opera staged. *Fanny Mendelssohn Hensel* (1805–1847) wrote over four hundred works. *Clara Schumann* (1819–1896) shaped Romantic piano music as pianist and composer.
- Outside Europe the same centuries made other closed systems. *Manuel de Zumaya* (c. 1678–1755) led music at Mexico City cathedral. *Yamada Kengyō* (1751–1817) founded a second school of koto.[91] *Hamparsum Limonciyan* (1768–1839) gave

[90] Kallel & Sinha (2009), see footnote above; Tuffley (2002): $\exp_4(S^1) \simeq S^3$, $\exp_5(S^1) \simeq S^5$; the step from $2k-1$ to $2k$ points multiplies homology by two.

[91] Yatsuhashi Kengyō (1614–1685) founded modern koto music; hirajōshi remained the standard tuning. Yamada Kengyō (1751–1817) founded the Yamada school. Britannica, “Japanese music: Koto music”.

Ottoman music a notation of its own.[92] After 1755 the courts of Surakarta and Yogyakarta built the great Javanese court gamelans.

- In Zimbabwe the Shona *mbira* is played in two interlocking parts, kushaura and kutsinhira, around a harmonic cycle that never cadences.[93] It is a closed harmony without a tonic: a turn that does not need to come home. [Known: the practice. Reading: the turn.]

[92]Dimitrie Cantemir (1673–1723) wrote an Ottoman music treatise with an alphabetic notation. Hamparsum notation recorded the Ottoman repertoire from the early 19th century. Wright, O. (1992–2000). Demetrius Cantemir: The Collection of Notations. SOAS.

[93]Berliner, P. (1978). The Soul of Mbira. University of California Press. Pérez Bocanegra, J. (1631). Ritual formulario. Lima.

Chapter 5 — Where Tonality Gets Stuck

c. 1850 – 1960 · Liszt, Wagner, Debussy, Schoenberg, Stravinsky, Bartók, Messiaen, jazz

5.1 The chord that does not resolve

Wagner composed Tristan und Isolde between 1857 and 1859. It was first performed in Munich on 10 June 1865. Its opening chord, F–B–D♯–G♯, is enharmonically a half-diminished seventh chord. What shocked listeners was not the chord but its motion: it resolves to another dissonance.[94] Cohn showed that Franck, Liszt, Mahler and Wagner moved through hexatonic cycles of triads that the diatonic system cannot explain.[95] [Known]

5.2 Debussy and the gamelan

At the Paris Exposition Universelle of 1889 Claude Debussy heard a Javanese gamelan.[96] His later music uses static harmony, layered textures, pentatonic and whole-tone scales. The whole-tone scale is maximally even and symmetric: it repeats after every whole tone.[97] [98] Debussy brought into the concert hall a world that had never closed the circle in the European way. [Known]

5.3 Schoenberg: the end of the centre

On 19 February 1909 Arnold Schoenberg finished the first of his Three Piano Pieces op. 11: the first work to do entirely without tonal organisation. Between 1921 and 1923 he wrote the Suite for Piano op. 25, the first work built throughout on a row of all twelve tones.[99] Alban Berg and Anton Webern followed. [Known]

The row is an ordered loop of twelve notes. It is a permutation: a level-1 object whose order matters. When the centre is gone, music returns to order. [Reading]

5.4 Stravinsky, Bartók and symmetric collections

The Rite of Spring caused a riot in Paris on 29 May 1913. Scholars have read much of Stravinsky's Russian music through the octatonic scale, which alternates whole and half

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[94] Wagner, R. Tristan und Isolde, composed 1857–59, first performed in Munich on 10 June 1865 under Hans von Bülow. The Tristan chord F–B–D♯–G♯ is enharmonically a half-diminished seventh chord.

[95] Cohn, R. (1996). Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions. Music Analysis 15(1), 9–40.

[96] Mueller, R. (1986). Javanese Influence on Debussy's Fantaisie and Beyond. 19th-Century Music 10(2), 157–186. On the gamelan at the Paris Exposition Universelle of 1889.

[97] Clough, J. & Douthett, J. (1991). Maximally Even Sets. Journal of Music Theory 35(1/2), 93–173.

[98] Computed for this book: of the 351 transposition classes of non-empty pitch-class sets in 12-tone equal temperament, 16 are symmetric under a transposition. With two notes there is exactly one (the tritone); with three notes exactly one (the augmented triad).

[99] Schoenberg finished the first of the Three Piano Pieces op. 11 on 19 February 1909, the first work to do without tonal organisation. The Suite for Piano op. 25 (1921–23) is the first work in the twelve-tone method throughout. Britannica, "Arnold Schoenberg".

steps. Tymoczko later argued that its role was smaller than claimed.[100] Béla Bartók turned folk music from Hungary and beyond into symmetric harmonic systems. [Known]

5.5 Messiaen: the symmetric modes

In 1944 Olivier Messiaen published his Technique de mon langage musical. He listed seven “modes of limited transposition”. A mode of limited transposition gives back the same notes after fewer than twelve transpositions. Messiaen believed the series was closed in the twelve-semitone system.[101] Mathematically these are exactly the sets with a non-trivial group of transpositions.[102]

There are sixteen such classes among the 351 transposition classes of twelve-tone equal temperament. They include the tritone, the augmented triad, the diminished seventh chord, the whole-tone scale and the octatonic scale.[103] [Computed]

In the geometry of Chapter 4 these are the chords where transposition does not act freely: the exceptional circles, and above three notes the singular layers. Messiaen built a whole language on the singular places of chord space. [Reading]

5.6 Jazz: building on the exceptional circles

Jazz took a different road through the same geometry. In Giant Steps (recorded 1959) John Coltrane moves through three keys, B, G and E♭, a major third apart. Together their roots form an augmented triad.[104] The common jazz practice of tritone substitution replaces a dominant chord by the one a tritone away. Coltrane’s harmony uses the circle of multiplicity 3. Tritone substitution uses the circle of multiplicity 2. [Known: the practice. Reading: the link.]

5.7 The barrier: no level to climb

Here the theory of this book gets stuck, and so did tonal music. Above three pitch classes the chord space is no longer a manifold.[105] With five it has the homotopy type of a five-

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[100]Van den Toorn, P.C. (1983). The Music of Igor Stravinsky. Yale University Press; Taruskin, R. (1996). Stravinsky and the Russian Traditions; Tymoczko, D. (2002). Stravinsky and the Octatonic: A Reconsideration. Music Theory Spectrum 24.

[101]Messiaen, O. (1944). Technique de mon langage musical. Paris: Leduc. Modes of limited transposition; Messiaen held their series closed in the 12-semitone system. Mathematically they are sets with a non-trivial group of transpositions.

[102]Messiaen, O. (1944). Technique de mon langage musical. Paris: Leduc. Modes of limited transposition; Messiaen held their series closed in the 12-semitone system. Mathematically they are sets with a non-trivial group of transpositions.

[103]Computed for this book: of the 351 transposition classes of non-empty pitch-class sets in 12-tone equal temperament, 16 are symmetric under a transposition. With two notes there is exactly one (the tritone); with three notes exactly one (the augmented triad).

[104]Coltrane, J. Giant Steps (recorded 1959). Key centres B, G and E♭ a major third apart: together an augmented triad.

[105]Kallel, S. & Sinha, D. (2009). Remarks on Finite Subset Spaces. arXiv:0902.3773, Lemma 7.4 (a result of Wagner).

sphere, but without the clean structure of the three-sphere.[106] The symmetric places multiply into whole layers. There is no single centre. Chapter 7 shows what does carry over: the symmetric chords form four windings, one of which the twelve-tone grid cannot reach.

The barrier rule names what happened. The rule was “return to the tonic”. When all twelve notes are equal, the undo fails: there is no tonic to return to. The new objects were the row, the symmetric mode and the substituted chord. The lost property was the centre.

5.8 Further figures at this barrier

The crisis of the centre had many voices.

- *Anton Bruckner* (1824–1896) and *Johannes Brahms* (1833–1897) kept classical form while their harmony grew chromatic. *Modest Mussorgsky* (1839–1881) wrote harmony that follows Russian speech, not the cadence. *Nikolai Rimsky-Korsakov* (1844–1908) used the octatonic scale so often that Russians named it after him.[107]
- *Alexander Scriabin* (1872–1915) built whole works on one six-note chord. *Maurice Ravel* (1875–1937) widened tonality with modal colour. *Charles Ives* (1874–1954) layered keys and wrote for pianos tuned a quarter tone apart. *Heitor Villa-Lobos* (1887–1959) joined Brazilian music to Bach. *Dmitri Shostakovich* (1906–1975) kept tonality under pressure. *Nadia Boulanger* (1887–1979) taught a whole generation, from Copland to Glass.
- Some left the twelve steps altogether. *Julián Carrillo* (1875–1965) began his Sonido 13 in 1895 while experimenting on the violin. *Alois Hába* (1893–1973) wrote operas in quarter and sixth tones. *Harry Partch* (1901–1974) built his own instruments for a just scale of 43 tones and set out his ideas in Genesis of a Music (1949).[108] They answered the barrier of Chapter 3 again, now by opening the closed circle.
- Jazz grew its own theory. *Duke Ellington* (1899–1974) used the orchestra as a harmonic laboratory. *Charlie Parker* (1920–1955) and *Thelonious Monk* (1917–1982) multiplied chord substitutions. *Miles Davis* (1926–1991) released Kind of Blue in 1959, built on modes instead of chord chains.
- In Cairo *Umm Kulthum* (c. 1904–1975) sang a maqam music that had just debated its own standard of 24 quarter tones.[109]
- In the Americas a new music crossed the barrier from below. *Scott Joplin* (1868–1917) published Maple Leaf Rag in 1899. *W. C. Handy* (1873–1958) published St. Louis Blues in 1914. Its blue notes are flattened thirds and sevenths, sometimes

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[106]Kallel & Sinha (2009), see footnote above; Tuffley (2002): $\exp_4(S^1) \simeq S^3$, $\exp_5(S^1) \simeq S^5$; the step from $2k-1$ to $2k$ points multiplies homology by two.

[107]Van den Toorn, P.C. (1983). The Music of Igor Stravinsky. Yale University Press; Taruskin, R. (1996). Stravinsky and the Russian Traditions; Tymoczko, D. (2002). Stravinsky and the Octatonic: A Reconsideration. Music Theory Spectrum 24.

[108]Carrillo developed Sonido 13 from 1895 while experimenting with his violin. Hába wrote operas in quarter and sixth tones. Partch, H. (1949). Genesis of a Music. University of Wisconsin Press.

[109]Mikhail Mishaqa (1800–1888), Essay on the Art of Music (c. 1840): 24 equal quarter tones. The Cairo Congress of Arab Music (1932) discussed standardisation on 24 equal steps.

pitches between the steps of the scale.[110] The blues reopened the circle that temperament had closed: a twist undone by the voice. *Bessie Smith* (1894–1937) and *Louis Armstrong* (1901–1971) carried it into jazz. [Known: the music. Reading: the reopening.]

- *Florence Price* (1887–1953) became the first Black woman to have a symphony played by a major American orchestra, in Chicago on 15 June 1933.[111] *Lili Boulanger* (1893–1918) was the first woman to win the Prix de Rome, in 1913. *Chiquinha Gonzaga* (1847–1935) shaped Brazilian choro. *Silvestre Revueltas* (1899–1940) and *Carlos Chávez* (1899–1978) built a Mexican modernism. *Astor Piazzolla* (1921–1992) joined tango with jazz and counterpoint.
- Elsewhere the same decades rebuilt local systems. *Vishnu Narayan Bhatkhande* (1860–1936) ordered Hindustani ragas into ten parent scales, the thaats. *Rabindranath Tagore* (1861–1941) wrote more than two thousand songs. *Sayed Darwish* (1892–1923) renewed Egyptian song. *Ephraim Amu* (1899–1995) wrote Ghanaian choral music in African rhythm. *Yamada Kōsaku* (1886–1965) and *Xian Xinghai* (1905–1945) brought the orchestra to Japan and China. In Bali the fast style *gong kebyar* appeared around 1915.

[110] Joplin's Maple Leaf Rag was published in 1899; Handy's St. Louis Blues in 1914. Blue notes are flattened thirds and sevenths, sometimes pitches between the notes of the scale.

[111] Price's Symphony in E minor was premiered by the Chicago Symphony Orchestra under Frederick Stock on 15 June 1933: the first symphony by a Black woman played by a major American orchestra. Britannica, "Florence Price".

Chapter 6 — New Objects: Series, Silence, Sound, Loop and Machine

1945 – 2026 · Messiaen and Boulez, Cage, Schaeffer, Xenakis, Hiller, Reich, Glass, Ligeti, Grisey, Pärt, Fela Kuti, Kool Herc, AI music

6.1 Ordering everything

In 1949 Messiaen wrote Mode de valeurs et d'intensités, with modes of 36 pitches, 24 durations, 12 attacks and 7 dynamics. Pierre Boulez took the idea further in Structures I for two pianos (1951–52), the most consistent work of integral serialism.[112] When every parameter is ordered, the result can sound like chance. That was the barrier of total order. [Known]

6.2 Silence: back to the parts

On 29 August 1952 David Tudor sat at a piano in Woodstock, New York, and played nothing for four minutes and thirty-three seconds. John Cage's 4′33″ made the sounds of the room the music.[113] In the language of the net this is a return to level 0: the parts, before any cycle. [Known: the event. Reading: the level.]

6.3 Sound itself as the object

In 1948 Pierre Schaeffer broadcast his Étude aux chemins de fer, made of recorded train sounds: the start of musique concrète. Iannis Xenakis, an engineer, used probability and mathematics in Metastaseis (1953–54). Lejaren Hiller and Leonard Isaacson let a computer compose the Illiac Suite (1956–57), using Fux's counterpoint rules and random choice.[114] The machine entered composition. [Known]

6.4 The loop returns

In Steve Reich's Piano Phase (1967) two pianists play the same twelve-note pattern. One slowly gets ahead. The two loops drift through every relative position and come back together. In 1970 Reich studied Ewe drumming in Accra with Gideon Alorwoyie. Drumming (1970–71) followed.[115] Philip Glass found his additive rhythm while notating Ravi Shankar's film music: the music fit only when he erased the bar lines.[116] György Ligeti heard Simha Arom's recordings of Central African music in 1982 and wrote the

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[112]Messiaen, O. Mode de valeurs et d'intensités (1949): modes of 36 pitches, 24 durations, 12 attacks and 7 dynamics. Boulez, P. Structures I (1951–52), the most consistent work of integral serialism.

[113]Cage, J. 4′33″, first performed by David Tudor at Woodstock, New York, on 29 August 1952. Museum of Modern Art.

[114]Schaeffer, P. Étude aux chemins de fer (1948), the start of musique concrète. Hiller, L. & Isaacson, L. Illiac Suite (1956–57): Fux's counterpoint rules and random choice by computer. Xenakis, I. Metastaseis (1953–54).

[115]Reich, S. Piano Phase (1967). In 1970 Reich studied Ewe drumming in Accra with Gideon Alorwoyie; Drumming (1970–71) premiered at MoMA on 3 December 1971. Reich, S. (2002), Writings on Music, Oxford University Press.

[116]Philip Glass notated Ravi Shankar's score for the film Chappaqua (1964–65). He found the music would not fit Western bars until he removed the bar lines; additive rhythm followed.

foreword to Arom's book. His piano études from 1985 on carry its polyrhythm.[117] [Known]

Chapter 1 returns here. The timelines of the forest enter the concert hall. Phasing is two equal turns with a slowly changing offset: a path on the torus of two cycles. [Reading]

6.5 The spectrum

Gérard Grisey's Partiels (1975) opens with the spectrum of a low E on the trombone, re-synthesised by the ensemble. With Tristan Murail and others he had founded L'Itinéraire in 1973.[118] Spectral music builds harmony from the overtone series. It goes back to the ratios of Pythagoras, but now all of them at once. [Known]

6.6 The triad reborn

In 1976 Arvo Pärt wrote the piano piece Für Alina. One voice arpeggiates the tonic triad. The other moves by step around a centre. Pärt called the style tintinnabuli, after bells.[119] After serialism, the triad came back as a bell: the three-sphere as a refuge. [Known: the style. Reading: the refuge.]

6.7 Loops in popular music

In Nigeria in the late 1960s Fela Kuti and drummer Tony Allen created Afrobeat from highlife, jùjú, Yoruba traditions and jazz.[120] On 11 August 1973, at 1520 Sedgwick Avenue in the Bronx, DJ Kool Herc used two copies of one record to loop the drum break. He called it the Merry-Go-Round. Hip hop grew from it.[121] The break is a timeline. As in the forest of Chapter 1, a loop again carries a whole culture. [Known: the history. Reading: the timeline.]

6.8 The machine composes: AI music to 2026

In the 2020s generative models began to produce complete songs from a text prompt. In June 2024 the three major labels sued the AI companies Suno and Udio for training on their recordings. Universal settled with Udio in October 2025. Warner settled with Udio

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[117]Taylor, S.A. (1997). Chopin, Pygmies, and Tempo Fugue: Ligeti's "Automne à Varsovie". Music Theory Online 3(3). Ligeti heard Arom's Banda-Linda recordings in 1982 and wrote the foreword to Arom's African Polyphony and Polyrhythm.

[118]Grisey, G. Partiels (1975), from Les Espaces acoustiques. Its opening re-synthesises the spectrum of a low trombone E. L'Itinéraire was founded in 1973.

[119]Pärt, A. Für Alina (1976). The first work in the tintinnabuli style: one voice arpeggiates the tonic triad, the other moves by step. Universal Edition; Arvo Pärt Centre.

[120]Afrobeat was developed in Nigeria in the late 1960s by Fela Anikulapo Kuti with drummer Tony Allen, from highlife, jùjú, Yoruba traditions and jazz.

[121]DJ Kool Herc (Clive Campbell), party at 1520 Sedgwick Avenue, the Bronx, 11 August 1973. His "Merry-Go-Round" used two copies of one record to loop the break. NPR (2023).

and then with Suno in November 2025. Licensed models are promised for 2026.[122] Millions of generated songs have reached streaming platforms.

The barrier rule applies again. The rule was "a composer writes a work". When a machine can generate any style, the undo fails: one cannot trace a song back to a single author. The new object is the licensed model and its training set. The lost property is authorship. [Reading]

6.9 What the net predicts

The twentieth century did not climb above level 3 in chords. It went sideways: back to the pulse, the loop, the spectrum and the machine. The net suggests where the next step could come from. Above level 3, new windings appear by *coupling*: the join and suspension of nets.[123] A reading follows: the next musical object will couple two whole musics, not enlarge the chord. Phasing, the gamelan's interlocking parts and hocket are small existing cases. The prediction fails if the next lasting development is a larger chord system. [Reading; a falsifiable prediction]

6.10 Further figures at this barrier

The new objects had many makers.

- *Edgard Varèse* (1883–1965) called music "organised sound" and prepared the electronic studio. *Karlheinz Stockhausen* (1928–2007) worked serially and electronically in Cologne. *Milton Babbitt* (1916–2011) gave twelve-tone writing a mathematical theory. *Luigi Nono* (1924–1990) and *Luciano Berio* (1925–2003) carried serialism into politics and the voice. *Morton Feldman* (1926–1987) slowed it down to quiet, very long forms.
- *Conlon Nancarrow* (1912–1997) punched canons into player-piano rolls. His voices run at complex tempo ratios, and from Study No. 21 their tempi slide against each other.[124] This is coupling of loops at different speeds: the nearest older case of the prediction in 6.9. [Known: the music. Reading: the coupling.]
- *Max Mathews* (1926–2011) wrote the program MUSIC at Bell Labs in 1957, the first widely used software to generate sound.[125] The machine now sounded as well as composed.
- *La Monte Young* (b. 1935) began minimalism with sustained tones. *Terry Riley* (b. 1935) wrote In C (1964), in which players move through short patterns at their own

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[122]The three major labels sued Suno and Udio in June 2024. Universal settled with Udio in October 2025; Warner settled with Udio and then with Suno in November 2025. Associated Press (26 February 2026); Rolling Stone (November 2025).

[123]Konstapel, J. (2026). Upward in the Windings: What the net makes by itself, and which coupling makes which level, v2.0. Constable Research, Leiden.

[124]Nancarrow composed mainly for player pianos from 1948 until his death in 1997. His canons set voices in complex tempo ratios; from Study No. 21 tempi slide within strata.

[125]Mathews wrote MUSIC at Bell Labs in 1957, the first widely used program for generating sound; it was demonstrated with a 17-second composition. Computer History Museum.

pace. *Ravi Shankar* (1920–2012) brought Indian rhythm to Western composers.[126] *Brian Eno* (b. 1948) made ambient and generative music: rules that keep producing sound.

- In Asia and beyond the new objects took other shapes. *Toru Takemitsu* (1930–1996) joined Japanese sound to Messiaen's modes. *Isang Yun* (1917–1995) turned Korean court music into modern form. *Sofia Gubaidulina* (1931–2025) built form from number and spirit. *Kaija Saariaho* (1952–2023) joined spectral colour to electronics. *Tan Dun* (b. 1957) brought Chinese ritual sound into the orchestra.
- Popular music made the loop global. *James Brown* (1933–2006) built funk on the downbeat, "the one". *Bob Marley* (1945–1981) carried the offbeat cycle of reggae around the world. Both are timelines in the sense of Chapter 1.
- Since 1980 the new objects multiplied. *Pauline Oliveros* (1932–2016) made listening itself the practice: Deep Listening. *Meredith Monk* (b. 1942) widened the voice. *Georg Friedrich Haas* (b. 1953) joined microtonal and spectral harmony. *Unsuk Chin* (b. 1961) built virtuosic orchestral colour. *David Cope* (b. 1941) began his Experiments in Musical Intelligence in 1981, a program that composes in a given style.[127] *José Maceda* (1917–2004) wrote Ugnayan (1974) for twenty radio stations playing at once across Manila.
- Dance music turned the loop into a world culture. In Chicago *Frankie Knuckles* (1955–2014) shaped house at the Warehouse. In Detroit *Juan Atkins*, *Derrick May* and *Kevin Saunderson* made techno.[128] In 2018 *Kendrick Lamar* (b. 1987) received the Pulitzer Prize for Music for DAMN., the first time it went outside classical music and jazz.[129]
- Worldwide, local musics joined the global loop. *Fairuz* (b. 1934) became the voice of Lebanese song. *Ali Farka Touré* (1939–2006) linked Malian music to the blues. *Youssou N'Dour* (b. 1959) carried Senegalese mbalax abroad. *Nusrat Fateh Ali Khan* (1948–1997) brought qawwali to a world audience. *Ryuichi Sakamoto* (1952–2023) joined electronics and pop with the Yellow Magic Orchestra. *A. R. Rahman* (b. 1966) joined Indian and electronic film music. *Burna Boy* (b. 1991) made Afrobeats a global genre.

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[126]Philip Glass notated Ravi Shankar's score for the film Chappaqua (1964–65). He found the music would not fit Western bars until he removed the bar lines; additive rhythm followed.

[127]Cope began Experiments in Musical Intelligence in 1981 (Leonardo Music Journal 9, 1999). Oliveros's Deep Listening practice trains attention to sound.

[128]Detroit techno is credited to Juan Atkins, Derrick May and Kevin Saunderson, the Belleville Three, who drew on Chicago house as played by Frankie Knuckles at the Warehouse.

[129]Kendrick Lamar received the 2018 Pulitzer Prize for Music for DAMN., the first time the prize went to a work outside classical music and jazz. NPR, 16 April 2018.

Chapter 7 — Conclusion: The Strand through the Barriers

7.1 The barriers in one table

Period	Rule that worked	Undo that failed	New object	Property lost
before 1400 BCE	sing by variation	carry a song back unchanged	fixed step, written string	living variation
1800 BCE–1300 CE	stack pure fifths	return around the circle	(open spiral; ch. 3)	—
1300–1750	pure intervals	close the circle of fifths	tempered fifth	purity
1600–1850	interval as unit	move between keys by intervals	chord, root, key	independence of voices
1850–1960	return to the tonic	find a tonic in the chromatic	row, symmetric mode	the centre
1945–2026	the composer orders	order everything / trace an author	chance, loop, spectrum, model	authorship

7.2 Where the lines converge

Collecting across forty thousand years, three points keep returning.

- *Maximal evenness*. The same arithmetic produces the African standard bell pattern and the major scale.[130] Its symmetric members, the tritone and the augmented triad, are the two exceptional circles of the three-note chord space.[131] [132]
- *The exceptional circles*. Cohn's hexatonic cycles surround the augmented triad. Debussy's whole-tone scale and Messiaen's modes live on the symmetric places. Coltrane and tritone substitution build jazz harmony on them.[133] [134] [135]

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[130]Computed for this book: the strokes of the 12-pulse standard pattern lie on pulses 0, 2, 4, 5, 7, 9, 11. The gaps are 2-2-1-2-2-2-1. These are the steps of the major scale. The Euclidean rhythm E(7,12) gives a rotation of the same cycle.

[131]Clough, J. & Douthett, J. (1991). Maximally Even Sets. Journal of Music Theory 35(1/2), 93–173.

[132]Tuffley, C. (2002). Finite subset spaces of S^1. Algebraic & Geometric Topology 2, 1119–1145. See the proof of his Theorem 3.

[133]Cohn, R. (1996). Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions. Music Analysis 15(1), 9–40.

[134]Messiaen, O. (1944). Technique de mon langage musical. Paris: Leduc. Modes of limited transposition; Messiaen held their series closed in the 12-semitone system. Mathematically they are sets with a non-trivial group of transpositions.

[135]Coltrane, J. Giant Steps (recorded 1959). Key centres B, G and E ♭ a major third apart: together an augmented triad.

- *The loop*. The forest timeline, the Babylonian retuning, isorhythm, the sonata's return, phasing and the breakbeat are all turns of level 1. Each new beginning in music starts again with a loop.

7.3 Where the theory got stuck, and how it moves on

Earlier drafts of this book left four points open. Each can be settled with the tools already in the book. This section gives the answers. The proofs are short; the computations are in Appendix B.

7.4 Level 5: the windings of five-note chords

Above three notes the chord space is no longer a manifold.[136] That looked like a dead end. It is not. The useful question is not the shape of the space but its symmetric places. A chord of at most five notes that a rotation of the octave leaves unchanged must consist of whole regular polygons. With at most five points the only possible orders are 2, 3, 4 and 5.[137]

So level 5 has four exceptional windings: the tritone (order 2), the augmented triad (order 3), the diminished seventh (order 4) and the regular pentagon (order 5). Level 3 had only the first two. The new invariant of level 5 is this set of orders, {2, 3, 4, 5}.

Now compare the grid. In twelve equal steps the orders 2, 3 and 4 exist, but 5 does not, because 5 does not divide 12. A computer check over all chords of up to five notes confirms this: twelve steps give {2, 3, 4}; five steps give {5}; ten give {2, 5}; fifteen give {3, 5}; 31 and 53 give none. The twelve-tone grid therefore cannot reach the fourth winding of level 5. This is the precise barrier behind the old feeling that Western harmony stops at the seventh chord.

7.5 Local tunings: the gamelan answer

The missing winding exists in practice. It is slendro. Javanese slendro has five steps of roughly equal size. In one measured gamelan the steps run from 213 to 270 cents; exact equality would be 240 cents.[138] Slendro is a tuning that sits near the order-5 winding which twelve steps cannot hold. Pelog, with seven very unequal steps, sits near no symmetric place at all. It is a tuning of pure difference.

Every gamelan is tuned as a set, and no two sets are tuned alike. The octaves are often stretched.[139] In the net this is a closure that is local: the circle closes within one ensemble, not across all ensembles. Western temperament chose one shared circle for

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[136]Kallel, S. & Sinha, D. (2009). Remarks on Finite Subset Spaces. arXiv:0902.3773, Lemma 7.4 (a result of Wagner).

[137]Derivation: a finite set fixed by a rotation of order m is a union of regular m-gons, so with at most five points m is 2, 3, 4 or 5. The check over the m-step grids is the function symmetry_orders in Appendix B. Background: Tuffley, C. (2002). Finite subset spaces of S^1. Algebraic & Geometric Topology 2, 1119–1145.

[138]Montagu, J. Gamelan Handbook: slendro steps of 213–270 cents and pelog steps of 88–295 cents in one gamelan. Surjodiningrat, W., Sudarjana, P.J. & Susanto, A. measured the central-octave saron of 76 Central Javanese gamelans (Tone Measurements of Outstanding Javanese Gamelans in Yogyakarta and Surakarta).

[139]Carterette, E.C. & Kendall, R.A. (1994). On the tuning and stretched octave of Javanese gamelans. Leonardo Music Journal 4, 59–68. Alves, B. notes that slendro and pelog exist in no standard form; each gamelan has its own tuning.

every instrument. The gamelan chose a private circle for each orchestra. Both are level-1 turns. They differ in where the turn closes.

7.6 The trefoil of single notes

Inside the three-note chord space, the single notes form a knot: the trefoil.[140] Its meaning can be computed. Measure how often the trefoil winds around each exceptional circle. Around the tritone circle it winds 3 times. Around the augmented circle it winds −2 times. The product is −6, the Hopf invariant of the whole map.[141]

So the single note, the part of level 0, carries the full invariant of level 3. It is knotted around both symmetric chords at once. In the language of the net: the part already contains the coupling. A melody of one voice moves on this knot, and every turn of the knot passes the tritone three times and the augmented triad twice.

7.7 Merging voices and the join

There are two roads from circles to the three-sphere. The join of two circles gives S^3 with the Hopf map, invariant 1. Merging voices, the rule that two equal notes count once, gives S^3 with the transposition map, invariant −6.[142] [143] The spaces are the same; the maps are not. Maps from S^3 to S^2 are classified completely by this invariant, so the two roads are genuinely different. Merging is not a form of the join. It is a second road to level 3, with its own invariant.

7.8 The minimal model

Bludov, Fomin and Musin prove that an essential map from S^3 to S^2 needs at least nine tetrahedra over a triangle of the target. They also build, for every non-zero invariant, a map that uses exactly nine.[144] [145] Since maps are classified by the invariant, one of these maps is a minimal model of the transposition map. A triangulated chord space with nine tetrahedra over a triangle therefore exists. Writing it out explicitly is a construction task, not an open question.

7.9 What remains disputed, and what is predicted

- Some historical claims are disputed in the sources: the shared root of San and Pygmy music, the priority of Zhu Zaiyu or Stevin, and the role of the octatonic in

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[140] Tuffley, C. (2002). Finite subset spaces of S^1. Algebraic & Geometric Topology 2, 1119–1145. See the proof of his Theorem 3.

[141] Computed with the Gauss linking integral on the (2,−3) circle action of S^3 (script trefoil.py): linking with the tritone circle 3.0, with the augmented circle −2.0, between the two circles −1.0.

[142] Tuffley, C. (2002). Finite subset spaces of S^1. Algebraic & Geometric Topology 2, 1119–1145. See the proof of his Theorem 3.

[143] Computed for this book: Gauss linking integral of two regular orbits of the (2,−3) circle action on S^3. Result −6.000. Script available on request.

[144] Bludov, M., Fomin, S.V. & Musin, O.R. (2025). On essential simplicial maps $S^3 \to S^2$. arXiv:2512.08584.

[145] Bludov, Fomin & Musin (2025), Theorem 2.3 and the constructions ζ_n. Maps $S^3 \to S^2$ are classified up to homotopy by the Hopf invariant (Hopf 1931); reversing the orientation of S^3 changes its sign.

Stravinsky.[146] [147] [148] No theory can settle them. The chapters give the evidence on each side.

- The book makes one prediction. The next lasting musical object will couple whole musics, each with its own circle, without forcing them onto one grid. The test is concrete: a practice in which, for example, a gamelan and a twelve-tone ensemble play together with both tunings intact, and which is still alive a generation later. If such practices appear and last, the prediction holds. If every coupling ends in one shared grid, it fails.

[146] Cross, I. (2006). Clues from our present peers? A response to Victor Grauer. The World of Music 48(2).

[147] Kuttner, F.A. (1975). Prince Chu Tsai-Yü's Life and Work. Ethnomusicology 19, 163–206; Robinson, K. (1980). A Critical Study of Chu Tsai-yü's Contribution to the Theory of Equal Temperament. Zhu Zaiyu published in 1584; Stevin's derivation came in 1585 or later.

[148] Van den Toorn, P.C. (1983). The Music of Igor Stravinsky. Yale University Press; Taruskin, R. (1996). Stravinsky and the Russian Traditions; Tymoczko, D. (2002). Stravinsky and the Octatonic: A Reconsideration. Music Theory Spectrum 24.

Part II — Composing with the Net

Part I told the history. Part II turns it into tools. Every barrier of Part I left a rule behind. Each rule can be used to make music. This part gives the rules exactly: in words, in tables and in code. A reader can follow them by hand. A programmer, or Claude Code, can follow them to build a full music book or instrument.

All tables in Part II were produced by one small program, vacuumnet_music.py, which comes with the book. A test file checks every value. An example piece shows all rules at work.[149]

[149]Reference implementation: vacuumnet_music.py (Constable Research 2026), delivered with this book together with test_vacuumnet_music.py and compose_example.py. All values in the tables of Part II were produced by this code; the test file checks each of them.

Chapter 8 — How to Use Part II

8.1 From level to material

The table maps each level of the net to the musical material it governs, the chapter where the rule is explained, and the function that carries it out.

Level	Musical material	Chapter	Function in the code
0 · parts	pulse, strikes, hocket voices	9, 11	balanced_ternary, trit_rhythm, hocket
1 · turns	timelines, octave circle, modulation	10, 11, 13	euclid, maximally_even, freq, winding_of_walk
1 · order	rotations of scales and timelines	11	rotate, same_necklace
1 · twist	temperament: the glued cell	10	pythagorean12, meantone12, werckmeister3, equal
3 · interlocked	triads, transposition, voice leading	12	triad, P, L, R, voice_leading, progression_winding
singular places	tritone, augmented triad, symmetric modes	12	stabilizer, symmetric_classes, MESSIAEN_MODES
coupling	phasing, loops at different speeds	13	phase_pairs

8.2 Conventions

- Pitch classes are the numbers 0 to 11, with 0 = C. A chord is a set of pitch classes.
- Intervals and tunings are measured in cents. An octave is 1200 cents. An equal semitone is 100 cents.
- Time is measured in beats. A pulse is a fixed fraction of a beat, chosen per piece.
- A trit is −1, 0 or +1. In this book it is read as a direction: down, stay, up.
- Labels follow Part I: [Known], [Computed], [Reading].

Chapter 9 — Trits and Directions: Level 0

9.1 Numbers as directions

Every whole number can be written with the digits −1, 0 and +1 in powers of three. This is balanced ternary. Five, for example, is 9 − 3 − 1: the trits are +1, −1, −1 from the highest power down. In the code the lowest power comes first, so 5 gives [−1, −1, +1].

Read each trit as a direction and a number becomes a path. +1 steps up, −1 steps down, 0 stays. This is the musical form of the start of the theory: at a crossing there is no value, only a direction. [Reading]

9.2 Three rules for level 0

- *Strike rule.* For rhythm, +1 is a high strike, −1 a low strike, 0 a rest. This is a convention of this book (function trit_rhythm).
- *Walk rule.* For melody, each trit moves one step on a scale (function trit_walk). The scale can be any list of degrees.
- *Tail rule.* The net joins every stream to its tail: the stream without its first trit (function tail). A piece can follow a stream, then its tail, then the tail of the tail. Each section is one trit shorter. The music ends when the stream closes.

9.3 Hocket

Spread one melody over several players, note by note (function hocket). No player has the tune. The tune exists only in the whole. This is the oldest form of level 0 in Chapter 1.

9.4 Exercises

- Write the numbers 1 to 16 in balanced ternary. Clap them with the strike rule. This is the first movement of the example piece.
- Take the trits of one large number. Play the walk rule on a pentatonic scale, then on the tail of the stream, until it closes.

Chapter 10 — Tunings: The Circle and the Twist

10.1 Five tunings of twelve

The table gives five historical tunings of the twelve pitch classes, in cents above C. All values are computed from their definitions.[150] [Computed]

Pitch class	Pythagorean	Meantone ¼	Werckmeister III	Just	12-ET
C	0.0	0.0	0.0	0.0	0.0
C♯	113.7	76.0	90.2	111.7	100.0
D	203.9	193.2	192.2	203.9	200.0
E ♭	294.1	310.3	294.1	315.6	300.0
E	407.8	386.3	390.2	386.3	400.0
F	498.0	503.4	498.0	498.0	500.0
F♯	611.7	579.5	588.3	590.2	600.0
G	702.0	696.6	696.1	702.0	700.0
G♯	815.6	772.6	792.2	813.7	800.0
A	905.9	889.7	888.3	884.4	900.0
B ♭	996.1	1006.8	996.1	1017.6	1000.0
B	1109.8	1082.9	1092.2	1088.3	1100.0

- *Pythagorean:* fifths k = −3 … +8 above C, folded into the octave. Pure fifths, wide thirds, one wolf fifth.
- *Quarter-comma meantone* (Aron 1523): each fifth narrowed by a quarter of the syntonic comma. Pure major thirds.
- *Werckmeister III* (1691): the fifths C–G, G–D, D–A and B–F♯ narrowed by a quarter of the Pythagorean comma, the rest pure. All keys usable, each with its own colour.
- *Just (senario):* the ratios 1, 16/15, 9/8, 6/5, 5/4, 4/3, 45/32, 3/2, 8/5, 5/3, 9/5, 15/8.
- *12-ET:* twelve equal steps of 100 cents (Zhu Zaiyu 1584).

10.2 Beyond twelve

The function equal(n) gives any equal division. Three values have history: 24 (Mishaqa, Arab quarter tones), 31 (Vicentino and Huygens, near quarter-comma meantone) and 53 (near Jing Fang's fifty-three fifths). The function pythagorean17 gives a seventeen-step chain of the type of al-Urmawi. Its steps are only limmas (90.2 cents) and commas (23.5 cents):

[150]Reference implementation: vacuumnet_music.py (Constable Research 2026), delivered with this book together with test_vacuumnet_music.py and compose_example.py. All values in the tables of Part II were produced by this code; the test file checks each of them.

0.0 · 90.2 · 180.4 · 203.9 · 294.1 · 384.4 · 407.8 · 498.0 · 588.3 · 678.5 · 702.0 · 792.2 · 882.4 · 905.9 · 996.1 · 1086.3 · 1176.5 cents. [Computed]

The function local_tuning takes measured cents. This is the gamelan principle: the tuning is what this ensemble measures. No standard is assumed.

10.3 The comma as material

Stack pure fifths and fold them back into the octave. After each fifth the pitch lies this many cents from an equal-tempered note:

-498.04 · +203.91 · -294.13 · +407.82 · -90.22 · -588.27 · +113.69 · -384.36 · +317.60 · -180.45 · +521.51 · +23.46. The twelfth value is the Pythagorean comma, 23.46 cents. [Computed]

This drift is audible. Play the twelfth note against the starting note and the two beat against each other. The third movement of the example piece does exactly this, and then closes the same chain in 12-ET.

10.4 Choosing a tuning

- Pure thirds and few keys: quarter-comma meantone.
- All keys with different colours: Werckmeister III.
- All keys alike, any modulation: 12-ET.
- Pure thirds and all keys: 31-ET, at the cost of more keys.
- A world of its own: a local tuning.

Chapter 11 — Rhythm: Timelines and Evenness

11.1 The Euclidean rule

To place k strikes in a cycle of n pulses as evenly as possible, follow Euclid. Start with k groups [1] and n – k groups [0]. Append the remainder groups to the first groups, one by one, as long as more than one remainder group is left. Repeat. Read the groups in order. The function euclid(k, n) does this.[151]

Rule	Pattern (x = strike)	Gaps between strikes
E(2,5)	x·x··	2 3
E(3,8)	x··x··x·	3 3 2
E(5,8)	x·xx·xx·	2 1 2 1 2
E(4,12)	x··x··x··x··	3 3 3 3
E(5,12)	x··x·x··x·x·	3 2 3 2 2
E(7,12)	x·xx·x·xx·x·	2 1 2 2 1 2 2
E(5,16)	x··x··x··x··x···	3 3 3 3 4
E(7,16)	x··x·x·x··x·x·x·	3 2 2 3 2 2 2
E(9,16)	x·xx·x·x·xx·x·x·	2 1 2 2 2 1 2 2 2

E(7,12) is a rotation of the 12-pulse West African standard pattern. Its gaps are the steps of the major scale. E(3,8) is the three-in-eight cell found across Africa and the Americas.[152] [Computed]

11.2 Maximal evenness in pitch

The same rule chooses notes. maximally_even(k, 12) gives the k pitch classes spread most evenly over the octave. With k = 7 it gives a diatonic scale, with k = 3 the augmented triad, with k = 2 the tritone.[153] One rule makes the bell pattern, the scale and the two exceptional chords of Chapter 4. [Computed]

11.3 Rotation and order

A timeline can start on any of its strikes. rotate gives the rotations. same_necklace tests if two patterns are rotations of each other. Rotations of a scale are its modes. This is the row “1 · order”: the same steps, a different start.

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[151]Toussaint, G. (2005). The Euclidean Algorithm Generates Traditional Musical Rhythms. Proceedings of BRIDGES, 47–56.

[152]Toussaint, G. (2005). The Euclidean Algorithm Generates Traditional Musical Rhythms. Proceedings of BRIDGES, 47–56.

[153]Clough, J. & Douthett, J. (1991). Maximally Even Sets. Journal of Music Theory 35(1/2), 93–173.

11.4 Phasing

Two players repeat the same loop. After every m repetitions the second moves one pulse ahead. phase_pairs gives the offset per repetition. After n moves the two coincide again. This is two equal turns with a slowly changing offset: a path on the torus.

11.5 Exercises

- Play E(5,12) against E(7,12). Find the pulse where both strike.
- Use E(7,12) as a bell and the major scale on the same twelve pulses as a melody. Hear them as one object.

Chapter 12 — Harmony on the Three-Sphere

12.1 Chords, transposition and symmetry

T(chord, t) moves every note up by t semitones. I(chord, t) mirrors the chord. tclass gives the transposition class. stabilizer lists the transpositions that leave a chord unchanged.

Most chords have an empty stabilizer. The sixteen classes that do not are listed below. They are Messiaen's territory: the symmetric places of chord space. [Computed][154]

Notes	Pitch classes (representative)	Repeats after (steps)
2	{0,6}	6
3	{0,4,8}	4
4	{0,1,6,7}	6
4	{0,2,6,8}	6
4	{0,3,6,9}	3
6	{0,1,2,6,7,8}	6
6	{0,1,3,6,7,9}	6
6	{0,1,4,5,8,9}	4
6	{0,1,4,6,7,10}	6
6	{0,2,4,6,8,10}	2
8	{0,1,2,3,6,7,8,9}	6
8	{0,1,2,4,6,7,8,10}	6
8	{0,1,3,4,6,7,9,10}	3
9	{0,1,2,4,5,6,8,9,10}	4
10	{0,1,2,3,4,6,7,8,9,10}	6
12	{0,1,2,3,4,5,6,7,8,9,10,11}	1

The first two rows are the tritone and the augmented triad: the two exceptional circles of the three-sphere. The four-note row {0,3,6,9} is the diminished seventh chord. The six-note {0,2,4,6,8,10} is the whole-tone scale. The eight-note {0,1,3,4,6,7,9,10} is the octatonic scale. Messiaen's seven modes are all in the list.[155]

[154]Computed for this book: of the 351 transposition classes of non-empty pitch-class sets in 12-tone equal temperament, 16 are symmetric under a transposition. With two notes there is exactly one (the tritone); with three notes exactly one (the augmented triad).

[155]Messiaen, O. (1944). Technique de mon langage musical. Paris: Leduc. Modes of limited transposition; Messiaen held their series closed in the 12-semitone system. Mathematically they are sets with a non-trivial group of transpositions.

12.2 Triads and the three moves

A triad is (root, quality). Three moves change one note each:[156]

- *P* (parallel): C major ↔ C minor. The third moves by a semitone.
- *L* (leading-tone exchange): C major ↔ E minor. One note moves by a semitone.
- *R* (relative): C major ↔ A minor. One note moves by a whole tone.

Alternating P and L gives a hexatonic cycle: six triads around one augmented triad. Alternating P and R gives an octatonic cycle: eight triads around one diminished seventh. The functions hexatonic and octatonic build them. [Computed]

12.3 Voice leading

voice_leading(a, b) finds the way to move chord a to chord b with the least total motion. Each voice takes the shortest way around the circle. The size is the sum of the absolute motions.[157]

12.4 The winding number of a progression

Add up all voice motions along a progression that ends where it began, and divide by twelve. The result is a whole number: how often the progression has turned around the space of chords.[158] This is a level-1 winding. progression_winding computes it with the smoothest voice leading.

Progression (smoothest voice leading)	Steps	Winding number
I–IV–V–I	3	1
Circle of fifths, 12 major triads	12	-3
Chromatic ascent, 12 major triads	12	3
Hexatonic cycle (P,L) from C	6	0
Octatonic cycle (P,R) from C	8	-1
Giant Steps keys B–G–E ♭ –B	3	0
Cyclic step on augmented triad (×1)	1	1

The simplest cadence, I–IV–V–I, turns exactly once. It is the generator of the space: every closed progression of triads is a whole number of such turns. The hexatonic cycle and the Giant Steps cycle turn zero times: their voices swing back and forth around the augmented triad without going round. The circle of fifths with smooth voices winds three times downward. [Computed]

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[156]The P, L and R operations are the standard transformations of neo-Riemannian theory. Cohn (1996, 2012) uses them to describe hexatonic and octatonic cycles.

[157]Tymoczko, D. (2011). A Geometry of Music. Oxford University Press. Voice-leading size measured as the sum of the absolute motions of the voices (the taxicab or L1 norm).

[158]Morton, H.R. (1967). Symmetric products of the circle. Proceedings of the Cambridge Philosophical Society. The sum of the pitch classes maps the space of n-note chords onto a circle and is a homotopy equivalence; the winding number of a closed progression is therefore the total voice motion divided by the octave.

One more rule comes from Chapter 4. On the augmented triad a single cyclic step, every voice up a major third, is already one full turn. At the centre of the three-sphere a third of the octave does what a whole cadence does elsewhere. [Computed]

12.5 Rules for composing

- Depart and return: plan the winding number of a section before writing it.
- Use the exceptional chords as poles. The augmented triad and the tritone are places where key dissolves and any direction is possible.
- Move between hexatonic cycles through the augmented triad. Move between octatonic cycles through the diminished seventh.
- Leave the three-sphere deliberately. A fourth note adds no new level; it doubles what the three-sphere already does.

Chapter 13 — Form: Windings, Barriers and Coupling

13.1 The winding plan

Write the form as a list of sections, each with a winding number. A classical sonata is +1 out, −1 back. A piece in the spirit of the hexatonic cycle stays at 0 and gains colour instead of distance. A piece can also climb: +1, +1, +1, each turn in a new tuning.

13.2 The barrier form

barrier_form turns the barrier rule into four sections:

- *Rule:* establish a material that works (for example pure fifths).
- *Undo fails:* try to return and fail audibly (the comma).
- *New object:* introduce the repair (temperament).
- *Loss:* let the listener hear what was lost (the pure interval).

Any barrier of Part I can be composed this way: the living variation against the fixed step, the tonic against the row, the author against the model.

13.3 Coupling: testing the prediction

Chapter 6 predicts that the next lasting musical object will couple whole musics instead of enlarging chords. A composer can test this. Take two complete musics, each with its own timeline, tuning and winding plan. Couple them in one of three ways.

- *Phase coupling:* the same loop, slowly shifting (Reich).
- *Tempo coupling:* two loops at different speeds whose ratio slides (Nancarrow).[159]
- *Join coupling:* every event of music A answered by every event of music B in turn, so that the pair becomes one larger cycle. This is the musical form of the join that makes the Hopf winding in the net. [Reading; open]

[159]Nancarrow composed mainly for player pianos from 1948 until his death in 1997. His canons set voices in complex tempo ratios; from Study No. 21 tempi slide within strata.

Chapter 14 — The Example Piece: Upward in the Windings

The program compose_example.py writes a piece of about two and a half minutes in five studies. It produces a MIDI file and a WAV file. Each study realises one level.[160]

Starts at	Study	Rule
0:00	I. Pulse	Trits of 1–16 in balanced ternary; +1 high strike, −1 low strike, 0 rest
0:15	II. Timeline	E(7,12) bell, six cycles; trit-walk melody on the major scale in hocket over three voices
0:39	III. Comma	Twelve pure fifths over a pure C drone, ending 23.46 cents sharp; then the same chain in 12-ET, closing exactly
1:01	IV. Three-sphere	I–IV–V–I twice in quarter-comma meantone (winding 1 each); hexatonic cycle (winding 0); augmented triad held; Giant Steps cycle; minimal voice leading
1:25	V. Coupling	One 12-note loop in two voices, the second one pulse ahead every two cycles; coda with a triad voice and a stepwise voice

The MIDI file gives each note its own channel and pitch bend, so meantone and the comma sound exactly.[161] The WAV file is rendered at exact frequencies. The piece is a demonstration of the rules, not a finished work of art. Its sound has not been judged by a listener.

[160]Reference implementation: vacuumnet_music.py (Constable Research 2026), delivered with this book together with test_vacuumnet_music.py and compose_example.py. All values in the tables of Part II were produced by this code; the test file checks each of them.

[161]Standard MIDI Files 1.0 (MIDI Association). Pitch bend with a range of ±2 semitones gives a resolution of about 0.05 cents; one channel per sounding note lets every note carry its own bend.

Chapter 15 — Specification for a Music Book or Instrument

This chapter is written for whoever builds on Part II, human or Claude Code. It states what must exist and how to check it.

15.1 Data model

- *Tuning:* a sorted list of cents within one octave; any length.
- *Pitch:* (tuning, degree), where the degree may exceed the octave; frequency = tonic × 2^(octave + cents/1200).
- *Chord:* a sorted tuple of pitch classes modulo the tuning length.
- *Pattern:* a list of 0/1 of length n; a timeline.
- *Note:* start and duration in beats, frequency in hertz, velocity 1–127, voice, drum flag.
- *Section:* (name, level, winding number, rule) — the unit of form.

15.2 Required functions

The reference implementation provides all of these. A new implementation must give the same results on the tests in 15.4.

Function	Contract
balanced_ternary(n)	trits of n, lowest power first; inverse is from_balanced_ternary
tail(stream)	stream without its first trit
trit_walk(trits, start)	positions after each direction
chain_of_fifths(lo, hi, fifth)	folded cents of fifths k = lo…hi
pythagorean12, meantone12, werckmeister3, just12, equal(n), pythagorean17, local_tuning	tunings as in Chapter 10
freq(tuning, degree, tonic)	frequency in hertz
euclid(k, n)	Euclidean pattern beginning with a strike
maximally_even(k, n)	Clough–Douthett set
rotate, same_necklace, gaps, onsets	pattern tools
hocket(melody, voices)	distribution note by note
phase_pairs(len, m, cycles)	offset of the second voice per cycle
T, I, tclass, stabilizer, symmetric_classes	pitch-class set tools
triad, P, L, R, hexatonic, octatonic	triads and cycles
voice_leading(a, b)	minimal total motion and the motion per voice
progression_winding(chords)	sum of motions / 12 for a closed progression
comma_drift(steps), barrier_form(…)	form tools
write_midi(notes, path, bpm)	format-0 SMF, one channel per note, pitch bend ±2 semitones
render_wav(notes, path, bpm)	mono 16-bit, exact frequencies

15.3 Building the music book

- One chapter per level, each with: the history from Part I, the rule from Part II, a playable example, and an exercise.
- Every example generated from code, never typed by hand, so that text and sound agree.
- Scores in standard notation where the tuning allows; otherwise cents tables next to the notes.
- Every claim keeps its label: [Known], [Computed] or [Reading].
- Open points stay open: level 5, local tunings in the net, the join coupling.

15.4 Acceptance tests

An implementation is correct when all of the following hold. The file test_vacuumnet_music.py checks them.

- balanced_ternary(5) = [−1, −1, +1]; round trip for −200…200.
- Pythagorean comma 23.460 cents; syntonic comma 21.506; 53 fifths minus 31 octaves 3.615.
- Quarter-comma meantone major third equals 5/4 (386.31 cents).
- 12-ET fifth is 1.955 cents narrow; 12-ET major third 13.686 cents wide.
- pythagorean17 has only steps of 90.22 and 23.46 cents.
- euclid(7,12) is a rotation of the major scale on twelve pulses.
- maximally_even(3,12) = {0,4,8}; maximally_even(2,12) = {0,6}.
- 351 transposition classes; 16 symmetric; with at most three notes only {0,6} and {0,4,8}.
- Every Messiaen mode has a non-empty stabilizer.
- hexatonic cycle has 6 triads on 6 pitch classes; octatonic cycle has 8 triads.
- Winding numbers: I–IV–V–I = 1; hexatonic = 0; Giant Steps = 0; octatonic = −1; circle of fifths = −3; chromatic ascent = +3.

Appendix A — Register of Composers, Theorists and Traditions

This register lists the 195 people, groups and traditions that mark the barriers in this book. For each it gives the chapter, the region and its place at the barrier. Dates follow standard reference works such as Grove Music Online; “c.” marks an uncertain date and “fl.” the years of known activity. The register is a guide for further reading, not a ranking.

Coverage by chapter and region

Each cell counts the entries for one chapter and one region. No cell is empty.

Chapter	Africa	Middle East & N. Africa	South Asia	East Asia	SE Asia & Oceania	Europe	Americas
1	2	1	1	1	1	1	1
2	1	9	1	3	1	4	1
3	1	2	3	3	1	24	1
4	1	1	3	1	1	28	2
5	1	2	2	2	1	21	19
6	3	1	3	5	1	16	22

Register

Name	Dates	Ch.	Region	Place at the barrier
San (Ju|'hoansi) healers and singers	living tradition	1	Africa	Polyphony with individual variation; songs received in dream or trance
Aka and other Central African forest peoples	living tradition	1	Africa	Four-voice contrapuntal polyphony; hocket; timelines
Makers of the Swabian flutes	c. 43,000–35,000 BP	1	Europe	First instruments that fix pitch in bone and ivory
Makers of the Jiahu flutes	c. 9,000 BP	1	East Asia	Oldest playable multi-note instruments
Urhiya and Ammurapi	c. 1400 BCE	1	Middle East & N. Africa	Collector and copyist of the Hurrian hymn h.6
Marquis Yi of Zeng and his bell-makers	d. 433 BCE	2	East Asia	Two-tone bells covering all twelve semitones
Pythagoreans (Philolaus, Archytas)	5th–4th c. BCE	2	Middle East & N. Africa	Intervals as ratios; the leimma
Aristoxenus	b. c. 375 BCE	2	Middle East & N. Africa	The ear against the numbers
Bharata (Natyashastra)	400 BCE–200 CE	2	South Asia	Twenty-two shrutis; mūrchanās as rotations
Jing Fang	78–37 BCE	2	East Asia	Sixty lü; 53 fifths close to 31 octaves
Ptolemy	c. 100–170	2	Middle East & N. Africa	Harmonics: ratios checked by the ear
Boethius	c. 477–524	2	Europe	Passed Greek ratio theory to medieval Europe
Kassia	c. 810–before 865	2	Middle East & N. Africa	Byzantine hymnographer; one of the first named women composers
Ziryab	789–857	2	Middle East & N. Africa	Brought Baghdad music to Córdoba
al-Farabi	872–950/51	2	Middle East & N. Africa	Great Book of Music: theory and practice
Ibn Sina (Avicenna)	980–1037	2	Middle East & N. Africa	Music theory within his encyclopaedia
Guido of Arezzo	c. 991–after 1033	2	Europe	Staff in thirds; ut-re-mi syllables
Hildegard of Bingen	1098–1179	2	Europe	Large body of monophonic song
Léonin and Pérotin	fl. c. 1150–1200	2	Europe	Notre-Dame organum
Safi al-Din al-Urmawi	c. 1216–1294	2	Middle East & N. Africa	Seventeen-tone Pythagorean system; maqam theory
Philippe de Vitry	1291–1361	3	Europe	Ars nova notation; isorhythm
Guillaume de Machaut	c. 1300–1377	3	Europe	First complete Mass setting as one work
Francesco Landini	c. 1325–1397	3	Europe	Italian trecento song
John Dunstable	c. 1390–1453	3	Europe	Thirds and sixths: the contenance angloise

Name	Dates	Ch.	Region	Place at the barrier
Gilles Binchois	c. 1400–1460	3	Europe	Burgundian chanson
Guillaume Dufay	c. 1397–1474	3	Europe	Took up the English sound on the Continent
Johannes Ockeghem	c. 1410–1497	3	Europe	Long-breathed polyphony
Heinrich Isaac	c. 1450–1517	3	Europe	Franco-Flemish polyphony across Europe
Josquin des Prez	c. 1450–1521	3	Europe	Balance of imitation and text
Jacob Obrecht	1457/58–1505	3	Europe	Large-scale cantus firmus Masses
Pietro Aron	c. 1480–after 1545	3	Europe	First description of quarter-comma meantone
Purandara Dasa	1484–1564	3	South Asia	Founder of Carnatic music teaching
Tansen	c. 1500–1589	3	South Asia	Leading musician of the Mughal court
Nicola Vicentino	1511–c. 1576	3	Europe	Keyboard with 31 steps to the octave
Gioseffo Zarlino	1517–1590	3	Europe	Exact meantone; the senario; primacy of the triad
Vincenzo Galilei	c. 1520–1591	3	Europe	Argued for equal division on the lute
Giovanni Pierluigi da Palestrina	c. 1525–1594	3	Europe	Model of smooth Renaissance polyphony
Orlando di Lasso	1532–1594	3	Europe	The most prolific polyphonist of his time
Zhu Zaiyu	1536–1611	3	East Asia	First exact calculation of equal temperament (1584)
William Byrd	c. 1540–1623	3	Europe	English polyphony and keyboard music
Simon Stevin	1548–1620	3	Europe	Equal temperament defined mathematically
Jan Pieterszoon Sweelinck	1562–1621	3	Europe	Keyboard variation in Amsterdam
Carlo Gesualdo	1566–1613	3	Europe	Extreme chromaticism within meantone
Marin Mersenne	1588–1648	3	Europe	Harmonie universelle (1636): tuning and acoustics
Christiaan Huygens	1629–1695	3	Europe	Recommended 31 equal steps
Andreas Werckmeister	1645–1706	3	Europe	Well temperament (1681, 1691)
Venkatamakhin	fl. c. 1630–1670	3	South Asia	The 72 melas
Gamelan makers of Java and Bali	living tradition	3	SE Asia & Oceania	Each ensemble its own tuning: local closure
Mikhail Mishaqa	1800–1888	3	Middle East & N. Africa	Twenty-four equal quarter tones
Giovanni Maria Artusi	c. 1540–1613	4	Europe	Attacked the new dissonance

Name	Dates	Ch.	Region	Place at the barrier
Claudio Monteverdi	1567–1643	4	Europe	Seconda pratica: words govern harmony
Jean-Baptiste Lully	1632–1687	4	Europe	French opera and orchestra
Arcangelo Corelli	1653–1713	4	Europe	Fixed the tonal cadence in the sonata
Henry Purcell	1659–1695	4	Europe	English opera and ground bass
Antonio Vivaldi	1678–1741	4	Europe	The concerto and its ritornello
Georg Philipp Telemann	1681–1767	4	Europe	Huge output across national styles
Jean-Philippe Rameau	1683–1764	4	Europe	Root, inversion, fundamental bass (1722)
Johann Sebastian Bach	1685–1750	3, 4	Europe	All 24 keys (1722); counterpoint over tonal harmony
George Frideric Handel	1685–1759	4	Europe	Opera and oratorio in tonal harmony
Domenico Scarlatti	1685–1757	4	Europe	Keyboard sonatas
Leonhard Euler	1707–1783	4	Europe	Lattice of fifths and thirds (1739)
Christoph Willibald Gluck	1714–1787	4	Europe	Opera reform
Carl Philipp Emanuel Bach	1714–1788	4	Europe	Expressive keyboard style
Joseph Haydn	1732–1809	4	Europe	The sonata form and the string quartet
Wolfgang Amadeus Mozart	1756–1791	4	Europe	Departure and return at its most balanced
Syama Sastri	1762–1827	4	South Asia	One of the trinity of Carnatic music
Tyagaraja	1767–1847	4	South Asia	Carnatic kriti; trinity of Carnatic music
Ludwig van Beethoven	1770–1827	4	Europe	Stretched the sonata to its limits
Muthuswami Dikshitar	1775–1835	4	South Asia	Carnatic trinity; systematic use of the melas
Gioachino Rossini	1792–1868	4	Europe	Italian opera
Franz Schubert	1797–1828	4	Europe	Keys a third apart
Hector Berlioz	1803–1869	4	Europe	The orchestra as colour
Felix Mendelssohn	1809–1847	4	Europe	Revived Bach
Frédéric Chopin	1810–1849	4	Europe	Chromatic piano harmony
Robert Schumann	1810–1856	4	Europe	Character pieces and song cycles
Giuseppe Verdi	1813–1901	4	Europe	Italian opera at its height
Franz Liszt	1811–1886	5	Europe	Augmented triads and hexatonic cycles
Richard Wagner	1813–1883	5	Europe	Tristan: dissonance resolving to dissonance
César Franck	1822–1890	5	Europe	Chromatic cycles of triads
Anton Bruckner	1824–1896	5	Europe	Symphonic chromaticism

Name	Dates	Ch.	Region	Place at the barrier
Johannes Brahms	1833–1897	5	Europe	Classical form at the edge of chromaticism
Modest Mussorgsky	1839–1881	5	Europe	Non-functional harmony from Russian speech
Nikolai Rimsky-Korsakov	1844–1908	5	Europe	The octatonic scale in Russian practice
Gustav Mahler	1860–1911	5	Europe	Hexatonic and progressive tonality
Claude Debussy	1862–1918	5	Europe	Gamelan, whole-tone and pentatonic scales
Alexander Scriabin	1872–1915	5	Europe	Works built on one six-note chord
Arnold Schoenberg	1874–1951	5	Europe	Atonality (1909); twelve-tone method (1921–23)
Charles Ives	1874–1954	5	Americas	Polytonality and quarter tones
Maurice Ravel	1875–1937	5	Europe	Modal and extended tonal colour
Julián Carrillo	1875–1965	5	Americas	Microtonal “Sonido 13”
Béla Bartók	1881–1945	5	Europe	Folk music and symmetric systems
Igor Stravinsky	1882–1971	5	Europe	The Rite of Spring (1913); octatonic debate
Anton Webern	1883–1945	5	Europe	Concentrated twelve-tone music
Edgard Varèse	1883–1965	5, 6	Americas	Music as organised sound
Alban Berg	1885–1935	5	Europe	Twelve-tone music with tonal echoes
Nadia Boulanger	1887–1979	5	Europe	Teacher of a whole generation
Heitor Villa-Lobos	1887–1959	5	Americas	Brazilian music meets Bach
Alois Hába	1893–1973	5	Europe	Quarter-tone composition
Duke Ellington	1899–1974	5	Americas	Jazz orchestra as harmonic laboratory
Harry Partch	1901–1974	5	Americas	Forty-three-tone just intonation
Umm Kulthum	c. 1904–1975	5	Middle East & N. Africa	Arab song at the time of the Cairo Congress
Dmitri Shostakovich	1906–1975	5	Europe	Tonality under pressure
Olivier Messiaen	1908–1992	5, 6	Europe	Modes of limited transposition; total serialism
Thelonious Monk	1917–1982	5	Americas	Whole-tone and angular jazz harmony
Charlie Parker	1920–1955	5	Americas	Bebop: chromatic chord substitution
John Coltrane	1926–1967	5	Americas	Giant Steps: keys on an augmented triad
Miles Davis	1926–1991	5	Americas	Kind of Blue (1959): modal jazz
Pierre Schaeffer	1910–1995	6	Europe	Musique concrète (1948)

Name	Dates	Ch.	Region	Place at the barrier
John Cage	1912–1992	6	Americas	4′33″ (1952): back to the parts
Conlon Nancarrow	1912–1997	6	Americas	Polyrhythm for player piano
Milton Babbitt	1916–2011	6	Americas	Combinatorial twelve-tone theory
Isang Yun	1917–1995	6	East Asia	Korean court music in modern form
Ravi Shankar	1920–2012	6	South Asia	Brought Indian rhythm to Western composers
Iannis Xenakis	1922–2001	6	Europe	Stochastic music; Metastaseis (1953–54)
György Ligeti	1923–2006	6	Europe	Micropolyphony; African polyrhythm in the études
Lejaren Hiller	1924–1994	6	Americas	Illiac Suite: the computer composes (1956–57)
Luigi Nono	1924–1990	6	Europe	Serialism with political voice
Luciano Berio	1925–2003	6	Europe	Collage and the voice
Pierre Boulez	1925–2016	6	Europe	Structures I: integral serialism
Max Mathews	1926–2011	6	Americas	MUSIC program for computer sound (1957)
Morton Feldman	1926–1987	6	Americas	Quiet, very long forms
Karlheinz Stockhausen	1928–2007	6	Europe	Electronic and serial music in Cologne
Toru Takemitsu	1930–1996	6	East Asia	Japanese sound world with Messiaen’s modes
Sofia Gubaidulina	1931–2025	6	Europe	Spiritual form and number
James Brown	1933–2006	6	Americas	Funk: the groove on the one
La Monte Young	b. 1935	6	Americas	Sustained tones; start of minimalism
Terry Riley	b. 1935	6	Americas	In C (1964): patterns in free phase
Arvo Pärt	b. 1935	6	Europe	Tintinnabuli (1976): the triad as bell
Steve Reich	b. 1936	6	Americas	Phasing; Drumming after Ghana
Philip Glass	b. 1937	6	Americas	Additive rhythm after Shankar
Fela Kuti	1938–1997	6	Europe	Afrobeat
Bob Marley	1945–1981	6	Americas	Reggae: the offbeat cycle worldwide
Gérard Grisey	1946–1998	6	Europe	Spectral music: Partiels (1975)
Tristan Murail	b. 1947	6	Europe	Spectral music; L’Itinéraire
Brian Eno	b. 1948	6	Europe	Ambient and generative music
Kaija Saariaho	1952–2023	6	Europe	Spectral colour and electronics
DJ Kool Herc	b. 1955	6	Americas	The looped break (1973)
Tan Dun	b. 1957	6	East Asia	Chinese ritual sound and Western orchestra

Name	Dates	Ch.	Region	Place at the barrier
Dmitri Tymoczko	b. 1969	4, 6	Americas	Composer and theorist of chord geometry
Suno, Udio and other model makers	2023–	6	Americas	Generated songs; licensing from 2025
Inuit katajjaq singers	living tradition	1	Americas	Vocal game of two women; interlocking breath sounds
Vedic chanters	from c. 1500 BCE	1	South Asia	Exact oral transmission: the opposite of variation
Aboriginal Australian song traditions	living tradition	1	SE Asia & Oceania	Song as a map of land (songlines); didjeridu drone
Saint Yared	6th c.	2	Africa	Ethiopian chant (zema) in three modes; later melekket notation
Hymnographers of the octoechos	6th–9th c.	2	Middle East & N. Africa	Eight modes as a cycle of the liturgical year
Andean siku players (ira and arka)	from c. 8th c.	2	Americas	Panpipe pairs that play a melody in hocket
Dong Son bronze-drum makers	c. 1000 BCE–100 CE	2	SE Asia & Oceania	Bronze drums of northern Vietnam
Gagaku musicians	from 7th c.	2	East Asia	Japanese court music from Tang China; oldest living orchestra
Griots of the Mali Empire	from 13th c.	3	Africa	Hereditary musicians; the Sundiata epic; kora and balafon
Kunqu performers	14th–17th c.	3	East Asia	Ming-dynasty opera of Kunshan
Juan Pérez Bocanegra	c. 1560–1645	3	Americas	Printed Hanacpachap cussicuinin (1631), polyphony in Quechua
Yatsuhashi Kengyō	1614–1685	3	East Asia	Father of modern koto; the hirajōshi tuning
Dimitrie Cantemir	1673–1723	3	Middle East & N. Africa	Ottoman music theory and alphabetic notation
Barbara Strozzi	1619–1677	4	Europe	Published more secular vocal music than any composer of her time
Élisabeth Jacquet de La Guerre	1665–1729	4	Europe	First French woman to have an opera staged
Manuel de Zumaya	c. 1678–1755	4	Americas	Mexico City cathedral; early opera in the Americas
Yamada Kengyō	1751–1817	4	East Asia	Yamada school of koto
Hamparsum Limonciyan	1768–1839	4	Middle East & N. Africa	Hamparsum notation for Ottoman music
Javanese court gamelan (Surakarta, Yogyakarta)	from 1755	4	SE Asia & Oceania	Court ensembles after the division of Mataram
Shona mbira players	living tradition	4	Africa	Interlocking cyclic parts (kushaura, kutsinhira) without cadence
Fanny Mendelssohn Hensel	1805–1847	4	Europe	Over 450 works, mostly songs and piano music

Name	Dates	Ch.	Region	Place at the barrier
Clara Schumann	1819–1896	4	Europe	Pianist and composer at the centre of Romantic tonality
Chiquinha Gonzaga	1847–1935	5	Americas	Brazilian choro and the first carnival march
Vishnu Narayan Bhatkhande	1860–1936	5	South Asia	Ten thaats: a parent-scale system for Hindustani music
Rabindranath Tagore	1861–1941	5	South Asia	Over two thousand songs (Rabindrasangeet)
Scott Joplin	1868–1917	5	Americas	Maple Leaf Rag (1899): ragtime syncopation over a steady pulse
W. C. Handy	1873–1958	5	Americas	St. Louis Blues (1914): blue notes between the twelve steps
Yamada Kōsaku	1886–1965	5	East Asia	Western orchestral music in Japan
Florence Price	1887–1953	5	Americas	First symphony by a Black woman played by a major American orchestra (1933)
Sayed Darwish	1892–1923	5	Middle East & N. Africa	Egyptian popular song and musical theatre
Lili Boulanger	1893–1918	5	Europe	First woman to win the Prix de Rome (1913)
Bessie Smith	1894–1937	5	Americas	Classic blues singer
Ephraim Amu	1899–1995	5	Africa	Ghanaian choral music in African rhythm and speech tone
Silvestre Revueltas	1899–1940	5	Americas	Mexican modernism
Carlos Chávez	1899–1978	5	Americas	Mexican nationalism and indigenous instruments
Louis Armstrong	1901–1971	5	Americas	Jazz improvisation as a soloist's art
Xian Xinghai	1905–1945	5	East Asia	Yellow River Cantata
Balinese gong kebyar	from c. 1915	5	SE Asia & Oceania	Sudden, virtuosic Balinese gamelan; studied by Colin McPhee
Astor Piazzolla	1921–1992	5	Americas	Nuevo tango: tango with jazz and counterpoint
Pauline Oliveros	1932–2016	6	Americas	Deep Listening: attention to sound as practice
Fairuz	b. 1934	6	Middle East & N. Africa	Voice of Lebanese song with the Rahbani brothers
Ali Farka Touré	1939–2006	6	Africa	Malian desert blues
David Cope	b. 1941	6	Americas	Experiments in Musical Intelligence (from 1981): style by machine
Meredith Monk	b. 1942	6	Americas	Extended voice and music theatre
José Maceda	1917–2004	6	SE Asia & Oceania	Philippine composer; Ugnayan (1974) for twenty radio stations
Nusrat Fateh Ali Khan	1948–1997	6	South Asia	Qawwali to a world audience

Name	Dates	Ch.	Region	Place at the barrier
Ryuichi Sakamoto	1952–2023	6	East Asia	Yellow Magic Orchestra; electronic pop and film music
Georg Friedrich Haas	b. 1953	6	Europe	Microtonal and spectral music
Frankie Knuckles	1955–2014	6	Americas	Chicago house at the Warehouse
Juan Atkins, Derrick May, Kevin Saunderson	b. 1962–64	6	Americas	Detroit techno (the Belleville Three)
Youssou N'Dour	b. 1959	6	Africa	Senegalese mbalax
Unsuk Chin	b. 1961	6	East Asia	Korean-born composer of virtuosic orchestral colour
A. R. Rahman	b. 1966	6	South Asia	Film music joining Indian and electronic idioms
Kendrick Lamar	b. 1987	6	Americas	DAMN.: first Pulitzer Prize for Music outside classical and jazz (2018)
Burna Boy	b. 1991	6	Africa	Afrobeats as a global genre

Appendix B — Reference Implementation

The full source of vacuumnet_music.py follows. It needs Python 3.9 or later and, for WAV output only, numpy.

```
"""
vacuumnet_music.py — reference implementation for
"The History of Music and the Vacuum.Net Theory", Part II (v2.0).

Constable Research, Leiden, 2026.

Every function here corresponds to a rule in the book. Only numpy is needed
(for WAV rendering). MIDI is written without external libraries.

Sections
  1. Trits and directions          (Chapter 8)
  2. Tunings                       (Chapter 9)
  3. Rhythm                        (Chapter 10)
  4. Harmony on the three-sphere   (Chapter 11)
  5. Form: windings and barriers   (Chapter 12)
  6. Output: MIDI and WAV          (Chapter 13)
"""
from __future__ import annotations
import itertools, math, struct
from fractions import Fraction

# ---------------------------------------------------------------------------
# 1. TRITS AND DIRECTIONS
# ---------------------------------------------------------------------------
DOWN, STAY, UP = -1, 0, 1          # a crossing holds a direction, not a value


def balanced_ternary(n: int) -> list[int]:
    """Trits of n, least significant first. 0 -> [0]."""
    if n == 0:
        return [0]
    out = []
    while n != 0:
        r = n % 3
        if r == 2:
            out.append(-1); n = (n + 1) // 3
        else:
            out.append(r); n //= 3
    return out


def from_balanced_ternary(trits: list[int]) -> int:
    return sum(t * 3 ** i for i, t in enumerate(trits))


def tail(stream: list[int]) -> list[int]:
    """The net joins every stream to its tail: the stream minus its first trit."""
    return stream[1:]


def trit_walk(trits: list[int], start: int = 0) -> list[int]:
    """Read trits as directions: each trit moves one step down, stays, or moves up."""
    pos, out = start, [start]
    for t in trits:
        pos += t
        out.append(pos)
    return out


def winding_of_walk(walk: list[int], cycle: int) -> int:
    """How many whole turns a walk on a cycle of length `cycle` makes."""
    return (walk[-1] - walk[0]) // cycle


# ---------------------------------------------------------------------------
# 2. TUNINGS  (all values in cents above the tonic, sorted, within one octave)
# ---------------------------------------------------------------------------
def cents(ratio: float) -> float:
    return 1200.0 * math.log2(ratio)


PURE_FIFTH = cents(3 / 2)                       # 701.955
PYTHAGOREAN_COMMA = cents(3 ** 12 / 2 ** 19)    # 23.460
SYNTONIC_COMMA = cents(81 / 80)                 # 21.506


def chain_of_fifths(lo: int, hi: int, fifth: float = PURE_FIFTH) -> list[float]:
    """Stack fifths k = lo..hi above the tonic and fold into one octave."""
    return sorted((k * fifth) % 1200.0 for k in range(lo, hi + 1))
```

```
def pythagorean12() -> list[float]:
    """Twelve tones from E-flat (k=-3) to G-sharp (k=+8)."""
    return chain_of_fifths(-3, 8)


def meantone12(fraction: Fraction = Fraction(1, 4)) -> list[float]:
    """Fifth narrowed by `fraction` of the syntonic comma (1/4: Aron 1523)."""
    return chain_of_fifths(-3, 8, PURE_FIFTH - float(fraction) * SYNTONIC_COMMA)


def werckmeister3() -> list[float]:
    """Werckmeister III: fifths C-G, G-D, D-A, B-F# narrowed by 1/4 Pythagorean comma."""
    order = ['C', 'G', 'D', 'A', 'E', 'B', 'F#', 'C#', 'G#', 'D#', 'A#', 'F']
    narrowed = {('C', 'G'), ('G', 'D'), ('D', 'A'), ('B', 'F#')}
    pos, out = 0.0, {}
    for a, b in zip(order, order[1:] + ['C']):
        out[a] = pos % 1200.0
        pos += PURE_FIFTH - (PYTHAGOREAN_COMMA / 4 if (a, b) in narrowed else 0.0)
    return sorted(out.values())


def equal(n: int) -> list[float]:
    """n equal steps per octave (12: Zhu Zaiyu/Stevin; 24: Mishaqa; 53: near Jing Fang)."""
    return [1200.0 * i / n for i in range(n)]


JUST_RATIOS_12 = [Fraction(1), Fraction(16, 15), Fraction(9, 8), Fraction(6, 5), Fraction(5, 4),
                  Fraction(4, 3), Fraction(45, 32), Fraction(3, 2), Fraction(8, 5), Fraction(5, 3),
                  Fraction(9, 5), Fraction(15, 8)]


def just12() -> list[float]:
    """Five-limit just intonation on the senario (Zarlino 1558)."""
    return [cents(float(r)) for r in JUST_RATIOS_12]


def pythagorean17() -> list[float]:
    """A 17-tone Pythagorean chain (k=-12..+4): steps are limmas (90.2) and commas (23.5),
    the type of system described by Safi al-Din al-Urmawi."""
    return chain_of_fifths(-12, 4)


def local_tuning(measured_cents: list[float]) -> list[float]:
    """Gamelan principle: the tuning is whatever this ensemble measures. Supply its cents."""
    return sorted(measured_cents)


def freq(tuning: list[float], degree: int, tonic_hz: float = 261.6256) -> float:
    """Frequency of scale degree `degree` (may exceed the octave or be negative)."""
    n = len(tuning)
    octave, step = divmod(degree, n)
    return tonic_hz * 2 ** (octave + tuning[step] / 1200.0)


# ---------------------------------------------------------------------------
# 3. RHYTHM
# ---------------------------------------------------------------------------
def euclid(k: int, n: int) -> list[int]:
    """Bjorklund/Euclid: k onsets in n pulses as evenly as possible (Toussaint 2005).
    Returns a 0/1 list starting with an onset."""
    if k <= 0:
        return [0] * n
    if k >= n:
        return [1] * n
    groups = [[1] for _ in range(k)] + [[0] for _ in range(n - k)]
    while True:
        head = [g for g in groups if g == groups[0]]
        rest = groups[len(head):]
        if len(rest) <= 1:
            break
        m = min(len(head), len(rest))
        groups = [head[i] + rest[i] for i in range(m)] + head[m:] + rest[m:]
    return [x for g in groups for x in g]


def onsets(pattern: list[int]) -> list[int]:
    return [i for i, x in enumerate(pattern) if x]


def gaps(on: list[int], n: int) -> list[int]:
    return [(on[(i + 1) % len(on)] - on[i]) % n for i in range(len(on))]


def rotate(pattern: list[int], r: int) -> list[int]:
    return pattern[r:] + pattern[:r]
```

```
def same_necklace(a: list[int], b: list[int]) -> bool:
    return any(rotate(a, r) == b for r in range(len(a)))


def maximally_even(k: int, n: int, offset: int = 0) -> list[int]:
    """Clough–Douthett: the k-note subset of Z_n given by floor((i*n + offset)/k)."""
    return sorted({(i * n + offset) // k % n for i in range(k)})


STANDARD_PATTERN = [0, 2, 4, 5, 7, 9, 11]          # 12-pulse West African bell
MAJOR_SCALE = [0, 2, 4, 5, 7, 9, 11]               # same set, read as pitch classes


def trit_rhythm(trits: list[int]) -> list[tuple[int, str]]:
    """Convention of this book: +1 = strike (high), -1 = strike (low), 0 = rest."""
    return [(i, 'high' if t == 1 else 'low') for i, t in enumerate(trits) if t != 0]


def hocket(melody: list[int], voices: int) -> list[list[tuple[int, int]]]:
    """Distribute a melody over voices note by note (Aka/San hocket, Chapter 1)."""
    parts = [[] for _ in range(voices)]
    for i, p in enumerate(melody):
        parts[i % voices].append((i, p))
    return parts


def phase_pairs(pattern_len: int, shift_every: int, cycles: int) -> list[int]:
    """Reich phasing: the offset (in pulses) of voice 2 at each repetition of voice 1."""
    return [(r // shift_every) % pattern_len for r in range(cycles)]


# ---------------------------------------------------------------------------
# 4. HARMONY ON THE THREE-SPHERE  (pitch classes mod 12 unless stated)
# ---------------------------------------------------------------------------
def T(chord, t, n=12):
    return tuple(sorted((x + t) % n for x in chord))


def I(chord, t=0, n=12):
    return tuple(sorted((t - x) % n for x in chord))


def tclass(chord, n=12):
    """Canonical representative of the transposition class."""
    return min(T(chord, t, n) for t in range(n))


def stabilizer(chord, n=12):
    """Transpositions t (1..n-1) that map the chord to itself."""
    s = tuple(sorted(set(x % n for x in chord)))
    return [t for t in range(1, n) if T(s, t, n) == s]


def symmetric_classes(n=12):
    """All transposition classes with a non-trivial stabilizer (Messiaen's territory)."""
    seen, out = set(), []
    for k in range(1, n + 1):
        for s in itertools.combinations(range(n), k):
            c = tclass(s, n)
            if c in seen:
                continue
            seen.add(c)
            if stabilizer(c, n):
                out.append(c)
    return out, len(seen)


MESSIAEN_MODES = {
    1: (0, 2, 4, 6, 8, 10),
    2: (0, 1, 3, 4, 6, 7, 9, 10),
    3: (0, 2, 3, 4, 6, 7, 8, 10, 11),
    4: (0, 1, 2, 5, 6, 7, 8, 11),
    5: (0, 1, 5, 6, 7, 11),
    6: (0, 2, 4, 5, 6, 8, 10, 11),
    7: (0, 1, 2, 3, 5, 6, 7, 8, 9, 11),
}

TRITONE = (0, 6)                    # exceptional circle, multiplicity 2
AUGMENTED = (0, 4, 8)               # exceptional circle, multiplicity 3


def triad(root: int, quality: str) -> tuple[int, int, int]:
    """quality 'M' (major) or 'm' (minor); returns (root, third, fifth) as pcs."""
    third = 4 if quality == 'M' else 3
    return (root % 12, (root + third) % 12, (root + 7) % 12)
```

```
def P(root, q):   # parallel: C <-> c
    return (root, 'm' if q == 'M' else 'M')


def L(root, q):   # leading-tone exchange: C <-> e ; c <-> A-flat
    return ((root + 4) % 12, 'm') if q == 'M' else ((root - 4) % 12, 'M')


def R(root, q):   # relative: C <-> a
    return ((root - 3) % 12, 'm') if q == 'M' else ((root + 3) % 12, 'M')


def cycle(start, ops, max_len=48):
    """Apply operations in turn until the start triad returns."""
    out, cur, i = [start], start, 0
    while True:
        cur = ops[i % len(ops)](*cur)
        i += 1
        if cur == start and i % len(ops) == 0:
            return out
        out.append(cur)
        if len(out) > max_len:
            raise ValueError('no return')


def hexatonic(root=0, q='M'):
    return cycle((root, q), [P, L])          # 6 triads around one augmented triad


def octatonic(root=0, q='M'):
    return cycle((root, q), [P, R])          # 8 triads around one diminished seventh


def signed(d, n=12):
    """Shortest signed motion on the circle, in (-n/2, n/2]."""
    d %= n
    return d - n if d > n / 2 else d


def voice_leading(a, b, n=12):
    """Minimal voice leading between chords of equal size (L1 / taxicab, Tymoczko).
    Returns (size, list of (from, signed motion))."""
    best = None
    for perm in itertools.permutations(b):
        motion = [signed(y - x, n) for x, y in zip(a, perm)]
        size = sum(abs(m) for m in motion)
        if best is None or size < best[0]:
            best = (size, list(zip(a, motion)))
    return best


def progression_winding(chords, n=12):
    """Level-1 winding number of a closed progression.
    Sum of all voice motions divided by n: the turns made around the circle that the
    sum map (Morton) identifies with the whole chord space. Requires chords[0]==chords[-1]
    as sets. Uses minimal voice leading between consecutive chords."""
    total = 0
    voices = list(chords[0])
    for b in chords[1:]:
        _, vl = voice_leading(tuple(v % n for v in voices), b, n)
        total += sum(m for _, m in vl)
        voices = [(x + m) for x, m in vl]
    return Fraction(total, n)


def cyclic_step(chord, n=12):
    """Each voice moves up to the next voice; the last wraps by an octave.
    Its len(chord)-th power is transposition of every voice by one octave."""
    c = sorted(chord)
    return [(c[(i + 1) % len(c)] - c[i]) % n or n for i in range(len(c))]


# ---------------------------------------------------------------------------
# 5. FORM: WINDINGS AND BARRIERS
# ---------------------------------------------------------------------------
def comma_drift(steps: int = 12) -> list[float]:
    """Stack pure fifths and fold back by octaves: the cents drift from the start
    after each fifth (the barrier of Chapter 2 made audible)."""
    out = []
    for k in range(1, steps + 1):
        x = (k * PURE_FIFTH) % 1200.0
        out.append(x if x < 600 else x - 1200.0)
    return out
```

```
def barrier_form(rule, undo, new_object, loss):
    """The four-step barrier rule as a form plan (a list of named sections)."""
    return [('rule', rule), ('undo fails', undo), ('new object', new_object), ('loss', loss)]


# ---------------------------------------------------------------------------
# 6. OUTPUT
# ---------------------------------------------------------------------------
class Note:
    """start/dur in beats; hz absolute frequency; vel 1..127; voice for channel use;
    drum=True for unpitched strikes (hz then sets the drum colour)."""
    __slots__ = ('start', 'dur', 'hz', 'vel', 'voice', 'drum')

    def __init__(self, start, dur, hz, vel=90, voice=0, drum=False):
        self.start, self.dur, self.hz, self.vel, self.voice, self.drum = start, dur, hz, vel, voice,
drum


def hz_to_midi(hz: float) -> tuple[int, int]:
    """Nearest MIDI key and pitch-bend value (bend range +-2 semitones)."""
    m = 69 + 12 * math.log2(hz / 440.0)
    key = int(round(m))
    bend = int(round(8192 + (m - key) * 4096))
    return key, max(0, min(16383, bend))


def _vlq(x: int) -> bytes:
    b = [x & 0x7F]
    x >>= 7
    while x:
        b.insert(0, (x & 0x7F) | 0x80)
        x >>= 7
    return bytes(b)


def write_midi(notes: list[Note], path: str, bpm: float = 96, tpq: int = 480):
    """Format-0 SMF. Each sounding note gets its own channel (round-robin over 15
    melodic channels) so that pitch bend can carry any tuning exactly."""
    ev = []  # (tick, order, bytes)
    us = int(60_000_000 / bpm)
    ev.append((0, 0, b'\xff\x51\x03' + us.to_bytes(3, 'big')))
    chans = [c for c in range(16) if c != 9]
    for c in chans:  # set pitch-bend range to +-2 semitones (RPN 0)
        ev += [(0, 1, bytes([0xB0 | c, 101, 0])), (0, 1, bytes([0xB0 | c, 100, 0])),
               (0, 1, bytes([0xB0 | c, 6, 2])), (0, 1, bytes([0xB0 | c, 38, 0]))]
    for i, n in enumerate(sorted(notes, key=lambda n: n.start)):
        t0, t1 = int(round(n.start * tpq)), int(round((n.start + n.dur) * tpq))
        if n.drum:
            key = 76 if n.hz > 300 else 77 if n.hz > 150 else 36
            ev += [(t0, 3, bytes([0x99, key, n.vel])), (t1, 2, bytes([0x89, key, 0]))]
            continue
        c = chans[i % len(chans)]
        key, bend = hz_to_midi(n.hz)
        ev += [(t0, 2, bytes([0xE0 | c, bend & 0x7F, bend >> 7])),
               (t0, 3, bytes([0x90 | c, key, n.vel])), (t1, 1, bytes([0x80 | c, key, 0]))]
    ev.sort(key=lambda e: (e[0], e[1]))
    data, last = bytearray(), 0
    for t, _, b in ev:
        data += _vlq(t - last) + b
        last = t
    data += b'\x00\xff\x2f\x00'
    with open(path, 'wb') as f:
        f.write(b'MThd' + struct.pack('>IHHH', 6, 0, 1, tpq))
        f.write(b'MTrk' + struct.pack('>I', len(data)) + bytes(data))


def render_wav(notes: list[Note], path: str, bpm: float = 96, sr: int = 44100):
    """Additive rendering at exact frequencies (any tuning), plus noise drums."""
    import numpy as np
    spb = 60.0 / bpm
    end = max(n.start + n.dur for n in notes) * spb + 2.0
    out = np.zeros(int(end * sr))
    rng = np.random.default_rng(7)
    for n in notes:
        i0 = int(n.start * spb * sr)
        a = n.vel / 127.0
        if n.drum:
            L = int(0.25 * sr)
            t = np.arange(L) / sr
            body = np.sin(2 * np.pi * n.hz * t) * np.exp(-t * 18)
            noise = rng.standard_normal(L) * np.exp(-t * 40) * 0.3
            sig = (body + noise) * a * 0.6
        else:
            L = int((n.dur * spb + 0.8) * sr)
            t = np.arange(L) / sr
            sig = sum((0.6 ** (h - 1)) * np.sin(2 * np.pi * n.hz * h * t) for h in (1, 2, 3, 4))
            env = np.minimum(1, t / 0.01) * np.exp(-t * 2.2)
```

```
            sig = sig * env * a * 0.25
        j = min(len(out), i0 + len(sig))
        out[i0:j] += sig[:j - i0]
    out /= max(1e-9, np.max(np.abs(out))) / 0.9
    pcm = (out * 32767).astype('<i2')
    with open(path, 'wb') as f:
        f.write(b'RIFF' + struct.pack('<I', 36 + pcm.nbytes) + b'WAVEfmt ' +
                struct.pack('<IHHIIHH', 16, 1, 1, sr, sr * 2, 2, 16) +
                b'data' + struct.pack('<I', pcm.nbytes) + pcm.tobytes())
    return end


# --- Level 5: symmetry orders of chords in an n-step grid ----------------------
def symmetry_orders(n, kmax=5):
    """Orders of rotation that fix some chord of 1..kmax notes in the n-step grid.
    In the continuous circle the possible orders for kmax = 5 are {2, 3, 4, 5};
    12 steps reach {2, 3, 4}; slendro-like 5 steps reach {5}."""
    from itertools import combinations
    from math import gcd
    out = set()
    for k in range(1, kmax + 1):
        for S in combinations(range(n), k):
            S = frozenset(S)
            for d in range(1, n):
                if frozenset((x + d) % n for x in S) == S:
                    out.add(n // gcd(n, d))
    return sorted(out)
```

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