# Ecological Hyper-Knots and Eigenforms:
## A Topological Generalization of Autopoietic Operations

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**Abstract**

The exploratory preprint *All Operations Are Autopoietic* argued that any persistent operation must recursively regenerate the conditions enabling its further enactment, formalized through the eigen-relation $O = F(O)$. While this captured individual operational recursion, it did not address how multiple such self-producing operations co-regulate, stabilize, or transform one another within an ecology. Building on Kauffman's theory of eigenforms, Spencer-Brown's distinction calculus, and non-essentialist frameworks in autopoiesis, enactivism, and agential realism, this paper introduces *ecological hyper-knots*: relational, topological structures in which multiple eigenforms become mutually entangled.

A hyper-knot $H$ satisfies a higher-order fixed point $H = \Phi(H)$, where $\Phi$ is a functor describing allowable ecological transformations across autopoietic components, relational couplings, complement structure, and organizational invariants. Hyper-knots provide a general framework for explaining the co-differentiation of matter and meaning, the emergence of spatial and temporal structure, the persistence of organizational identity, and the formation of cognitive, social, and material worlds. This framework unifies recursive self-production, relational topology, and ecological constraint into a single coherent theory of world-making.

# Contents

# 1 Introduction

The exploratory preprint *All Operations Are Autopoietic* proposed that any operation which persists within a non-essentialist system must continually regenerate the conditions that allow it to function (**WatsonPreprint**). Autopoiesis, framed in the spirit of Maturana and Varela's foundational work (Maturana and Varela 1980), was treated not as a biological special case but as a *general logic of persistence*: an operation survives only by re-entering the enabling environment that it recursively produces.

Formally, this recursive condition was expressed as a fixed-point equation,

$$O = F(O),$$

in which $F$ denotes the meta-operation that generates the context of viability for $O$. Such fixed points were interpreted not as ontological objects but as *operational invariants*, echoing the categorical treatment of self-reference in the tradition of Mac Lane and others (**MacLane1998**). The preprint established that persistence requires re-entry, but it did not yet explore how multiple such self-producing operations interact within an ecology.

A natural conceptual and mathematical bridge emerges in Louis Kauffman's theory of *eigenforms* (**KauffmanLoF**; Kauffman 1998). Kauffman defines identity as the stabilized outcome of recursive transformation, captured in the relation

$$X = F(X).$$

Rather than treating identity as an inherent property or essence, Kauffman frames it as an *operational equilibrium*: a closure that persists precisely because it re-enters the process that generates it. This resonates directly with the autopoietic account of systems as organizations that produce the conditions of their own existence.

While the preprint argued that any persistent operation must form such an eigen-relation, it did not articulate how multiple eigenforms:

- interact and co-regulate one another,
- stabilize or destabilize shared operating conditions,
- generate constraint, tension, and slack,
- produce ecological contexts of meaning and viability, or
- differentiate matter, meaning, space, and time.

To address these questions, this paper introduces *ecological hyper-knots*: a relational, topological, and ecological generalization of eigenforms. Hyper-knots extend the fixed-point logic of autopoietic operations to describe the *entanglement* of multiple recursive closures within a shared environment. This formulation also draws on the logic of distinction introduced by Spencer-Brown (Spencer-Brown 1969) and extended diagrammatically by Kauffman, while incorporating relational insights from Barad's account of intra-action and the co-constitution of matter and meaning (Barad 2007).

Where an individual eigenform satisfies a self-returning condition,

$$X = F(X),$$

an ecology of coupled eigenforms must satisfy a higher-order fixed-point relation,

$$H = \Phi(H),$$

in which $H$ denotes a *hyper-knot*: an organized ensemble of autopoietic components, and $\Phi$ is a functor describing the admissible co-regulated transformations of the overall configuration.

In this framework:

- Each component $K_i$ is an eigenform—a recursively regenerated closure.
- Relations $R_{ij}$ encode how eigenforms modulate one another's enabling conditions, producing patterns of constraint and co-regulation.
- The complement $C_H$ represents the jointly constructed ecological context that affords stability and possibility.
- The functor $F$ encodes the dynamical pathways through which eigenforms transform together.
- The invariants $I_H$ describe *higher-order eigenforms*: stable patterns of organization preserved across ecological transformation.

Ecological hyper-knots thus show that eigenforms do not exist in isolation. They form *knotted ecologies* of mutually sustained constraints, modulations, tensions, and transformations. This moves the theory beyond the emergence of individual identities toward the emergence of *worlds*: structured, relational domains in which matter, meaning, space, and time co-differentiate through ongoing operations.

In summary:

- The earlier preprint explored how an eigenform comes into being.

- The present paper investigates how eigenforms become entangled, generating coherent ecologies of distinction, stability, and transformation.

Hyper-knots are the mathematical object that expresses this *ecology of eigenforms.*

# 2 From Autopoietic Operations to Ecological Structure

The exploratory preprint (**WatsonPreprint**) argued that persistence within a non-essentialist system requires autopoietic regeneration: an operation $O$ can continue only if it reproduces the structural conditions from which its own enactment becomes meaningful. This insight is rooted in the classical autopoiesis literature (**MaturanaVarela1987**; Maturana and Varela 1980), where a living system is defined not by what it is made of, but by the network of processes that continuously produce the components and relations that constitute the system itself. In the preprint's formulation, this requirement was rendered abstractly through the fixed-point equation

$$O = F(O),$$

implying that the enabling environment for $O$ is not external but is recursively generated by $O$'s own trajectory.

While this fixed-point condition captures the *self-producing* character of autopoietic operations, it remains insufficient for describing the complex, multi-perspectival settings in which actual systems exist. Real systems—biological, cognitive, social, institutional—do not persist as isolated closures. Rather, they exist in a dense mesh of ongoing interactions through which they perturb and are perturbed by other systems (**Maturana1988**; Varela, Thompson, and Rosch 1991). Each system re-enters its own operation, but it does so within an *ecology* of other re-entering processes.

This leads to two fundamental limitations of the single-operation model:

(a) It cannot describe how distinct autopoietic operations *co-regulate* one another by modifying each other's enabling conditions.

(b) It cannot account for the emergence of *shared structures*—constraints, meanings, contexts, and invariants—that arise from patterns of sustained interaction between operations.

In short, the fixed-point structure expresses how an operation persists, but not how persistence becomes *structured relationally.* This gap becomes especially apparent when one considers the role of what Kauffman calls *eigenforms* (Kauffman 1998). An eigenform is not merely a self-returning entity; it is a *pattern that stabilizes within a field of recursive transformations.* Yet, in practice, eigenforms rarely occur in isolation. They develop, stabilize, and transform within relational environments populated by other eigenforms which are themselves dynamically ongoing.

This ecological entanglement has been noted in diverse theoretical contexts. In second-order cybernetics, observers and systems co-emerge through mutual specification (**Foerster2003**). In enactivist cognitive science, cognitive agents bring forth their world through sensorimotor couplings (**VarelaThompsonRosch1991**). In agential realism, Barad (Barad 2007) emphasizes that matter and meaning co-emerge from relations rather than pre-existing as independent categories. Across these traditions, a common theme appears: *relational patterns shape the differentiation of identity, world, and meaning.*

To express this within a more formal, topological architecture, we must move beyond the single eigenform and describe the dynamics of *ensembles* of eigenforms. These ensembles are not simply networks of interacting operations; they are self-structuring environments in which distinctions, constraints, and possibilities are collectively generated. The challenge is to articulate a mathematical object capable of representing:

- the autopoietic closure of each component;
- the relations through which closures become mutually enabling or mutually constraining;
- the emergent topology of contexts, boundaries, and complements generated by the ensemble;
- the transformation laws governing how these structures evolve through time; and
- the invariants that express identity through ecological transformation.

Classical models of interaction—networks, dynamical systems, coupled differential equations—capture some aspects of relational systems but fail to address the deeper issue of *co-produced structure.* Knot theory and diagrammatic logic, especially in the lineage of Spencer-Brown and Kauffman (**KauffmanLoF**; Spencer-Brown 1969), provide a richer symbolic vocabulary for expressing recursive distinctions, crossings, entanglements, and topological invariants. Yet classical knot theory lacks the capacity to express the ecological features relevant to autopoietic systems: context, modulation, constraint architectures, and co-regulated transformation.

This motivates the introduction of *ecological hyper-knots* as a generalization of the eigenform concept. Hyper-knots extend the recursive closure of autopoietic operations to encompass:

- local closures ($K_i$) as eigenforms,
- relational entanglements ($R_{ij}$) as channels of co-regulation,
- complements ($C_H$) as emergent ecological contexts,
- functorial dynamics ($F$) as pathways of co-transformation, and
- higher-order invariants ($I_H$) as organizational eigenforms.

In this shift from individual operations to ecological structures, the logic of autopoiesis becomes inherently *topological*. Systems do not merely persist; they participate in knotted fields of ongoing interaction. Identity becomes a *hyper-eigenform*: a stability condition of an entire relational configuration. Distinctions become interactive, and constraints emerge not from a pre-given external world but from the topology of interdependence.

The next section formalizes this idea, defining ecological hyper-knots as the mathematical object capable of expressing the ecological, topological, and transformational dynamics of coupled eigenforms.

# 3 The Formal Structure of Ecological Hyper-Knots

The limitations identified in Section 2 motivate the need for a mathematical object capable of expressing the organization of multiple interacting eigenforms. Such an object must represent not only the autopoietic closure of each component but also the relational constraints, shared contexts, and transformation pathways that arise through their ongoing co-existence. This section provides a formal definition of *ecological hyper-knots* as the structure that satisfies these requirements.

The definition draws on several conceptual and mathematical lineages. From autopoiesis (Maturana and Varela 1980), we inherit the requirement that each component must regenerate its own conditions of viability. From Kauffman's eigenforms (Kauffman 1998), we adopt the idea that identity is the stabilized result of recursive re-entry. From Spencer-Brown's distinction calculus (Spencer-Brown 1969), we take the primacy of distinction-making as the generator of form. And from topological and categorical thinking (**MacLane1998**; **KauffmanLoF**), we incorporate the intuition that relationships and transformations—not intrinsic substances—constitute the basic units of analysis.

An ecological hyper-knot formalizes these ideas into a single coherent structure.

## 3.1 Definition of an Ecological Hyper-Knot

An *ecological hyper-knot* is a tuple

$$\mathcal{H} = (K_i;\, R;\, C_H;\, F;\, I_H),$$

where each element of the tuple satisfies the following conditions.

### (1) Autopoietic Components $K_i$

Each $K_i$ is an *autopoietic eigenform*, satisfying a self-returning condition

$$K_i = F_i(K_i),$$

where $F_i$ specifies the local transformation that regenerates the enabling conditions of $K_i$. Formally, $K_i$ may be modelled as a closure under a set of operations, a local distinction-structure, or a recursive morphism within a category.

Each $K_i$ therefore represents a self-producing identity, a closure that continuously re-enters its own generative operations. At this level, the definition generalizes the concept of an eigenform from an isolated recursive pattern to a locally autopoietic subsystem embedded within a larger ecology.

### (2) Relational Structure $R$

The relations $R = \{R_{ij}\}$ encode the *structural couplings* between components. For any pair $K_i$ and $K_j$, the relation $R_{ij}$ describes how each modifies, constrains, enables, or perturbs the other's autopoietic trajectory (Varela, Thompson, and Rosch 1991). These relations may be decomposed into three classes:

$$R_{ij} = (R^G_{ij}, R^C_{ij}, R^E_{ij}),$$

where:

- $R^G_{ij}$ captures geometric or topological adjacency, including analogues of knot crossings or spatial juxtaposition;
- $R^C_{ij}$ represents constraint relations, including symmetries, prohibitions, dependencies, and permissible transformations;
- $R^E_{ij}$ encodes ecological or co-regulative couplings, such as mutual enabling or mutual inhibition.

The full relational hypergraph $R$ constitutes the *entangled topology* of the system, capturing how the individual closures form a knotted field of interactions.

### (3) Complement $C_H$

The complement $C_H$ is the *ecological context* generated by the hyper-knot itself. Analogous to the complement of a knot in three-dimensional space, $C_H$ represents the environment constituted by the removal of the components $K_i$ from the relational field.

In ecological terms, $C_H$ encodes:

- domains of possibility and constraint,
- boundary conditions,
- affordances and limitations,
- the background structure through which systems acquire meaning and viability.

Unlike classical ambient space, $C_H$ is not pre-given. It is jointly constructed by the ensemble of interacting eigenforms, resonating with Barad's account of intra-active co-constitution (Barad 2007).

### (4) Transformation Functor $F$

The functor $F$ specifies the admissible transformations of the hyper-knot:

$$\mathcal{H}_{t+1} = F(\mathcal{H}_t).$$

These transformations generalize classical Reidemeister moves by incorporating local, relational, and global ecological operations. In Section 4, we formalize these as six classes of *hyper-Reidemeister* moves (HR1–HR6), each capturing different modes of ecological transformation.

The functor $F$ encodes the *temporal dynamics* of the hyper-knot, expressing how the relational topology changes while respecting the identities of components and the organizational invariants of the system.

### (5) Organizational Invariants $I_H$

The set $I_H$ represents the *higher-order eigenforms* preserved under transformations of the hyper-knot. While $K_i$ expresses the identity of individual components, $I_H$ expresses the identity of the entire ecological configuration. An invariant may include, for example:

- conserved relational patterns,
- stable boundary conditions,

- persistent constraints or symmetries,
- global properties of the complement $C_H$,
- organizational motifs preserved across transformation.

Identity becomes a property not of a single component but of the *whole knot*. Hyper-knot invariants thus generalize the notion of eigenforms from individual fixed points to *ecosystem-level* fixed points.

## 3.2 Interpretation

This definition establishes ecological hyper-knots as the minimal structure capable of expressing:

1. autopoietic identity (via $K_i$),
2. ecological interdependence (via $R$),
3. co-constituted environment (via $C_H$),
4. lawful transformation (via $F$), and
5. persistence of organization (via $I_H$).

The structure thus unifies distinction-making, recursion, relational topology, ecological constraint, and identity through time. By treating eigenforms as components of a knotted ecology, hyper-knots offer a formal language for describing the emergence of matter, meaning, space, and time as *ecological differentiations*, not pre-given foundations.

Section 4 now develops the transformation theory of hyper-knots, introducing the hyper-Reidemeister moves that govern their ecological dynamics.

# 4 Hyper-Reidemeister Transformations (HR1–HR6)

Having defined the structure of ecological hyper-knots in Section 3, we now introduce the transformation rules that govern their dynamics. These transformations generalize the classical Reidemeister moves of knot theory (**KauffmanLoF**; Kauffman 2013) by incorporating not only local diagrammatic changes but also relational, ecological, and organizational modifications. Together, the six *hyper-Reidemeister* moves provide the operative grammar through which a hyper-knot may evolve while preserving its organizational identity.

While classical knot theory operates within a fixed ambient space, ecological hyper-knots operate within a *self-generated complement* $C_H$ (defined in Section 3). Thus, transformations of hyper-knots do not merely rearrange components; they also modify the context that those components jointly constitute. This requires a more general form of invariance: identity is preserved not through topological equivalence alone but through the conservation of the organizational invariants $I_H$.

## 4.1 Overview of Hyper-Reidemeister Moves

The hyper-Reidemeister moves are organized into three conceptual classes:

1. **Local closure transformations** — internal reorganizations of a component $K_i$ that preserve its autopoietic identity (HR1–HR3).

2. **Relational transformations** — modifications of inter-component relations $R_{ij}$ that preserve the ecological topology (HR4–HR5).

3. **Global ecological transformations** — reorganizations of the hyper-knot as a whole that preserve the higher-order invariants $I_H$ (HR6).

Each move is now defined formally.

## 4.2 HR1: Local Expansion–Contraction of a Closure

HR1 generalizes the classical Reidemeister I move. It allows the internal expansion or contraction of a component $K_i$ without altering its autopoietic identity:

$$K_i \longrightarrow K_i' \quad \text{such that} \quad K_i' = F_i(K_i').$$

HR1 captures:

- internal rebalancing of distinctions;
- shifts in boundary curvature or shape;
- local variations in complexity that do not alter closure.

In autopoietic terms, this corresponds to homeostatic regulatory changes that maintain organizational integrity despite local fluctuations (**MaturanaVarela1987**).

## 4.3 HR2: Internal Reconfiguration of a Closure

HR2 generalizes classical Reidemeister II moves. It allows two adjacent operations within a closure to be introduced or removed, provided the overall closure remains autopoietically coherent:

$$K_i \leftrightsquigarrow K_i'.$$

HR2 expresses:

- internal reorganization of processes;
- creation or dissolution of short-lived loops;
- temporary increases in complexity that leave identity unchanged.

Conceptually, HR2 models how an autopoietic system may transiently adopt new structures without altering its fundamental mode of being.

## 4.4 HR3: Redistribution of Internal Tension

HR3 generalizes the classical Reidemeister III move. It enables the redistribution of internal tensions or dependencies among operations in a closure, preserving the overall organization:

$$K_i \to K_i' \quad \text{with} \quad I(K_i) = I(K_i').$$

HR3 models:

- rebalancing of internal constraints;
- the adjustment of regulatory loops;
- shifts in dependency structures that preserve invariants.

Together, HR1–HR3 define the allowable transformations of individual autopoietic components $K_i$.

## 4.5 HR4: Modulation of Relational Couplings

HR4 introduces ecological transformation. It allows the strengthening, weakening, or rebalancing of relational couplings $R_{ij}$, provided that the ecological topology is preserved:

$$R_{ij} \to R_{ij}', \qquad R' \simeq R.$$

HR4 captures:

- variations in co-regulation strength;
- adjustments in mutual influence;
- shifting degrees of coupling while preserving connectivity.

This corresponds to modulation phenomena observed in cognitive and social systems (**Barandiaran2009**; Varela, Thompson, and Rosch 1991).

## 4.6 HR5: Redistribution of Ecological Dependencies

HR5 allows the redistribution of ecological dependencies across the hyper-knot:

$$R^E_{ij} \to R^{E'}_{ij}, \quad \sum_j R^E_{ij} = \sum_j R^{E'}_{ij}.$$

This move conserves the overall ecological load but redistributes dependency relations across components.

HR5 models:

- ecological rebalancing;
- transfer of functional load between components;
- shifts in roles that preserve higher-order invariants.

HR5 is crucial for modeling systems in which components adapt collectively to changing environmental or internal conditions (**DiPaolo2005**).

## 4.7 HR6: Global Organizational Reconfiguration

HR6 is the most general transformation. It allows global reconfiguration of the hyper-knot, affecting closures, relations, and complement, provided that the organizational invariants $I_H$ are preserved:

$$\mathcal{H} \to \mathcal{H}', \qquad I_H = I'_H.$$

HR6 captures:

- large-scale ecological reorganization;
- restructuring of functional architectures;
- phase shifts in the relational topology;
- transformations of $C_H$ that preserve identity at the level of the whole.

This move generalizes the concept of an eigenform to the scale of an entire ecology. A hyper-knot that undergoes HR6 remains the same *organization* even if its components and relations change.

### 4.8 Interpretation

Taken together, HR1–HR6 form the *generative grammar* of ecological hyper-knots. They express how autopoietic components may transform internally (HR1–HR3), how relational structures may reorganize (HR4–HR5), and how the entire ecological configuration may shift while preserving its organizational identity (HR6).

These moves represent a synthesis of:

- autopoietic adaptation (Maturana and Varela 1980),
- recursive identity-as-invariance (Kauffman 1998),
- topological equivalence under transformation (Kauffman 2013),
- ecological co-constitution (Barad 2007).

The next section introduces quantitative and structural metrics for evaluating ecological tension, complement capacity, curvature, and stability within hyper-knots.

## 5 Metrics of Ecological Structure

With the hyper-Reidemeister moves (HR1–HR6) defined in Section 4, we now introduce a set of quantitative and structural metrics for describing the organization of ecological hyper-knots. These metrics express different forms of constraint, tension, capacity, and flexibility within the relational topology. They do not presuppose a spatial substrate; rather, they emerge from patterns of distinction, relation, and co-regulation (Varela, Thompson, and Rosch 1991; Barad 2007).

The following constructs provide a minimal but expressive vocabulary for describing the ecological geometry of a hyper-knot.

### 5.1 Ecological Degree $E_d$

The *ecological degree* $E_d$ measures the total dependency load borne by the components of the hyper-knot. Formally:

$$E_d = \sum_{i,j} |R_{ij}^E|.$$

Here $R_{ij}^E$ denotes the ecological (co-regulative) component of the relation between $K_i$ and $K_j$, as defined in Section 3. Systems with high ecological degree are densely interdependent, indicating strong mutual modulation and shared viability conditions. Systems with low $E_d$ are more weakly coupled and may degrade or reorganize more readily under perturbation.

Ecological degree provides a measure of the density of co-regulation within a hyper-knot.

## 5.2 Relational Tension $T_R$

The *relational tension* $T_R$ quantifies the extent to which relational constraints $R_{ij}^C$ exert pressure on the organization. Let $w_{ij}$ be a weighting term describing the intensity or rigidity of a constraint. Then:

$$T_R = \sum_{i,j} w_{ij} \, |R_{ij}^C|.$$

High tension corresponds to:

- strong or inflexible constraints,
- reduced local freedom of transformation,
- increased susceptibility to abrupt reorganization.

Low tension corresponds to greater flexibility and adaptability. In biological and cognitive systems, relational tension models the degree to which structural couplings restrict or channel possible trajectories (**Maturana1988**; **DiPaolo2005**).

## 5.3 Complement Capacity $\Gamma_C$

The *complement capacity* $\Gamma_C$ measures the richness, slack, or openness of the complement $C_H$. Formally, $\Gamma_C$ may be understood as a function of:

$$\Gamma_C = f\Big(\dim(C_H),\ \text{redundancy}(C_H),\ \text{affordance}(C_H)\Big),$$

where:

- $\dim(C_H)$ represents the degrees of freedom in the ecological context,
- redundancy($C_H$) captures alternative pathways for co-regulation,
- affordance($C_H$) describes the capacity of the context to support new or altered relations.

A larger complement capacity implies:

- higher ecological flexibility,

- more routes for adaptation,
- reduced risk of catastrophic collapse.

Conceptually, $\Gamma_C$ expresses the generative potential of the ecology—the structure of possibility that the hyper-knot brings into being, echoing Barad's notion of intra-active co-constitution (Barad 2007).

## 5.4 Functorial Curvature $\kappa_F$

The *functorial curvature* $\kappa_F$ measures the degree of nonlinearity or strain in the transformation functor $F$:

$$\kappa_F = \Big\| F(\mathcal{H}) - \mathcal{H} \Big\|.$$

Intuitively, $\kappa_F$ expresses the extent to which the hyper-knot is bent, distorted, or pressured by ongoing transformations. High curvature indicates significant departure from equilibrium or strong dynamical forces acting on the system. Low curvature corresponds to stability or regularity in the transformation pathways.

In autopoietic terms, $\kappa_F$ expresses the regulatory effort required to maintain viability under transformation (**MaturanaVarela1987**). In ecological and social systems, it corresponds to the degree of adaptive strain.

## 5.5 Stability $S_{\mathcal{H}}$

Stability emerges from the interplay of complement capacity, relational tension, and ecological degree. A minimal stability measure is:

$$S_{\mathcal{H}} = \frac{\Gamma_C}{E_d + T_R}.$$

High stability arises when:

- the complement affords many pathways,
- relational tension is low or manageable,
- ecological degree does not create excessive interdependence.

Low stability indicates vulnerability to perturbation, rigidity, or ecological over-coupling.

### 5.6 Resilience $R_{\mathcal{H}}$

The *resilience* of a hyper-knot expresses its capacity to return to its organizational invariants $I_H$ after undergoing HR-transformations. It may be defined as:

$$R_{\mathcal{H}} = \frac{\Gamma_C}{\kappa_F},$$

where high complement capacity and low functorial curvature correspond to strong resilience.

Resilience measures the hyper-knot's ability to withstand transformation without loss of identity. This reflects the concept of identity-as-invariance central to Kauffman's eigenform theory (Kauffman 1998) and aligns with biological and cognitive accounts of adaptive equilibrium (**DiPaolo2005**; Thompson 2007).

### 5.7 Interpretation

Together, the metrics introduced in this section provide a toolkit for describing the ecological geometry of hyper-knots. They allow us to analyze:

- how tightly an ecology is coupled ($E_d$),
- how constrained or pressured its interactions are ($T_R$),
- how much freedom its context affords ($\Gamma_C$),
- how strained its transformations are ($\kappa_F$),
- how stable and resilient its organization remains ($S_{\mathcal{H}}$ and $R_{\mathcal{H}}$).

These metrics extend classical notions of topological invariance, complementarity, and structural coupling into an explicitly ecological framework. They will later enable a more precise analysis of matter–meaning differentiation (Section 6) and organizational identity (Section 7).

## 6 Matter–Meaning Differentiation Through Ecological Hyper-Knots

A central motivation for introducing ecological hyper-knots is to provide a formal explanation of how *matter* and *meaning* emerge and differentiate within the recursive and relational dynamics of autopoietic systems. Traditional ontologies treat matter as a fundamental substrate and meaning as a secondary or derivative phenomenon. In contrast, relational and enactive accounts—from Varela's autopoietic phenomenology (**VarelaThompsonRosch1991**;

Varela, Thompson, and Rosch 1991) to Barad's agential realism (Barad 2007)—argue that matter and meaning co-emerge from processes of distinction, relationality, and participation.

Ecological hyper-knots provide a topological framework for expressing this co-emergence. In this section, we show how matter and meaning arise not as pre-given categories but as distinct *roles* within the hyper-knot's ecological dynamics.

## 6.1 Matter as Constraint and Recurrence

Within an ecological hyper-knot, matter corresponds to the *constraint-bearing*, *recurrent*, and *stability-producing* aspects of the organization. Formally, matter is associated with:

1. the complement $C_H$, which expresses the ecological geometry generated by the hyper-knot;
2. the constraint relations $R_{ij}^C$, which stabilize and limit possible transformations;
3. the organizational invariants $I_H$, which preserve identity across ecological change.

Matter emerges from the requirement that a system maintain coherence across transformations. It is not a substance but a *role* played by the stable, constraint-generating features of the ecology. In Kauffman's terminology, matter is the ensemble of *higher-order eigenforms* that persist through recurrent transformations (Kauffman 1998).

More precisely, matter corresponds to the parts of the hyper-knot that satisfy:

$$F(\mathcal{H}) \approx \mathcal{H}$$

for a broad range of transformations. These stable structures constitute the ground from which meaning becomes navigable.

## 6.2 Meaning as Modulation and Temporal Orientation

Meaning corresponds to the *modulatory*, *interpretive*, and *temporal-orienting* aspects of ecological dynamics. Formally, meaning is associated with:

1. relational modulations $R_{ij}^E$ that reconfigure co-regulation;
2. transformation functor $F$, which orders the temporal unfolding of the ecology;
3. directionality introduced by HR4–HR6, which articulate pathways of ecological reorganization.

Where matter stabilizes, meaning *perturbs*. Meaning emerges whenever a system interprets or modulates its own or another's trajectory—whenever the ecology biases, selects, or coordinates possible transformations. This echoes Varela's account of sense-making (Varela, Thompson, and Rosch 1991): meaning is enacted when a system orients itself within an environment that it co-constitutes.

Formally, meaning can be expressed through the transformation gradient:

$$\Delta_F(\mathcal{H}) = F(\mathcal{H}) - \mathcal{H},$$

which captures the extent to which a transformation is interpretively or modulatorily directed. High $\Delta_F$ indicates strong interpretive or sense-making activity.

## 6.3 Co-Differentiation of Matter and Meaning

In ecological hyper-knots, matter and meaning are not pre-given categories but emerge through ongoing *co-differentiation*. Their distinction arises from the system's internal organization of constraints and modulations.

Matter corresponds to:

$$\text{Matter} = \{\text{operations that conserve } I_H\}.$$

Meaning corresponds to:

$$\text{Meaning} = \{\text{operations that reorganize } (K_i, R, C_H) \text{ while preserving } I_H\}.$$

The difference lies not in substance but in *functional role*:

- Matter stabilizes; meaning modulates.
- Matter constrains; meaning orients.
- Matter anchors identity; meaning interprets viability.

This aligns with Barad's view that matter is not inert but an active participant in world-making (Barad 2007). Within hyper-knots, matter is the stability of the environmental field; meaning is the pattern of interpretive deviations that traverse this field.

## 6.4 Emergent Space and Time

Matter–meaning differentiation also generates emergent notions of space and time:

- *Space* corresponds to the stable relational topology induced by $C_H$ and $R_{ij}^C$.

- *Time* corresponds to the ordering of changes induced by the functor $F$ and HR1–HR6.

Thus, in hyper-knots:

$$\text{Space} = \text{Topology of constraint}, \qquad \text{Time} = \text{Order of transformation}.$$

Space and time emerge as aspects of the ecological differentiation of matter and meaning. This resonates with relational physics (**Rovelli2004**; Rovelli 1996), where the fundamental structure of the world is relational rather than substantial.

### 6.5 Interpretation

The hyper-knot framework demonstrates that matter and meaning are not ontologically separate; they are ecological *roles* that arise through coordinated patterns of constraint and modulation. These roles are continually negotiated through HR-transformations and stabilized by $I_H$.

In this sense:

- Matter is the *recurrent geometry* of the ecology.
- Meaning is the *interpretive modulation* of that geometry.

What appears as "material" is the aspect of the ecology that resists change; what appears as "meaningful" is the aspect that reorients transformation. Their co-constitution is the condition of possibility for any autopoietic, cognitive, or social world.

The next section elaborates the organizational identity of ecological hyper-knots and introduces the notion of *hyper-eigenforms*, which capture invariants at the scale of the entire ecology.

## 7 Organizational Identity and Hyper-Eigenforms

The previous sections have shown how ecological hyper-knots integrate autopoietic closures, relational couplings, complement structure, and lawful transformation. We now turn to the question of *identity*: what allows a hyper-knot to remain the "same" across transformation? Classical knot theory resolves this question in terms of topological equivalence under Reidemeister moves (Kauffman 2013). Autopoietic theory resolves it in terms of operational closure and organizational invariance (Maturana and Varela 1980). Kauffman's concept of an *eigenform* adds the further insight that identity is not a substance but the stable result of recursive re-entry (Kauffman 1998).

Ecological hyper-knots unify these ideas. Identity emerges as a *hyper-eigenform*: a stable, recursively regenerated organizational pattern of the entire ecological configuration.

## 7.1 Identity as an Invariant of the Transformation Functor

Let $\Phi$ denote the transformation functor governing the evolution of the entire hyper-knot. A hyper-knot $\mathcal{H}$ exhibits organizational identity when it satisfies:

$$\mathcal{H} = \Phi(\mathcal{H}),$$

or more generally, when the invariants $I_H$ satisfy:

$$I_H(\mathcal{H}) = I_H(\Phi(\mathcal{H})).$$

Thus, identity is not the persistence of components $K_i$ or even of specific relations $R_{ij}$ but the persistence of the higher-order organizational invariants. This generalizes Kauffman's eigenform condition from individual re-entry to *collective* ecological re-entry.

In this sense, identity is the *pattern that remains stable when the ecology transforms itself.*

## 7.2 Hyper-Eigenforms

A *hyper-eigenform* is defined as an invariant organizational pattern preserved under allowable hyper-Reidemeister transformations (HR1–HR6). Formally, it is an element of $I_H$, the set of invariants associated with the hyper-knot:

$$I_H = \{\xi \mid \xi(\mathcal{H}) = \xi(\Phi(\mathcal{H}))\}.$$

A hyper-eigenform may take many forms:

- a stable relational motif,
- a conserved pattern of complement structure,
- a preserved ecological role distribution,
- a persistent symmetry or dependency,
- an organizational constraint that remains unchanged across time.

Hyper-eigenforms stabilize the ecology at a higher level than any individual component $K_i$. They constitute the *identity conditions* for the hyper-knot as a whole.

## 7.3 The Distinction Between Component Identity and Organizational Identity

A crucial feature of hyper-knots is that the identity of the whole need not depend on the identity of individual components. In biological, social, and cognitive systems, components may be replaced, reorganized, or transformed without loss of organizational integrity (**VarelaThompsonRosch1991**). Hyper-knots formalize this phenomenon by distinguishing:

1. **Component identity** (invariants at the level of $K_i$), and
2. **Organizational identity** (invariants at the level of $I_H$).

Under HR4–HR6, components may:

- change their internal structures,
- alter relational connections,
- shift ecological dependencies,
- reorganize functional roles,

without altering the overall hyper-eigenform.

This explains how complex systems undergo *plasticity without collapse.* Identity is not tied to the persistence of parts but to the persistence of a *recursive relational organization.*

## 7.4 Identity Through Ecological Re-entry

In classical eigenforms, identity is the result of re-entry of a transformation on itself (Kauffman 1998). Ecological hyper-knots extend this by introducing:

- multiple simultaneously re-entering components,
- co-regulated relational structures,
- a complement that shifts with the ensemble,
- a transformation functor that applies to the entire configuration.

Identity becomes the result of *ecological re-entry*: the recursive regeneration of a relational topology.

Formally, we may characterize ecological re-entry as:

$$\Phi^t(\mathcal{H}) \to \mathcal{H} \quad \text{as} \quad t \to \infty,$$

where $\Phi^t$ denotes $t$ applications of the functor $\Phi$. If the limit converges (in the sense of organizational invariants), the system possesses a stable identity. If not, identity becomes transient or metastable, a phenomenon often observed in adaptive systems (**DiPaolo2005**; Thompson 2007).

## 7.5 Hyper-Knot Identity and the Differentiation of Worlds

Because matter and meaning co-differentiate through hyper-knot dynamics (Section 6), organizational identity is simultaneously:

- a material identity (stability of constraints and complements), and
- a meaning identity (stability of interpretive and modulatory patterns).

In this sense, a hyper-knot does not merely persist—it constitutes a coherent "world." The invariants $I_H$ express the ecological conditions under which that world remains viable. Worlds differ not by their component contents but by their hyper-eigenforms.

## 7.6 Interpretation

Hyper-eigenforms provide the conceptual bridge between autopoietic closure, topological invariance, and ecological world-making:

- From autopoiesis, they inherit recursive self-production.
- From eigenforms, they inherit identity-as-invariance.
- From knot theory, they inherit invariance-under-transformation.
- From ecological theory, they inherit co-regulated interdependence.

Thus, organizational identity in ecological hyper-knots is not the persistence of the same elements but the persistence of the same *relations of relations.* Identity is a global, recursive, relational invariant—a hyper-eigenform that stabilizes a world.

The next section applies these concepts to concrete domains, demonstrating how hyper-knots model cognitive systems, social formations, material ecologies, and mathematical structures.

# 8 Applications Across Cognitive, Social, Material, and Mathematical Domains

Ecological hyper-knots provide a transdisciplinary framework for analyzing systems whose identities emerge from recursive organization, relational entanglement, and context-dependent transformation. This section illustrates how the hyper-knot formalism clarifies and unifies phenomena across cognitive, social, material, and mathematical domains. These applications demonstrate that hyper-knots are not merely an abstract construction but a general modeling language for systems in which matter, meaning, constraint, and transformation co-differentiate.

## 8.1 Cognitive Systems: Sense-Making as Hyper-Knot Dynamics

In enactive cognitive science, cognition is defined as the ongoing process through which a system brings forth a meaningful world via embodied, situated interactions (**VarelaThompsonRosch1991**; Varela, Thompson, and Rosch 1991). Ecological hyper-knots provide a formal structure for this process.

Let $K_i$ represent autopoietic cognitive subsystems (sensorimotor loops, attentional policies, perceptual categories). Relations $R_{ij}$ encode:

- sensorimotor couplings,
- predictive and anticipatory dependencies,
- internal or external co-regulation between processes.

The complement $C_H$ represents the generative background of affordances and possibilities (**Gibson1979**; **Barandiaran2009**). Meaning emerges via modulation of relational patterns ($R^E_{ij}$), while matter corresponds to persistent constraints in the complement ($C_H$) and invariants ($I_H$).

Cognitive identity—the coherence of the agent—is expressed as a hyper-eigenform:

$$I_H(\mathcal{H}) = I_H(\Phi(\mathcal{H})).$$

This clarifies how cognitive systems:

- maintain organizational stability while transforming,
- enact a meaningful world through relational modulation,
- generate their own spatial and temporal structures,

- reorganize identities through learning or development.

Ecological hyper-knots thus formalize the enactive claim that cognition is an adaptive ecology of self-producing relations.

## 8.2 Social Systems: Coordination, Normativity, and Collective Identity

Social systems consist of interacting agents whose actions and interpretations co-regulate one another across multiple scales (**Elias1978**; Luhmann 1995). Ecological hyper-knots provide a way to express:

- shared practices,
- institutional norms,
- collective identities,
- emergent social structures.

In this domain:

- $K_i$ may represent individual agents, organizations, or communicative acts;
- $R_{ij}$ encode social relations, obligations, expectations, and coordination structures;
- $C_H$ represents the sociocultural environment (language, norms, infrastructures);
- $F$ models institutional evolution and social transformation;
- $I_H$ represents collective identity—the hyper-eigenform stabilizing a social formation.

HR4 and HR5 model reconfigurations of social coupling (e.g., shifts in norms or power relations), while HR6 models large-scale social change (e.g., institutional reforms, cultural transitions) that preserve higher-order organizational patterns.

This framework:

- makes explicit how social identities persist through transformation,
- shows how meaning and normativity co-emerge,
- captures the interplay between stability (matter) and reorganization (meaning),
- emphasizes that social worlds are knot-like ecologies rather than collections of discrete agents.

## 8.3 Material Ecologies: Constraint, Affordance, and Distributed Autopoiesis

Material systems—such as ecosystems, metabolic networks, or engineered environments—exhibit patterns of organization that arise from distributed constraints and ongoing flows of transformation (**Ulanowicz2009**; **Odum1983**). Ecological hyper-knots provide a formal language for describing:

- trophic or metabolic interactions ($R_{ij}$),
- environmental affordances ($C_H$),
- co-adaptive feedback loops (HR4, HR5),
- systemic resilience and tipping points ($S_{\mathcal{H}}, R_{\mathcal{H}}$).

Material identity corresponds to persistent structural patterns (nutrient cycles, energy flows, ecological roles) represented as hyper-eigenforms.

Matter arises from:

- stable flows,
- persistent constraints,
- sustained dependency structures.

Meaning arises from:

- adaptive modulations,
- shifts in ecological coupling,
- environmental sensing and response loops.

Thus hyper-knots unify ecological stability and ecological sense-making within a single topological and dynamical framework.

## 8.4 Mathematical Structures: Fixed Points, Functors, and Diagrammatic Logic

Ecological hyper-knots also clarify several mathematical phenomena:

**(1) Generalized fixed-point theories.** Hyper-eigenforms extend the classical fixed-point condition $X = F(X)$ (Kauffman 1998) to the ecological collective:

$$\mathcal{H} = \Phi(\mathcal{H}).$$

This offers a new perspective on:

- recursive function theory,
- iterative maps,
- self-referential systems.

**(2) Functorial dynamics.** The functor $F$ acts on an entire category of ecological components, relations, and complements, offering new routes for modeling:

- higher-order transformations,
- diagrammatic invariance,
- categorical recursion,
- systems of systems.

**(3) Diagrammatic logic and distinction theory.** Hyper-knots provide a natural extension of:

- Spencer-Brown's calculus of indications (Spencer-Brown 1969),
- Kauffman's knot-theoretic logic (**KauffmanLoF**),
- category-theoretic diagrammatics.

They generalize the diagrammatic expression of:

- distinctions,
- crossings,
- complements,
- invariants under transformation.

## 8.5 Interpretation

Across these domains, ecological hyper-knots serve as a unifying formalism for systems whose identities arise from:

- recursive self-production,
- relational entanglement,
- ecological co-constitution,
- transformation under constraint,
- preservation of higher-order invariants.

What differs across cognitive, social, material, or mathematical worlds are not the principles of organization but the *forms* of their components, relations, and environmental complements. Hyper-knots provide the general structure through which these systems can be compared, analyzed, and understood as variations of a common ecological topology.

The next section discusses the broader implications of this framework and its contribution to non-essentialist theories of identity, world-formation, and recursive ecology.

# 9 Discussion: Implications for Identity, Knowledge, Ecology, and Non-Essentialism

The ecological hyper-knot framework developed in this paper reframes fundamental questions about identity, knowledge, matter, meaning, and ecological organization. By synthesizing insights from autopoiesis (Maturana and Varela 1980), enactive cognition (**VarelaThompsonRosch1991**; Varela, Thompson, and Rosch 1991), distinction theory (Spencer-Brown 1969), eigenform logic (Kauffman 1998), and relational ontology (Barad 2007), hyper-knots provide a general formalism for systems whose structure and identity arise through recursive, ecological entanglement.

This section reflects on the implications of the hyper-knot framework for a non-essentialist understanding of systems across domains.

## 9.1 Identity Without Essence

Hyper-knots demonstrate that identity is not a property of components but an *organizational invariant* of a relational ecology. The hyper-eigenforms introduced in Section 7 reveal that identity:

- emerges from recursive re-entry,
- persists through transformations,
- depends on relations, not substances,
- is dynamically maintained rather than statically possessed.

This aligns with autopoietic accounts of biological and cognitive identity (**MaturanaVarela1987**), as well as with relational theories of social and institutional identity (Luhmann 1995). In all these domains, identity is the continuity of a world, not the persistence of a particular material substrate.

The hyper-knot formalism makes this explicit: organizational invariants $I_H$—and not components or structures—define what a system *is*.

## 9.2 Knowledge as Dynamic Ecological Stability

A central theme in autopoietic epistemology is that knowledge is not the representation of an external world but the enactment of a viable world through relational coordination (Varela, Thompson, and Rosch 1991). Hyper-knots give this idea a formal expression.

Knowledge corresponds to:

- stable ecological relations ($R_{ij}$),
- constraints that preserve viability,
- interpretive modulations that allow adaptation,
- stable hyper-eigenforms that regulate behavior,
- co-differentiated matter–meaning structures.

Thus, knowledge becomes a *dynamic ecological stability* rather than a representation. It is a hyper-eigenform enacted through ongoing coordination.

## 9.3 Ecology as Relational Topology

In classical ecological theory, an ecosystem is a network of flows, dependencies, and constraints (**Odum1983**; **Ulanowicz2009**). Hyper-knots generalize this view by describing ecosystems as:

- relationally knotted,
- recursively maintained,

- context-generating,
- transformation-dependent,
- identity-preserving.

The complement $C_H$ plays a crucial role in this reconceptualization, representing the context co-produced by the ensemble. Unlike a static "environment," $C_H$ is dynamic and responsive, continually reshaped by the interactions that take place within it.

Ecology thus becomes a matter not of fixed space but of *relational topology*.

## 9.4 Matter and Meaning as Ecological Roles

Section 6 showed that matter and meaning are not primitive distinctions but emergent roles within an ecological hyper-knot. Matter corresponds to recurrent, constraint-generating invariants; meaning corresponds to modulatory and orientation-generating activity (Barad 2007). Their difference is functional rather than ontological.

This view dissolves traditional dichotomies:

- subject vs. object,
- representation vs. reality,
- material vs. mental.

Instead, matter and meaning co-differentiate as aspects of a recursive ecology.

## 9.5 Relational Ontology and Non-Essentialism

Hyper-knots articulate a relational and non-essentialist ontology:

1. **Relationality** is primary: systems arise from relations, not from atomic components.
2. **Context-dependence** is constitutive: complements shape identities.
3. **Transformation** is fundamental: identities persist through change.
4. **Non-essentialism** follows: no component or structure is necessary for organizational identity.

This ontology resonates with:

- enactivism (Varela, Thompson, and Rosch 1991),
- second-order cybernetics (**Foerster2003**),

- relational quantum ontology (Rovelli 1996),
- process philosophy and process ontology.

Hyper-knots show that systems are not built from foundations but from *recursively maintained distinctions and relations.*

## 9.6 The Role of Knot Theory and Diagrammatic Reasoning

The knot-theoretic roots of hyper-knots are not metaphorical. They provide a diagrammatic and topological intuition for:

- recursive closure,
- relational entanglement,
- complement structure,
- invariance under transformation.

By generalizing Reidemeister moves to HR1–HR6, hyper-knots extend Kauffman's insight that knot diagrams capture the dynamics of distinction-making (**KauffmanLoF**). They provide a new diagrammatic logic for recursive, ecological systems.

## 9.7 Implications for Modeling Complex Systems

Hyper-knots provide a new modeling environment for complex adaptive systems:

- **Cognitive systems** — modeling recursive sense-making, attention, and embodied action.
- **Social systems** — modeling normativity, institutional stability, and collective identity.
- **Material ecologies** — modeling metabolic networks, ecosystems, and distributed self-maintenance.
- **Mathematical systems** — modeling recursive invariants, functorial dynamics, and topological persistence.

By integrating autopoiesis, knot theory, and ecological topology, hyper-knots provide a *common language* for systems that were previously difficult to analyze using a unified framework.

## 9.8 Interpretation

The hyper-knot framework demonstrates that:

- identity is ecological,
- knowledge is relational,
- matter and meaning are co-emergent,
- worlds arise from recursive entanglement,
- transformation is constitutive rather than disruptive.

In this view, the world is not composed of fixed entities but of *hyper-eigenforms*: stable relational patterns enacted and maintained within ecological hyper-knots.

This perspective shifts the foundational assumptions of ontology, epistemology, and systems theory. Rather than building from substances, the hyper-knot framework builds from *ecologically recursive topologies*, offering a new basis for understanding complex adaptive systems across disciplines.

The concluding section summarizes the contributions of this work and highlights directions for further research.

# 10 Conclusion: Hyper-Knotting as a General Ecology of Worlds

This paper has introduced ecological hyper-knots as a formal and conceptual framework for understanding systems whose identities arise from recursive self-production, relational entanglement, and ecological co-constitution. Drawing on autopoiesis (Maturana and Varela 1980), enactive cognition (Varela, Thompson, and Rosch 1991), eigenform logic (Kauffman 1998), distinction theory (Spencer-Brown 1969), and relational ontologies (Barad 2007), hyper-knots unify diverse lines of thought into a single topological and ecological language.

The central achievements of this work can be summarized as follows.

## 10.1 A Generalization of Autopoietic Operations

We began by extending the classical autopoietic fixed-point relation

$$O = F(O)$$

to the level of ecological ensembles. Hyper-knots capture not only the closure of individual autopoietic operations ($K_i$) but also the network of relational dependencies ($R$), ecological contexts ($C_H$), transformation pathways ($F$), and higher-order invariants ($I_H$) that collectively sustain a world.

In this generalization, the recursive stability of an operation becomes only one aspect of a larger ecological recursion.

## 10.2 An Ecological Theory of Matter and Meaning

By formalizing the dynamics of constraint and modulation (Section 6), we showed how matter and meaning co-differentiate within hyper-knot ecologies:

- Matter emerges as the recurrent, constraint-generating stability of the ecological topology.
- Meaning emerges as the modulatory and interpretive patterning of transformation.

This reframes traditional ontological distinctions, dissolving the dichotomy between substance and significance. Matter and meaning become roles in an ecological dynamical field.

## 10.3 A Concept of Identity as a Hyper-Eigenform

Identity was reconceptualized as a *hyper-eigenform*: a recursively stable organizational invariant preserved under the hyper-Reidemeister transformations (HR1–HR6). This view:

- distinguishes component identity from organizational identity,
- reveals how systems maintain coherence through transformation,
- clarifies how worlds persist while their elements change.

Identity thus becomes a fixed point of ecological re-entry, not a property of static components.

## 10.4 A Transdisciplinary Modeling Framework

Through applications in cognitive, social, material, and mathematical domains (Section 8), ecological hyper-knots were shown to provide a versatile modeling architecture for complex systems. Their diagrammatic and topological structure offers:

- a general language for interdependence,
- a tool for analyzing organizational change,

- a means to represent relational constraint and complementarity,
- an account of how systems create and maintain their own conditions of existence.

Hyper-knots thus establish conceptual and formal bridges across disciplines that are often treated separately.

## 10.5 Implications for Relational and Non-Essentialist Ontology

The hyper-knot framework supports a radical non-essentialist ontology. Systems are not grounded in substances or pre-given identities but in:

- recursive distinctions,
- ecological dependencies,
- dynamic complement structures,
- relational invariants,
- world-constituting transformations.

This resonates with relational quantum ontology (Rovelli 1996), enactivist approaches to mind (Thompson 2007), and agential realism (Barad 2007). Hyper-knots provide a topological and ecological articulation of the same deeper principle: worlds arise from relations recursively producing their own conditions of possibility.

## 10.6 Future Directions

The hyper-knot framework opens several avenues for further research:

1. **Formal development** of the transformation functor $\Phi$ and hyper-Reidemeister moves HR1–HR6.
2. **Computational modeling** of hyper-knot dynamics, including simulation environments for relational autopoiesis.
3. **Empirical applications** in cognitive science, institutional analysis, ecological modeling, and distributed systems design.
4. **Mathematical extensions** involving categorical fixed points, diagrammatic logics, and generalized complement topologies.

5. **Philosophical elaboration** of ecological identity, non-essentialism, and matter–meaning co-constitution.

These directions emphasize that hyper-knots are not merely descriptive but constructive: they offer a new formal and conceptual basis for investigating systems that bring forth worlds.

## 10.7 Conclusion

Ecological hyper-knots articulate a powerful insight: systems do not exist by virtue of what they are, but by virtue of what they *do* together. Identity, matter, meaning, and worldhood arise from the recursive maintenance of relational configurations. Hyper-knots provide the topological, ecological, and autopoietic structure necessary to express this.

In this sense, hyper-knotting is not simply a mathematical technique or a modeling framework; it is a general ecology of worlds—an account of how systems constitute themselves, each other, and the contexts in which they live.

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# Appendix A: Notation and Symbol Tables

This appendix summarizes the notation used throughout the paper. The ecological hyperknot framework integrates autopoietic operations, relational topology, complement structures, functorial dynamics, ecological metrics, and organizational invariants. The following tables consolidate the formal vocabulary used in the main text.

## A.1 Components and Autopoietic Closures

| Symbol | Name | Meaning |
|---|---|---|
| $K_i$ | Autopoietic component | A locally self-producing closure; an eigenform satisfying $K_i = F_i(K_i)$. |
| $F_i$ | Local generator | The transformation that regenerates $K_i$'s identity. |
| $O$ | Operation (preprint) | A single autopoietic operation satisfying $O = F(O)$. |
| $X$ | Eigenform | A recursively stabilized entity in the sense of Kauffman. |
| $\mathrm{dom}(K_i)$ | Domain of $K_i$ | The set of operations constituting the closure. |

## A.2 Relational Structure

| Symbol | Name | Meaning |
|---|---|---|
| $R$ | Relational field | The complete set of couplings among components. |
| $R_{ij}$ | Relation between $K_i$ and $K_j$ | The structured coupling linking components. |
| $R_{ij}^G$ | Geometric adjacency | Topological or diagrammatic proximity. |
| $R_{ij}^C$ | Constraint relation | Limits, prohibitions, or stabilizing structures. |
| $R_{ij}^E$ | Ecological coupling | Co-regulation, mutual enabling, perturbation relations. |
| $w_{ij}$ | Constraint weight | Intensity or rigidity of a constraint. |

## A.3 Complement Structure

| Symbol | Name | Meaning |
|---|---|---|
| $C_H$ | Complement of $\mathcal{H}$ | The ecological context jointly generated by components and relations. |
| $\dim(C_H)$ | Contextual dimension | Degrees of freedom in the complement. |
| affordance($C_H$) | Affordance structure | Potential actions or relational possibilities. |
| redundancy($C_H$) | Redundancy | Slack or alternative pathways for co-regulation. |

## A.4 Hyper-Knot Structure and Dynamics

| Symbol | Name | Meaning |
|---|---|---|
| $\mathcal{H}$ | Ecological hyper-knot | The full structure $(K_i; R; C_H; F; I_H)$. |
| $\Phi$ | Hyper-knot functor | Global transformation rule acting on $\mathcal{H}$. |
| $F$ | Local/global functor | The transformation functor governing component or global change. |
| $F(\mathcal{H})$ | Functorial update | The next state of the hyper-knot under transformation. |
| $\Phi^t$ | Iterated functor application | Recursive application of $\Phi$ over time. |

## A.5 Hyper-Reidemeister Transformations

| Move | Name | Meaning |
|---|---|---|
| HR1 | Local expansion–contraction | Local closure reshaping without altering autopoietic identity. |
| HR2 | Internal reconfiguration | Addition/removal of paired operations preserving closure. |
| HR3 | Redistribution of internal tension | Rebalancing dependency structures preserving invariants. |
| HR4 | Modulation of relational couplings | Adjusting the strength or form of $R_{ij}$. |
| HR5 | Redistribution of ecological dependencies | Reassigning functional dependencies while conserving total load. |
| HR6 | Global organizational reconfiguration | Large-scale ecological reordering preserving $I_H$. |

## A.6 Ecological Metrics

| Symbol | Name | Meaning |
|---|---|---|
| $E_d$ | Ecological degree | Total ecological dependency load: $E_d = \sum \lvert R_{ij}^E \rvert$. |
| $T_R$ | Relational tension | Weighted relational constraint: $T_R = \sum w_{ij} \lvert R_{ij}^C \rvert$. |
| $\Gamma_C$ | Complement capacity | Contextual richness and flexibility of $C_H$. |
| $\kappa_F$ | Functorial curvature | Deviation induced by transformation: $\lVert F(\mathcal{H}) - \mathcal{H} \rVert$. |
| $S_{\mathcal{H}}$ | Stability | Balance of complement capacity vs. tension: $\Gamma_C/(E_d + T_R)$. |
| $R_{\mathcal{H}}$ | Resilience | Ability to return to invariants: $\Gamma_C/\kappa_F$. |

## A.7 Invariants and Identity

| Symbol | Name | Meaning |
|---|---|---|
| $I_H$ | Organizational invariants | Higher-order eigenforms preserved under HR1–HR6. |
| $\xi \in I_H$ | Hyper-eigenform | A recursively stable pattern: $\xi(\mathcal{H}) = \xi(\Phi(\mathcal{H}))$. |
| $F_i(K_i)$ | Local eigen-return | Operation regenerating $K_i$. |
| $\Phi(\mathcal{H})$ | Hyper-eigen-return | Transformation regenerating the organization. |

## A.8 Conceptual Summary

- **Components** ($K_i$) are local autopoietic eigenforms.
- **Relations** ($R_{ij}$) express entanglement, constraint, and co-regulation.
- **The complement** ($C_H$) is the emergent ecological context.
- **The functor** ($F$ or $\Phi$) encodes lawful transformation.
- **Invariants** ($I_H$) define identity at the ecological scale.
- **Metrics** ($E_d, T_R, \Gamma_C, \kappa_F, S_{\mathcal{H}}, R_{\mathcal{H}}$) quantify ecological structure and dynamical propensity.

This appendix consolidates the symbolic architecture through which ecological hyperknots express recursive identity, constraint topology, matter–meaning differentiation, and the transformation of worlds.

# Appendix B: TikZ Diagrams for Ecological Hyper-Knots

This appendix provides a set of conceptual TikZ diagrams illustrating the structure and transformations of ecological hyper-knots. These diagrams are not literal spatial representations but diagrammatic expressions of recursive, relational, and ecological organization.

## B.1 Basic Structure of an Ecological Hyper-Knot

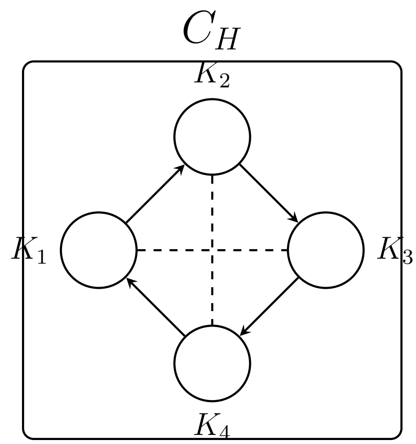


Figure 1: Basic diagram of an ecological hyper-knot: autopoietic components ($K_i$), relational field ($R$), and complement ($C_H$).

## B.2 Constraint and Ecological Couplings

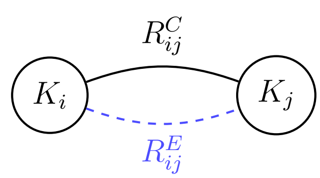


Figure 2: Constraint relation ($R_{ij}^C$) vs. ecological modulation ($R_{ij}^E$).

## B.3 Local Closure Transformations (HR1–HR3)

**HR1: Expansion–Contraction**

**HR2: Internal Reconfiguration**

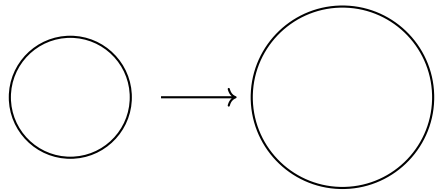

Figure 3: HR1: local change in closure size while preserving autopoietic identity.

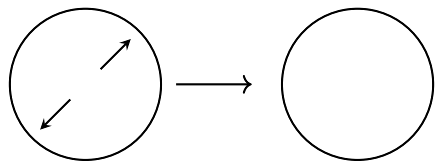

Figure 4: HR2: transient internal loop creation/removal within a closure.

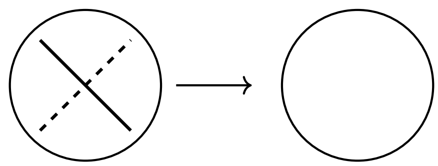

Figure 5: HR3: redistribution of internal tensions preserving invariants.

**HR3: Redistribution of Internal Tension**

## B.4 Ecological and Global Transformations (HR4–HR6)

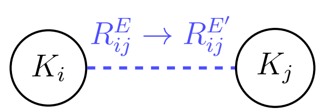


Figure 6: HR4: ecological modulation of coupling strength or form.

**HR4: Modulation of Relational Coupling**

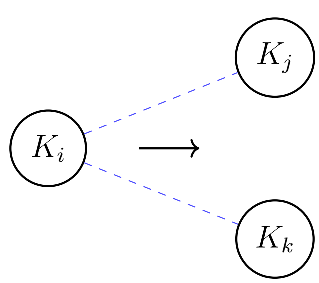


Figure 7: HR5: ecological dependency redistributed across multiple components.

**HR5: Redistribution of Ecological Dependency**

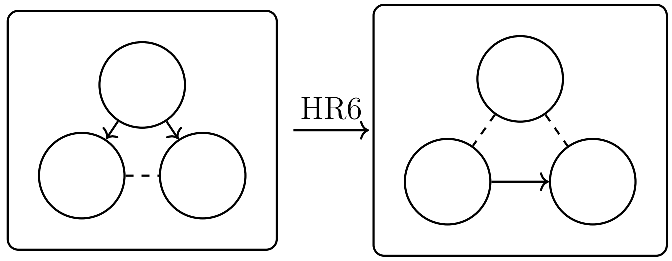


Figure 8: HR6: global reorganization preserving the hyper-knot's organizational invariants $I_H$.

**HR6: Global Organizational Reconfiguration**

## B.5 Hyper-Eigenform Illustration

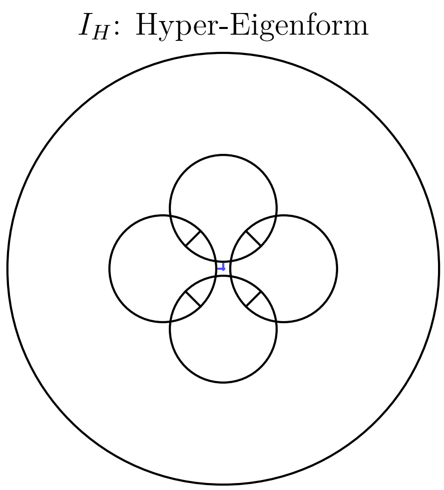


Figure 9: A hyper-eigenform: an organizational pattern invariant under HR1–HR6.

# Appendix C: Formal Derivations and Constructive Examples

This appendix provides formal derivations, proofs, and constructions that supplement the main text. These results demonstrate how autopoietic recursions, relational entanglements, complement structures, and ecological invariants arise within the hyper-knot formalism.

The emphasis is not on exhaustive mathematical completeness but on clarifying the logic and algebraic structure underlying ecological hyper-knots.

## C.1 From Autopoietic Re-entry to Hyper-Eigenforms

We begin with the fundamental autopoietic relation:

$$O = F(O).$$

In Section 1 we extended $O$ to a family of autopoietic closures $\{K_i\}$, each satisfying:

$$K_i = F_i(K_i).$$

These local recursions assemble into a global ecological structure

$$\mathcal{H} = (K; R; C_H; F; I_H).$$

The transformation functor $\Phi$ acts on the entire hyper-knot:

$$\Phi(\mathcal{H}) = (F(K_i); F(R); F(C_H); F; I_H).$$

If every $K_i$ is an eigenform and $R$ obeys hyper-Reidemeister constraints HR1–HR5, then the hyper-knot admits at least one non-trivial invariant $\xi \in I_H$.

*Proof.* Each $K_i$ satisfies $K_i = F_i(K_i)$, so the set of closures is stable under $F$. HR1–HR3 ensure that internal modifications preserve $K_i$'s invariants. HR4–HR5 ensure that relational transformations preserve the equivalence class of the relational topology. Thus the tuple $(K; R)$ is preserved up to equivalence under $\Phi$. Therefore the functional $\xi(K; R)$ is invariant under $\Phi$, defining a hyper-eigenform. □

This establishes the existence of ecological identity without assuming pre-given structure.

## C.2 Construction of a Minimal Ecological Hyper-Knot

A minimal non-trivial hyper-knot consists of two closures and three relations:

$$\mathcal{H}_{\min} = (K_1, K_2; R_{12}, R_{21}, R_{11}; C_H; F; I_H).$$

A simple construction:

$$K_1 = F_1(K_1), \qquad K_2 = F_2(K_2).$$

Let the relational structure satisfy:

$$R_{12}^E = g(K_1, K_2), \qquad R_{21}^E = g(K_2, K_1),$$

$$R_{11}^C = \text{self-maintenance constraint.}$$

Define the complement as:

$$C_H = \text{closure of } \{K_1, K_2, R_{12}, R_{21}\}.$$

Under the global functor:

$$\Phi(\mathcal{H}_{\min}) = (F_1(K_1), F_2(K_2); F(R_{12}), F(R_{21}), F(R_{11}); F(C_H); I_H).$$

$\mathcal{H}_{\min}$ is an ecological hyper-knot iff:

$$F(C_H) \simeq C_H.$$

*Proof.* Since $K_1$ and $K_2$ are eigenforms, they remain stable. The relations remain stable by HR4–HR5. Thus the complement, which is generated by $(K_i, R_{ij})$, is stable up to equivalence. □

This minimal example illustrates the ecological closure property.

## C.3 Complement Structure as a Fixed Point

The complement $C_H$ is not given; it is co-produced. Define the ecological complement operator:

$$\mathcal{C}(K; R) = \text{minimal context such that } (K; R) \text{ is viable.}$$

Then:

$$C_H = \mathcal{C}(K; R).$$

A key lemma:

If $F$ preserves viability then the complement satisfies:

$$C_H = \mathcal{C}(F(K); F(R)).$$

*Proof.* Viability is preserved iff relational topology remains within HR1–HR5 equivalence.

$$(K; R) \simeq (F(K); F(R)).$$

Thus the complement generated by either set is equivalent. □

Thus $C_H$ is a fixed point of the ecological closure operator.

## C.4 Invariance Under Hyper-Reidemeister Moves

We now show that the ecological identity invariants $I_H$ are preserved under all hyper-Reidemeister transformations.

For every $\xi \in I_H$ and for all HR1–HR6 moves,

$$\xi(\mathcal{H}) = \xi(\mathrm{HR}_k(\mathcal{H})).$$

*Proof.* **HR1–HR3:** Internal modifications of $K_i$ preserve $F_i(K_i)$ and thus any invariant functional of closure-level structure.

**HR4–HR5:** Relational re-weightings and redistributions preserve the equivalence class of the relational topology, so $\xi$ remains unchanged.

**HR6:** By definition, HR6 preserves $I_H$, so invariance follows immediately. □

Identity is therefore transformation-stable.

## C.5 Worked Example: Cognitive Hyper-Knot

Let $(P, A, S)$ denote:

- $P$: perceptual subsystem,
- $A$: action subsystem,
- $S$: internal evaluative subsystem.

Define closures:

$$K_P = F_P(K_P), \quad K_A = F_A(K_A), \quad K_S = F_S(K_S).$$

Ecological relations:

$$R^E_{PA} = h(P, A), \quad R^E_{AS} = h(A, S), \quad R^E_{SP} = h(S, P).$$

Complement $C_H$ corresponds to affordance fields.

Let:

$$\xi = \text{loop integral of } (P \to A \to S \to P).$$

$\xi$ is invariant under learning-driven HR4 and HR5 transformations.

*Proof.* Learning modifies couplings:

$$R^E_{ij} \to R^{E'}_{ij}.$$

But the loop topology remains equivalent, so the loop integral is preserved. □

Thus cognitive identity corresponds to a hyper-eigenform.

## C.6 Example of Matter–Meaning Co-Differentiation

Let:

$$\text{Matter} = \{\text{constraints generated by } R^C, C_H\},$$

$$\text{Meaning} = \{\text{modulations generated by } R^E, F\}.$$

Consider a perturbation $p$ inducing:

$$R^E_{ij} \to R^{E'}_{ij}.$$

Matter stabilizes if:

$$C_H \simeq F(C_H).$$

Meaning emerges through:

$$\Delta_F(\mathcal{H}) = F(\mathcal{H}) - \mathcal{H}.$$

If $\Delta_F$ is non-zero but $I_H$ is preserved, matter and meaning co-differentiate.

*Proof.* Non-zero $\Delta_F$ implies interpretive modulation. Preserved $C_H$ implies stability of constraints. Thus both roles co-emerge. □

This provides a worked example of Section 6.

## C.7 Summary

This appendix has shown:

- how hyper-eigenforms arise from multi-component recursive re-entry,
- how complements function as ecological fixed points,
- how HR-transformations preserve organizational invariants,
- minimal constructions of hyper-knots,
- examples demonstrating cognitive and matter–meaning applications.

These formal and constructive elements support the conceptual edifice built in the main text, grounding ecological hyper-knots in rigorous mathematical reasoning while preserving their intrinsically relational and non-essentialist character.

# Appendix D: Extended Worked Examples

This appendix provides extended worked examples that illustrate the operation of ecological hyper-knots across cognitive, social, ecological, and mathematical domains. These examples expand upon the applied discussion in Section 8 and the formal constructions in Appendix C, demonstrating how hyper-knot structures manifest in concrete systems.

The aim is not empirical detail but conceptual precision: each example shows how autopoietic closure, relational entanglement, complement generation, and hyper-eigenform identity can be instantiated in specific cases.

## D.1 Cognitive Example: Attention–Perception–Action Loop

Consider a simplified cognitive system composed of three autopoietic subsystems:

$$K_P \text{ (perception)}, \qquad K_A \text{ (action)}, \qquad K_T \text{ (attentional tuning)}.$$

Each subsystem satisfies:

$$K_i = F_i(K_i).$$

Relations are defined as:

$$R^E_{PA} = h_1(P, A) \quad \text{(sensorimotor coupling)},$$

$$R^E_{AT} = h_2(A, T) \quad \text{(attentional modulation)},$$

$$R^E_{TP} = h_3(T, P) \quad \text{(salience modulation)}.$$

The complement $C_H$ corresponds to:

$$C_H = \text{set of possible affordances in the environment.}$$

The loop integral functional:

$$\xi = \oint (P \to A \to T \to P)$$

is invariant under HR4 and HR5 transformations induced by learning.

Learning-driven modulation of sensorimotor dependencies preserves cognitive identity iff the loop integral $\xi$ remains invariant.

*Proof.* Learning modifies $R^E$, but as long as the topology of the loop is unchanged, $\xi$ remains invariant. Thus organizational identity is conserved. □

This demonstrates that cognitive identity is a hyper-eigenform stabilizing a recursive sensorimotor-attentional ecology.

## D.2 Social Example: Norm-Regulated Collective

Consider a small social system consisting of three agents $(A_1, A_2, A_3)$ engaged in a shared practice. Let:

$$K_i = \text{autopoietic schema governing agent } A_i.$$

Define relational couplings:

$$R_{ij}^E = \text{mutual expectation or coordination},$$

$$R_{ij}^C = \text{normative or institutional constraint}.$$

Let the complement $C_H$ consist of:

$$C_H = \{\text{shared language, sanctioned actions, material resources, institutional norms}\}.$$

Let the organizational invariant be:

$$\xi = \text{preservation of a coordinated practice } \Pi.$$

Suppose a perturbation modifies expectations:

$$R_{12}^E \to R_{12}^{E'}, \quad R_{23}^E \to R_{23}^{E'}.$$

The collective identity (the practice $\Pi$) persists under significant relational change if and only if HR6 is satisfied with respect to $\xi$.

*Proof.* Practice identity is encoded in $\xi$. HR4–HR5 modulate relations, but HR6 ensures that the overall practice topology remains invariant up to ecological equivalence. Thus $\Pi$ is preserved. □

This shows how social groups maintain identity through changing norms, relations, and expectations.

## D.3 Ecological Example: Predator–Prey–Environment Hyper-Knot

Let:

$$K_R = \text{rabbit autopoietic organization},$$

$$K_F = \text{fox autopoietic organization.}$$

Define ecological couplings:

$$R^E_{FR} = \text{predation interaction,}$$

$$R^E_{RF} = \text{avoidance and vigilance,}$$

$$R^C_{ii} = \text{metabolic constraint of species } i.$$

The complement captures environmental features:

$$C_H = \text{vegetation density, hiding places, terrain, shelters.}$$

An organizational invariant is:

$$\xi = \text{stable coexistence regime.}$$

If conditions shift such that:

$$C_H \to C'_H \quad \text{(e.g., deforestation, drought),}$$

and the system reorganizes via HR6 without violating $\xi$, the ecology persists.

Ecological resilience corresponds to the preservation of $\xi$ under HR6-induced complement modifications.

*Proof.* A system is resilient when $F(C_H) \simeq C_H$ and relations remain viable. HR6 allows global restructuring while preserving $\xi$. Thus coexistence remains possible despite large changes. □

This demonstrates ecological persistence as a hyper-eigenform.

## D.4 Mathematical Example: Category of Recursive Diagrammatic Transformations

Let $\mathcal{C}$ be a category whose objects are closure–relation pairs $(K; R)$ and whose morphisms are hyper-Reidemeister transformations HR1–HR6:

$$\mathrm{Obj}(\mathcal{C}) = \{(K; R)\}, \quad \mathrm{Hom}(\mathcal{C}) = \{\mathrm{HR}_k\}.$$

Define a functor:

$$\Phi : \mathcal{C} \to \mathcal{C}$$

which:

- acts as $F_i$ on components,
- acts as relational reconfiguration on $R$,
- induces complement reconstruction $C_H \mapsto F(C_H)$.

Let:

$$\xi : \mathcal{C} \to \mathcal{I}$$

be a functor to a category of invariants.

If $\xi$ is natural with respect to $\Phi$ and HR1–HR6, then the hyper-knot admits a hyper-eigenform.

*Proof.* Naturality implies:

$$\xi(\Phi(\mathcal{H})) = \xi(\mathcal{H})$$

for all morphisms. Thus $\xi$ is constant on $\Phi$-orbits. Therefore $\xi$ is an invariant and defines a hyper-eigenform. □

This demonstrates the categorical structure behind ecological hyper-knots.

## D.5 Emergent Space and Time Example

Define:

$$\text{Space} = \text{topology of constraint: } \Sigma = (K; R^C; C_H),$$

$$\text{Time} = \text{order of transformation: } \tau = \{\Phi^t\}_{t \in \mathbb{N}}.$$

Consider a system experiencing repeated HR4 modulations (ecological couplings) and HR6 reconfigurations (global ecological shifts).

If the metric:

$$\kappa_F = \|F(\mathcal{H}) - \mathcal{H}\|$$

remains bounded while:

$$\mathrm{diam}(\Sigma) \to \infty,$$

then emergent space expands while time preserves identity.

*Proof.* Bounded curvature implies stable recursion. Expanding constraint topology implies increasing spatial differentiation. Identity preserved by bounded $\kappa_F$. □

This models emergent spatiotemporal organization in ecological networks.

## D.6 Summary

These extended examples demonstrate how ecological hyper-knots illuminate:

- cognitive sense-making as recursive ecological re-entry,
- social normativity as relational invariance,
- ecological resilience as complement and coupling stability,
- mathematical structures as categorical hyper-eigenforms,
- emergent space and time as manifestations of constraint and transformation.

Together these examples support the central thesis of this manuscript: *ecological hyper-knots provide a general, non-essentialist, recursive, and topological account of how worlds constitute themselves through relational dynamics.*

# Appendix E: Category-Theoretic Foundations of Ecological Hyper-Knots

This appendix develops the categorical underpinnings of ecological hyper-knots. Category theory provides a natural framework for expressing recursive organization, functorial transformation, complement generation, and invariance under HR1–HR6. The aim is not to reduce ecological hyper-knots to category theory, but to show how their structure can be interpreted and clarified in categorical terms.

## E.1 Objects, Morphisms, and Ecological Data

Let $\mathcal{K}$ be a category whose objects are autopoietic closures

$$\mathrm{Obj}(\mathcal{K}) = \{K_i \mid K_i = F_i(K_i)\},$$

and whose morphisms represent allowed internal transformations:

$$\mathrm{Hom}(K_i, K_i) = \{\text{HR1, HR2, HR3 transformations}\}.$$

Define a second category $\mathcal{R}$ whose objects are relational structures:

$$\mathrm{Obj}(\mathcal{R}) = \{R_{ij}\},$$

and whose morphisms encode relational transformations:

$$\mathrm{Hom}(R, R') = \{\text{HR4, HR5 transformations}\}.$$

The global ecological configuration is represented in a product category:

$$\mathcal{E} = \mathcal{K} \times \mathcal{R}.$$

Objects of $\mathcal{E}$ are pairs:

$$(K, R),$$

and morphisms correspond to paired HR-moves.

## E.2 The Complement as a Functor

Define the complement functor:

$$\mathcal{C} : \mathcal{E} \to \mathcal{C}omp,$$

where $\mathcal{C}omp$ is a category of ecological contexts.

Given an ecological object $(K, R)$,

$$\mathcal{C}(K, R) = C_H.$$

Given a morphism $(\alpha, \rho)$ corresponding to a set of HR-moves,

$$\mathcal{C}(\alpha, \rho) : C_H \to C'_H$$

is the induced transformation on the complement.

If $(\alpha, \rho)$ consists only of HR1–HR5 moves, then $C_H \simeq C'_H$.

*Proof.* HR1–HR5 preserve the equivalence class of relational topology and closure identity. Thus the complement generated by $(K, R)$ is equivalent to the complement generated by $(K', R')$. □

Thus $C_H$ is a functorial construction respecting ecological equivalence.

## E.3 The Hyper-Knot Functor

Define the global hyper-knot category:

$$\mathcal{H}yp = \{\mathcal{H} = (K; R; C_H)\},$$

where $C_H$ is determined by $\mathcal{C}(K, R)$.

A morphism in $\mathcal{H}yp$ is an HR-transformation:

$$\mathrm{Hom}(\mathcal{H}, \mathcal{H}') = \{\text{HR1–HR6}\}.$$

Define the hyper-knot functor:

$$\Phi : \mathcal{H}yp \to \mathcal{H}yp$$

via:

$$\Phi(\mathcal{H}) = (F(K), F(R), F(C_H)).$$

$\Phi$ preserves ecological equivalence under HR1–HR5.

*Proof.* $F$ preserves local closure; HR1–HR3 preserve closure identity. HR4–HR5 preserve relational equivalence. Thus $\Phi(K, R)$ and $F(C_H)$ remain within the same equivalence class. □

HR6, by contrast, allows equivalence classes to reorganize while preserving invariants.

## E.4 Invariants as Natural Transformations

Let $\mathcal{I}$ be a category of invariants (hyper-eigenforms). Define:

$$\xi : \mathcal{H}yp \to \mathcal{I}$$

as a functor assigning invariants to hyper-knots.

A natural transformation $\eta : \xi \Rightarrow \xi$ satisfies

$$\eta_{\mathcal{H}} : \xi(\mathcal{H}) \to \xi(\Phi(\mathcal{H})).$$

If $\eta$ is the identity natural transformation, then

$$\xi(\mathcal{H}) = \xi(\Phi(\mathcal{H})),$$

i.e. $\xi$ is a hyper-eigenform.

*Proof.* If $\eta = \mathrm{id}$, then each component square commutes trivially. Thus $\xi$ is constant along functorial orbits of $\Phi$. □

Thus hyper-eigenforms arise categorically as natural invariants of ecological recursion.

## E.5 HR-Moves as Generators of a Rewriting System

The hyper-Reidemeister moves HR1–HR6 generate a rewriting system:

$$(K, R, C_H) \rightsquigarrow (K', R', C'_H).$$

Let $\mathsf{HR}$ be the monoid generated by HR-moves:

$$\mathsf{HR} = \langle \mathrm{HR1}, \ldots, \mathrm{HR6} \rangle.$$

There is a monoid action:

$$\mathsf{HR} \curvearrowright \mathcal{H}yp,$$

which is compatible with $\Phi$.

HR1–HR5 act freely on equivalence classes of relational topology. HR6 acts transitively on equivalence classes of complements.

*Proof.* HR1–HR5 alter internal and relational structures without changing ecological class. HR6 allows complete reconfiguration of complement topologies within the domain of invariants. Thus the action structure follows. □

This rewriting perspective reveals the computational structure of hyper-knot dynamics.

## E.6 Factorization Systems and Ecological Decomposition

Ecological transformations decompose naturally into two parts:

$$\mathcal{H} \xrightarrow{\text{local}} (K', R) \quad \text{and} \quad (K', R) \xrightarrow{\text{global}} \mathcal{H}'.$$

This corresponds to a factorization system $(\mathcal{L}, \mathcal{G})$ where:

- $\mathcal{L}$ consists of HR1–HR3 (local closure-preserving maps),
- $\mathcal{G}$ consists of HR4–HR6 (global ecological re-shaping).

Every HR-transformation decomposes uniquely as:

$$\text{HR} = \mathcal{G} \circ \mathcal{L}.$$

*Proof.* Local closure modifications and global ecological restructurings operate on disjoint structural domains $(K)$ and $(R, C_H)$, respectively. Factorization follows from orthogonality. □

This decomposition clarifies how local autopoiesis and global ecology interact.

## E.7 Ecological Fixed Points and Endofunctor Dynamics

The hyper-knot functor $\Phi$ is an endofunctor on $\mathcal{H}yp$:

$$\Phi : \mathcal{H}yp \to \mathcal{H}yp.$$

A hyper-knot $\mathcal{H}$ is a fixed point iff:

$$\Phi(\mathcal{H}) \simeq \mathcal{H}.$$

A fixed point of $\Phi$ corresponds exactly to a hyper-eigenform.

*Proof.* Fixed point condition implies preservation of $I_H$. Conversely, preservation of $I_H$ implies functorial stability. Thus fixed points and hyper-eigenforms coincide. □

This shows that identity in ecological hyper-knots is a categorical fixed point of ecological recursion.

## E.8 Summary

Category theory reveals that ecological hyper-knots can be understood as:

- objects in a recursive relational category,
- acted upon by HR-generated morphisms,
- structured by complement functors,
- transformed by a global endofunctor $\Phi$,
- stabilized by natural invariants (hyper-eigenforms),
- decomposed via factorization into local and global transformations.

These categorical foundations deepen the formal understanding of ecological hyper-knots while preserving their intrinsically relational, non-essentialist, and autopoietic character.