Mathematical Theory Generated with the Vacuum.Net Theory

From the eternal round to closure as oscillation: twenty-two rungs, checked in cubical Agda and by computation

J. Konstapel
Constable Research, Leiden, The Netherlands · Version 3.0 · 16 November 2026

Preface

This book grows mathematics from below. It starts with movement. In the beginning there is a direction, <- . ->: a point with a way out and a way back. Without movement nothing happens, so the movement has no start: it is an eternal round. The Foundations of the theory begin with a first distinction [1]. This version follows the author's correction: the first is the round, and a distinction is a still picture of it. It then lets the net grow, one rung at a time. At each rung it asks one question. Which field of mathematics already has a name for what the net has just made?

The order of the book is the order of the net, not the order of history. History found these structures in no fixed order. A study of the theory against the history of mathematics found no significant relation between the order of the theory's chain and the dates of the results it uses [2]. Here the order comes from the net. The names come from the fields.

Working from below solves a problem that working from above could not. When a named concept is taken first and then written as trits, its trit word depends on how it is written down [3]. When the net generates the structure first, the word comes first. The name is added afterwards.

How to read this book

Each chapter is one rung. It has four parts: what the net does, which field names it, what is established, and what is open. Every statement in the third part carries one of four labels.

- **Proved in Agda.** A theorem in one of the cubical Agda files of this project, accepted by the type checker under --safe, without postulates [15]. The Agda name is given.
- **Checked by computation.** A finite fact recomputed in Python, with the script named.
- **Known theorem.** An established result of mathematics, cited to its source. It is not reproved here.
- **Reading.** An interpretation that links the net to the field. It is not a theorem. Readings are listed under their own heading, after what is established.

The goal is derivation. The book tries to derive mathematics step by step from the net. Where the derivation stops, it names the smallest assumption that is still needed. These stopping points are results in their own right: they show exactly what the net does not yet give.

A word on method. Matching a rung to a field shows where mathematics has already put a name on a structure of the net. It does not show that the theory is true. The book claims only this: the structures arise in this order from one rule, and each already has a name. Where a rung needs a hypothesis, the hypothesis is quoted from the theory with its number. The book adds two of its own at Rung 11 (H1, H2). At Rung 14 it takes the real numbers as an input; the chapter "Beyond Rung 21" builds the real numbers of an interval from the net, up to Bishop's equality.

The Agda files are VacuumNetValue.agda, VacuumNetValue2.agda, VacuumNetArith.agda, VacuumNetLimit.agda, VacuumNetComma.agda, VacuumNetPhase.agda, VacuumNetRound.agda, VacuumNetHoTT.agda, VacuumNetFoundation.agda, VacuumNetDictionary.agda, VacuumNetLadder.agda, VacuumNetLadder2.agda, VacuumNetLadder3.agda, VacuumNetPairRing.agda, VacuumNetLadderPath.agda, VacuumNetBT0.agda,

VacuumNetEigenform.agda, VacuumNetTrefoil.agda, VacuumNetStream.agda, VacuumNetAverage.agda, VacuumNetProduct.agda, VacuumNetDigitWinding.agda, VacuumNetOscillator.agda, VacuumNetHelix.agda, VacuumNetSLNode.agda, VacuumNetPSLNode.agda, VacuumNetOpenPair.agda and VacuumNetHopf.agda. They type-check with Agda 2.8.0 and the cubical library v0.9 [7, 14].

Rung	The net	Field that names it	Status
0	Movement: the eternal round	Circle S^1, connectedness	Agda + library
1	Three directions	windings −1, 0, +1 in $\mathbb{Z}$; Boolean algebra as a still picture	Agda
2	The return (N1)	Circle, integers as windings	Agda + library
3	Direction (trit rule)	$\mathbb{Z}_3$, F_3, Kleene logic	Agda
4	Words (T12)	Balanced ternary	Agda
5	Window (T2–T4)	Fox colouring, AG(n,3), cap sets	Agda + computation
6	Three closures (T7)	$\mathbb{Z}_3$ with mirror, S_3	Agda
7	Closing everywhere (T8–T10)	Linear cellular automata, circulants, $d^3 = 0$	Agda + computation + proof
8	Seams (T11)	Sheaves (reading)	Agda
9	Closed without a start (T13)	Necklaces, Lyndon words	Computation
10	The plane (T14)	Eisenstein integers	Computation
11	Mirror and ring in one node	Modular group, braids	Agda + computation + cited
12	Two ladders (T15–T18)	Continued fractions, the comma	Computation
13	Into three dimensions (T19–T22)	Knots, SU(2), Hopf	Agda (maps to Rung 11 and S^1); complement cited; + embedding for the rest
14	Coupling (T23)	Analysis, Laplace's method	Computation; + energy
15	Free path (T24)	Random walks, Pólya	Computation; + walk
16	Closure on itself	Clifford algebra; $d \circ d = 0$; $d^3 = 0$	Computation; + Dirac; Agda via 7
17	Two registers	Quaternions, spectral theory	Computation; + Maxwell
18	Measure	Dimensional analysis	Computation; + Larson
19	Loss has an address	Onsager, PSD matrices	Cited; + thermodynamics
20	Numbers without end	Signed-digit reals, computable analysis	Agda (average at finite depth), computation, cited; + infinite streams
21	Closure as oscillation	Difference equations, crystallographic restriction	Agda, computation, cited; + integer sum 0

Table 1. The twenty-two rungs. "+" names the hypothesis a rung adds to the net.

Before Euclid, and after

Many have tried to order mathematics. Table 0 sorts the attempts by the form of their order. The ladder of this book is a list in the order of generation: each item comes from the previous one by one rule, and the place in the list is the depth.

Ordering	Form	Examples
From axioms	Tree with one root	Euclid; Frege and Peano; Russell and Whitehead; Zermelo
From structures	Lattice of mother structures	Bourbaki
From relations	Network of arrows	Eilenberg and Mac Lane; Grothendieck; Lawvere
From paths	Space of types	Martin-Löf; Voevodsky (homotopy type theory)
By strength or symmetry	Ranking	Klein (Erlangen programme); reverse mathematics
By subject	Catalogue	Mathematics Subject Classification
By generation	List, one rule from item to item	Pāṇini and Piṅgala in method; this book

Table 0. Ways of ordering mathematics. Sources: Euclid [74], Russell and Whitehead [75], Bourbaki [76], Klein [77], Eilenberg and Mac Lane [78], Lawvere [79], Grothendieck [80], Martin-Löf [81], reverse mathematics [50], MSC [56].

The oldest generative work is Indian. Pāṇini's grammar, around the fourth century BCE, produces the words of Sanskrit from about four thousand short rules [67]. Computer science later saw a formal grammar in it [68]. Piṅgala, in the third or second century BCE, gives a procedure to list all metres of a given length as words of short and long syllables. He also gives a rule to find the place of any pattern in that list [67, 69]. Brahmagupta, in 628, gives rules for three kinds of number: fortune, debt and zero [73]. These are the method, the address and the three values of this book, found independently of the Greek line of axioms.

Two lines run through the history. The Indian and Chinese line generates: rules produce words, and each word gets a place. The Greek and European line deduces: theorems follow from axioms, and later the whole is ordered into trees and lattices. Grothendieck stands between them. He looked at relations rather than objects. Through topos theory and Lawvere, that line contributes to homotopy type theory, the language of this book. Its main roots are Martin-Löf's type theory and Voevodsky's univalence [79, 80, 81]. The two lines were never separate. Indian numerals with zero reached Europe through the Islamic world, and the Yijing reached Leibniz through the Jesuits [70]. Whether the Kerala series reached Europe is not established [71].

Each chapter names its precursors where they are known.

Rung 0. Movement: the eternal round

The net at this rung

In the beginning there is movement. It has a direction: <- . ->, a point with a way out and a way back. Nobody was there at the start, but without movement nothing happens. So the movement never stops and never began. It is an eternal round.

The field that names it

Algebraic topology and homotopy type theory (MSC 55, 03). The circle S^1 is exactly this: one point, base, and one movement, loop, that returns to it [4]. The circle is connected: every place on it is reached from the point. The type fixes one point, base, as a reference; it has no time in it.

Precursors. Kauffman reads the imaginary unit as an oscillation, a value that alternates in time [82]. Spencer-Brown's re-entry gives such an oscillating value [10]. In this book the oscillation is not reached at the end but given at the start.

Martin-Löf, who founded the type theory beneath homotopy type theory, traced the order of conceptual priority backwards. From mathematics he went to proposition and truth, then to judgement, and ended at act and object. Rightness belongs first to the act, and an object is actual if an act with it has been performed [83]. Read this way, this book starts at the same place: the movement is the act, and a point or a trit is what the act leaves when seen still.

What is established

- Proved in Agda: a point without movement has only standing still; every return of the unit type is trivial (zonder-beweging, VacuumNetRound).
- Proved in Agda: every place on the round is reached from the point, base (geen-begin, VacuumNetRound, from the library's isConnectedS^1).

Reading

- the movement has no start and is eternal. S^1 has no time in it and fixes a base point as reference; that this point is no beginning is a reading, not a theorem.

Open

The Foundations 9.2 begin with a first distinction [1]. This rung follows the author's correction of 16 November 2026. The written theory has not yet been brought into line with it.

Rung 1. Three directions

The net at this rung

Seen at one place, the round shows three directions: back (<-), the point (.), and out (->). These are the three trits T, 0 and 1. A trit is not a value but a still picture of the movement.

The field that names it

Group theory and logic (MSC 20, 03, 06). On the circle the three directions are the reversed loop, the constant path and the loop. Their windings are −1, 0 and +1. A distinction in the sense of Spencer-Brown cuts a space into two sides [10]. It is the still picture with the middle left out. Its algebra is Boolean.

Precursors. Spencer-Brown begins Laws of Form with the instruction to draw a distinction [10]. In this version his mark is not the start but a view of the round.

What is established

- Proved in Agda: the three directions have windings −1, 0 and +1, and out is not back (w-terug, w-punt, w-heen, heen≠terug, VacuumNetRound).
- Proved in Agda: the two sides of a mark do not coincide, ¬ (true ≡ false) (markering, VacuumNetLadder).
- Known theorem: the primary algebra of Laws of Form is a Boolean algebra [10].

Reading

- these three directions are the trits T, 0 and 1 of the theory.

Open

A still picture alone has no motion. The next rung shows why the motion must be a round.

Rung 2. The return

The net at this rung

The theory states N1: there is one strand, and every distinction is a loop [1]. A movement that goes out and comes back along the same way leaves nothing behind. Only a round that comes back by another way leaves a trace. That is why the eternal movement is a round.

The field that names it

Algebraic topology and group theory (MSC 55, 20). The windings of the circle are the integers. Licata and Shulman proved this inside homotopy type theory [5].

What is established

- Proved in Agda: out and back along the same way cancels, $\mathrm{loop} \cdot \mathrm{loop}^{-1} = \mathrm{refl}$ (heen-en-terug, VacuumNetRound).
- Proved in Agda: one round is not standing still, $\neg (\mathrm{loop} \equiv \mathrm{refl})$ (rondgang-blijft, VacuumNetRound; also onderscheid, VacuumNetFoundation).
- Proved in Agda: every number of rounds is kept; n rounds equal m rounds only if n = m (rondes-verschillen, VacuumNetRound).
- Known theorem: the loops of the circle are the integers, $\Omega S^1 \simeq \mathbb{Z}$ [5, 14].

Reading

- only a round leaves a trace, so the eternal movement is a round.

Open

The loops are shown to be the trit words directly (Rung 4, VacuumNetHelix), without the library theorem $\Omega S^1 \simeq \mathbb{Z}$. The integers still serve as a measuring rod inside that proof.

Rung 3. Direction

The net at this rung

At a return, the trit rule of the theory reads one of three answers [1]. The passage leaves as the first passage left (+1, written R), leaves the way the first passage came in (−1, written L), or does neither (0, written O). These are the three trits, read as digits of the trinity (T1).

The field that names it

Group theory, field theory and logic (MSC 20, 12, 03). The three trits add as the cyclic group of order three, $\mathbb{Z}_3$. With multiplication they form the field with three elements. As truth values with an open middle they form Kleene's strong three-valued logic [11].

Precursors. Brahmagupta (628) gave rules for fortune (dhana, +), debt (ṛṇa, −) and zero (śūnya): a debt taken from zero is a fortune, a fortune taken from zero is a debt [73]. That is the mirror of T12.1. Schemes with more than two answers are older still: the Jain sevenfold predication and the Buddhist four corners (catuṣkoṭi), used by Nāgārjuna. Reading them as many-valued logics is debated [72].

What is established

- Proved in Agda: the addition ⊕ of trits is commutative and associative (⊕-comm, ⊕-assoc, VacuumNetLadder).
- Known theorem: $\mathbb{Z}_3$ is the unique group of order three.

Open

Foundations III.1 says: "The trits are therefore not directions that compose but digits that add." Its argument is aimed at one reading: "the same as the first" as neutral and "the reverse" as its own inverse, a structure of two values. This book does not use that reading. III.2 reads the trits as winding (1), rest (0) and counterwinding (T). The Agda map ℓ follows III.2: R is the loop, O is the constant path, L is the loop reversed.

For whole words the two readings meet. The map ρ sends a word to the winding of its value, and composing windings adds integers [5]. The carry of T1 is the scale step ×3 between places. For single trits they meet through the carry. ℓ R · ℓ R · ℓ R is three windings, while R ⊕ R ⊕ R is O in $\mathbb{Z}_3$. The digit is the winding modulo three; the rest, (s − d)/3, passes to the next scale (T1). Agda proves the digit part in general. The balanced residue modulo three is additive on $\mathbb{Z}$ (rest3-+). With the winding homomorphism of the library this gives, for any two loops p and q: the digit of p · q is the digit of p ⊕ the digit of q (cijfer-hom). For single trits the digit of ℓ d is d (cijfer-van-ℓ). So composing loops and adding digits agree for all loops (VacuumNetDigitWinding). The carry (s − d)/3 to the next scale is stated, not proved. This settles the question for the III.2 reading; III.1's objection concerns a different, two-valued reading. The physical reading of the trit rule stays open (IX.5).

Rung 4. Words

The net at this rung

Trits in a row form a word. Every integer has exactly one balanced-ternary word (T12) [1]. Mirroring R and L gives the negative (T12.1).

The field that names it

Number theory and computer science (MSC 11, 68). Knuth analyses balanced-ternary arithmetic and its history [13].

Precursors. Piṅgala listed all words of two kinds of syllable and gave rules to find the place of a given word in the list: a unique address, with two values instead of three [67, 69]. Pāṇini generated words from rules [68]. Leibniz wrote numbers as words of 0 and 1 (1679, published 1703). Only later, through Bouvet, did he see the Yijing hexagrams as the same [70].

What is established

- Proved in Agda: the words are exactly the windings of the loop, BT ≃ ΩS¹, and by univalence BT ≡ ΩS¹ (VacuumNetHoTT).
- Proved in Agda: the loop of a word is the winding of its value, ρ w ≡ intLoop (val w) (ρ≡intLoop, VacuumNetHoTT). The digit at each place is added by path composition; the rest of the word is traversed three times (×3), which is the carry to the next place.
- Proved in Agda, without the library theorem ΩS¹ ≃ ℤ and without the library fact that ΩS¹ is a set: the loops of the circle are the trit words (ΩS¹≃BT′). Over the base point lie the words; one turn around the loop is one step sucBT, an equivalence with inverse predBT (helixBT). A loop is sent to the word that ε becomes after transport (encodeBT). Going round the loop adds R with carry (encode-loop). This is the Licata–Shulman argument [5] with the net's own words as the fibre; decoding goes via intLoop ∘ val. Since the words form a set, it follows that the loops form a set too, in VacuumNetHelix.
- Proved in Agda: the set-truncation of the word type is not needed. The type BT^{o}, with only ε, _◁_ and stand, is a set by itself (isSetBT^{o}), because every word reads to a list with no O on top and reads back to itself (terug). It is equivalent to the truncated type and to ℤ (BT^{o}≃BT, BT^{o}≃ℤ, VacuumNetBT0). So the net's own rule, that a lone O on top falls away, is enough to make the words a set equal to ℤ.
- Proved in Agda: mirroring a word reverses its winding, ρ (mirror w) ≡ sym (ρ w) (spiegeling, VacuumNetFoundation).
- Checked by computation with the Agda value function val, least significant trit first: thirteen is R R R, three is O R, minus thirteen is L L L.

Open

The integers are still used as a measuring rod (val, toBT, intLoop). A proof with the words alone, without ℤ, is not done.

Rung 5. The window

The net at this rung

Three successive trits form a window. A window closes when its sum is zero modulo three (T2). Exactly 9 of the 27 windows close (T3). A window closes exactly when its three trits are all equal or all different (T4) [1].

The field that names it

Knot theory, combinatorics and finite geometry (MSC 57, 05, 51). The same rule appears three times in mathematics. In knot theory it is Fox's three-colouring: at a crossing the arcs carry one colour or all three [12]. In finite geometry, three points of the space F_3^n lie on a line exactly when they sum to zero. In additive combinatorics, the cap set problem asks for the largest set in F_3^n with no three distinct points that sum to zero, that is, with no line. Ellenberg and Gijswijt proved in 2017 that such a set has at most about 2.756^n points [16].

What is established

- Proved in Agda: a window closes exactly when its trits are all equal or all different, for all 27 windows (venster, VacuumNetLadder).
- Checked by computation: the closed triples in F_3^2 are its 12 lines, and in F_3^3 its 117 lines (bottomup_checks.py).
- Known theorem: the cap set bound of Ellenberg and Gijswijt [16].

Reading

- the rule is identical in the three fields, as a condition. That the fields therefore share more than this condition is a reading.

Open

The cap set problem asks for the largest set without a closed window of three distinct trits. The theory has no answer to it yet.

Rung 6. Three closures

The net at this rung

Every closed word reduces to nothing by removing three kinds of closure (T7) [1]. A standing is a single 0. A meeting is a trit and its mirror. A ring is three equal trits. Beside them stands the open pair: two equal nonzero trits. It does not close. It keeps a nonzero digit and may carry: 1 + 1 = 1T (T7.1).

The field that names it

Group theory (MSC 20). In $\mathbb{Z}_3$ the standing is the zero and the meeting is the inverse. The ring is the fact that every element has order dividing three. The open pair gives the mirror of x: x + x = −x. The mirror itself has order two. Rotation of order three and mirror of order two together are the symmetries of the trinity, the group S_3.

What is established

- Proved in Agda: x ⊕ spiegel x ≡ O (ontmoeting$_3$); x ⊕ x ⊕ x ≡ O (ring$_3$); x ⊕ x ≡ spiegel x (open-paar$_3$); spiegel (spiegel x) ≡ x (spiegel^2), all in VacuumNetLadder.
- Proved in Agda, on the loop: forward then back is no winding (meeting); a digit a scale up is three windings (ring); one step with carry is one winding more (openPair), in VacuumNetHoTT.
- Known theorem: $\mathbb{Z}_3$ with its mirror forms S_3.

Open

The equation x + x = −x is the arithmetic form of the open pair. That this is what T7.1 means is a reading.

Rung 7. Closing everywhere

The net at this rung

A word may close in every window. The sum of the next window is the sum of this one, plus the trit that enters, minus the trit that leaves (T8) [1]. So a word closes everywhere exactly when its first window closes and it repeats with period three. For every length $n \geq 3$ there are exactly nine such words. On a ring the count is nine when three divides n, and three otherwise (T9).

The theory then rewrites a ring. Each trit becomes minus the sum of its two neighbours (T10). Since $2 \equiv -1$ modulo three, each trit becomes the mean of its neighbours. A ring stays fixed exactly when every window closes. For lengths 3 to 10, rings that are not fixed end in cycles, except 24 rings of length 4 and 24 of length 8, which come to rest.

The field that names it

Linear algebra over F_3, symbolic dynamics and cellular automata (MSC 15, 37B, 68Q80). The rewrite is a linear cellular automaton over $\mathbb{Z}_3$. Martin, Odlyzko and Wolfram worked out the algebra of such automata [25]. Its fixed points form the kernel of a circulant matrix with rows (1, 1, 1) [26]. The kernel has dimension two when three divides n, and one otherwise. The reason lies in the polynomial $x^2 + x + 1$, which over F_3 equals $(x - 1)^2$. The words that close everywhere form a shift of finite type with no freedom left: zero entropy [27].

What is established

- Proved in Agda: the window sum moves by the new trit minus the old one (propagatie); two closed windows in a row force the trit to repeat after three (periode); a trit is fixed by the rewrite exactly when its window closes (vast-punt), all in VacuumNetLadder2.
- Checked by computation: nine words for n = 3 to 9 (upper_checks.py).
- Checked by computation: nine or three rings, and five or three rotation classes, for n = 3 to 11.
- Checked by computation: 24 rings come to rest for n = 4 and for n = 8, and none for the other n from 3 to 10.
- Checked by computation: for n = 12, 6,552 rings come to rest (rest12.py). This goes beyond the range of T10.
- Checked by computation: the fixed space has size 9 or 3 for n = 3 to 12.
- Checked by exact linear algebra: the count of rings that come to rest follows the rule below for every n from 3 to 48 (rest_rule.py). For n = 16 it is 24, for n = 36 it is 3,486,784,392.
- Proved in Agda: closing is a second difference. With $D\,t = t(i+1) - t(i)$ over $\mathbb{Z}_3$, D^2 of three trits is their window sum (venster-is-D^2). D^3 of four trits is the last minus the first (D^3-is-periode), both in VacuumNetLadder3. So a constant word has $D = 0$, a word that closes everywhere has $D^2 = 0$, and a word of period three has $D^3 = 0$. This joins T8 and T9 in one chain: $3 \subset 9 \subset 27$ rings when three divides n (ncomplex.py).
- Proved in Agda: among all 27 local linear rules $t(i) \mapsto a{\cdot}t(i-1) + b{\cdot}t(i) + c{\cdot}t(i+1)$ over $\mathbb{Z}_3$, exactly two leave a trit in place precisely when its window closes (enige-twee, VacuumNetEigenform). One is the rewrite of T10, $-(t(i-1) + t(i+1))$. The other is $t(i-1) - t(i) + t(i+1)$. The rewrite of T10 is the only one in which a trit does not read itself.
- Checked by computation: the same two rules, and no others, have as fixed points exactly the rings that close everywhere for all lengths 3 to 9 taken together (rule_search.py). For a single length this holds when three divides n; other lengths admit up to eight rules.

- Note: the 27-window result is close to a tautology. A rule (a, b, c) works exactly when (a, b − 1, c) is a nonzero multiple of (1, 1, 1). The real choice is to allow only linear rules of radius one. Among those, the rule is forced up to a scalar.
- Known theorem: the kernel of a circulant matrix has dimension equal to the degree of the greatest common divisor of its polynomial with $x^n - 1$ [26].

Open

The Agda proofs are local: they concern one window. The counts for all n are proved on paper and computed here, not formalised.

Rings that come to rest are not limited to lengths 4 and 8. They occur exactly when four divides n. The count is 3^(2·3^v) · F − F, with F the number of fixed rings and 3^v the power of three in n. The proof is short. Read a ring as a polynomial modulo $x^n - 1$ over F_3. The rewrite is multiplication by $-(x^2 + 1)/x$, and R − I by $-(x - 1)^2/x$. Over F_3 the factor $x^2 + 1$ is irreducible. Its roots are the fourth roots of one, so it divides $x^n - 1$ exactly when four divides n, with multiplicity 3^v. That factor is where the rewrite is nilpotent. On all other factors the rewrite is invertible [25, 26]. Coming to rest means $(R - I)R^m t = 0$ for some m. That kernel is the fixed space plus the $x^2 + 1$ part, of dimension 2·3^v. The count follows.

This corrects the statement of T10, which lists only lengths 4 and 8 because it was computed up to length 10.

Rung 8. Seams

The net at this rung

Two closed windows can be joined. Across the joint lie two new windows. The first closes exactly when the two first trits agree. The second closes exactly when the two last trits agree. So the seam closes only when both windows are the same (T11) [1]. Of the 81 ordered pairs, 9 seams close, 36 have one open window and 36 have two. Two different closed windows cannot be joined without tension.

The field that names it

Sheaf theory and cohomology (MSC 18F20, 55N30). A sheaf asks when local pieces fit into one global piece. The failure to fit is measured by cohomology [28]. Here each closed window is a local piece. The open windows at the seam are the failure to fit.

What is established

- Proved in Agda: a closed window is fixed by its first and last trit (this follows from vast-punt; midden names the middle); the seam closes exactly when first and last trits agree, for all 81 cases (naad, VacuumNetLadder2).
- Checked by computation: 9, 36 and 36 seams (upper_checks.py).

Reading

- the open windows at a seam as a cohomological obstruction.

Open

The sheaf reading is untested. A real test would build the cohomology of a word and show that it counts the open seams.

Rung 9. Closed without a start

The net at this rung

A closed strand has no first place. Read from a start, a word has length n. Read without a start, only its rotation class remains (T13) [1].

The field that names it

Combinatorics and algebra (MSC 05, 17, 68). Rotation classes of words are necklaces, counted by Burnside's lemma [24]. The primitive ones, which are not a repetition of a shorter word, are Lyndon words. Lyndon words form a basis of the free Lie algebra, by theorems of Chen, Fox and Lyndon and of Shirshov, both 1958 [21, 22].

Precursors. Piṅgala's tradition counted words of fixed length by kind; Halāyudha's commentary (10th century) gives the arithmetic triangle as meru-prastāra [67, 69].

What is established

- Checked by computation: ternary necklaces of length 1 to 6 number 3, 6, 11, 24, 51, 130, as in T13; primitive ones (Lyndon words) number 3, 3, 8, 18, 48, 116 (rungs_checks.py).
- Known theorem: Lyndon words give a basis of the free Lie algebra [21, 22].

Open

That the primitive closed strands of the net are a basis of a Lie algebra is a reading worth testing.

Rung 10. The plane

The net at this rung

The three digits can also be used in the plane. Take the Eisenstein integers $a + b\omega$, with ω a cube root of one, and the base $\beta = 1 - \omega$. Subtract the balanced residue and divide by β. This gives a map T (T14) [1]. Every orbit of T ends either at 0 or at one of two fixed points, $\pm(1 + \omega)$. These fixed points are sixth roots of one: a turn of 60°. An element has a finite word only if its orbit reaches 0. Not every element does: 2 goes to $2 + \omega$ and then to $1 + \omega$, where it turns for ever.

The field that names it

Algebraic number theory (MSC 11R, 11A63). The Eisenstein integers are the integers of the field $\mathbb{Q}(\sqrt{-3})$. The element $1 - \omega$ is the prime above three. Its residue field is $\mathbb{F}_3$, so its three residues are exactly the three trits. Three is, up to a unit, the square of this prime: $3 = -\omega^2(1 - \omega)^2$. Cubic reciprocity lives in this ring [29]. Number systems with bases in such rings are studied as canonical number systems [30].

What is established

- Checked by computation: $2 \to 2 + \omega \to 1 + \omega$ (upper_checks.py).
- Checked by computation: for $|a|, |b| \le 40$, 1,788 of 6,561 elements (27.3 %) end in the turn; for $|a|, |b| \le 100$, 16,546 of 40,401 (41.0 %), as in T14.
- Known theorem: $\mathbb{Z}[\omega] / (1 - \omega) \cong \mathbb{F}_3$ and $3 = -\omega^2(1 - \omega)^2$ [29].
- Checked by computation: the bases β for which $-1, 0, 1$ are a complete residue system are exactly the elements of norm 3: the six associates of $1 - \omega$ (base_search.py). A complete residue system of size three needs $|\mathbb{Z}[\omega]/(\beta)| = N(\beta) = 3$, and three has one prime above it. So the residue system forces β up to a unit. The behaviour of T14 depends on the unit.
- Checked by computation: a nonzero fixed point $z = d/(1 - \beta)$ exists only when $1 - \beta$ is a unit. Among the six associates that holds for $1 - \omega$ and its complex conjugate $2 + \omega = 1 - \omega^2$ only (base_units.py). The other four associates end in cycles of period 2 or 4 instead. Complex conjugation leaves the digit unchanged and carries the dynamics of one base to the other. On conjugation-symmetric regions (norm at most 300 or 1,500) the two give the same share ending in the turn: 332 of 1,075 and 1,676 of 5,437. So requiring a turn that is a fixed point (period 1) forces the base up to complex conjugation. The differences seen earlier came from the square box, which conjugation does not preserve.

Open

Whether the share settles at a limit is open (IX.3). This rung is not formalised in Agda. It shows the limit of the address: in the line the trits give every number a word, in the plane they do not.

Rung 11. Mirror and ring in one node

The net at this rung

Rung 6 gave a symmetry of order two and a ring of order three. On the three trit values they obey one more relation and give S_3. Now let both act at one node of the net, on everything that passes it. Write the node as a type with one point and two loops: S, which closes after two turns, and the ring, which closes after three. Nothing else is imposed. S is read here as a mirror; it is not the mirror of T12.1 (see Reading).

Rung 12 gives the ladder path. Up to sign, each R step is S followed by the ring, and each L step is S followed by the ring backwards. Does the net force the node to carry no further relation? Two hypotheses make it so. (H1) The steps can be undone. The ladder itself gives only forward steps. (H2) The node keeps distinct ladder paths apart. This is the unique address of T12, applied to the paths: distinct continued fractions stay distinct. Under H1 and H2 the argument runs as follows. A further relation would make a nontrivial normal subgroup of PSL(2, $\mathbb{Z}$) trivial at the node. Such a subgroup contains a hyperbolic element. Every hyperbolic element is conjugate to a positive R/L word. That word would then be the identity at the node, which H2 forbids. So, under H1 and H2, the free pairing is required by the ladders.

The field that names it

Number theory, hyperbolic geometry and knot theory (MSC 11F06, 20F05, 57K10). The free combination of a group of order two and a group of order three is the modular group PSL(2, $\mathbb{Z}$) [17]. Every element of it is written in one way only, as an alternation of the mirror and the ring turned forward or back. The braid group on three strands, B_3, is a central extension of it by $\mathbb{Z}$: B_3 divided by its centre is PSL(2, $\mathbb{Z}$). B_3 is also the knot group of the trefoil [18]. Modular forms on its congruence subgroups carry Wiles's proof of Fermat's Last Theorem [19]. Continued fractions are paths in this group [48].

What is established

- Proved in Agda: in the node, twice the ring is the ring backwards ($\text{ring}^2 \equiv \text{ring}^{-1}$); S is its own inverse ($\text{spiegel} \equiv \text{spiegel}^{-1}$); the L step equals S followed by the ring twice (stap-L), in VacuumNetPairRing.
- Defined in Agda: a trit ±1 as the step "S, then the ring forward or back" (stap); every trit word, and the comma path of $\log_2 3$ in particular, as a loop at the node (pad, kommapad).
- Proved in Agda by computation with integer matrices: $S^2 = -I$ (S-paar); $(ST)^3 = -I$ (U-ring); $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ (vlecht); $a^2 = b^3$ (klaverblad), in VacuumNetLadder.
- Checked by computation: $R = S \cdot U$ and $L = S \cdot U^2$, up to sign, with $U = ST$ the ring; the comma path of 41 steps is the same matrix whether written in R and L or in mirror and ring; $T^{-1}LT^{-1} = S$, so R and L generate SL(2, $\mathbb{Z}$) (pair_ring_path.py, sb_path.py).
- Known theorem: PSL(2, $\mathbb{Z}$) $\cong C_2 * C_3$, with a unique alternating normal form [17]; the fundamental group of a node with two such loops is $C_2 * C_3$ by Seifert–van Kampen, also in homotopy type theory [49]; $B_3 / Z(B_3) \cong$ PSL(2, $\mathbb{Z}$) and B_3 is the trefoil group [18].
- Proved in Agda: every ladder path has a unique address, its Stern–Brocot fraction. Reading the address back gives the path (lees-adres), so distinct paths have distinct pairs (p, q) (adres-injectief). Mirroring a path, R ↔ L, turns p/q into q/p (spiegel-pad), all in VacuumNetLadderPath. This is H2 for positive words, at the level of pairs: the classical Stern–Brocot uniqueness, now formalised. It also gives the inversion half of S.

- Proved in Agda: H2 holds in a node whose group is SL(2, $\mathbb{Z}$), a central ±1 extension of PSL(2, $\mathbb{Z}$) (the group is cited via van Kampen [49], not proved). The node SLKnoop has loops s and u with $s^4 = 1$ and $s^2 = u^3$. Above it lies a family of integer 2 × 2 matrices; s and u act by right multiplication with S and U = ST (F, built with univalence). The ladder steps are exact words: R is s s s u and L is s s s u u. The loop of a ladder word, applied to the identity and to (1, 1), gives its Stern–Brocot pair (vector-pad, vw-adres). So distinct ladder words give distinct loops (H2-knoop). H1 holds because loops are paths. The node of Rung 11 is this node with s·s set to rest (naar11), all in VacuumNetSLNode.
- Proved in Agda: H2 holds in the node of Rung 11 itself. The fibre is the set of integer 2 × 2 matrices with M and −M identified (a set quotient). On it the mirror closes after two turns and the ring after three, so the family G over the node exists. Transport along a ladder word gives the product of its matrices, up to sign (tr-pad11). Starting from the identity, the absolute values of M·(1, 1) ignore the sign and give back the Stern–Brocot pair. So distinct ladder words give distinct loops in the node of Rung 11 (H2-knoop11, VacuumNetPSLNode).
- Argument (not formalised): under H1 and H2 no relation beyond the two closures can hold. It uses two known facts about PSL(2, $\mathbb{Z}$): every nontrivial normal subgroup contains a hyperbolic element, and every hyperbolic element is conjugate to a positive R/L word [48].

Reading

- the matrix S, $z \mapsto -1/z$, as a mirror of the net. The mirror of T12.1 is negation, $z \mapsto -z$, which has determinant −1 and lies outside PSL(2, $\mathbb{Z}$). With the ring it gives PGL(2, $\mathbb{Z}$). S is negation followed by inversion. Inversion has a place in the net: mirroring a ladder path, R ↔ L, sends x to 1/x (proved in Agda, spiegel-pad). So S is the mirror of T12.1 applied twice: once to the value (negation) and once to the path (inversion).

Open

H1 and H2 are proved, for SLKnoop and for the node of Rung 11 itself. For the node of Rung 11, any further relation would merge two addresses; that is argued from cited facts, not formalised. What stays open is that the net's node is this minimal node. S is read as the mirror applied to value and path together, and that double reading is not yet proved. That the node is the theory's first knot is still a reading.

Rung 12. Two ladders

The net at this rung

Merging meetings doubles; merging rings triples (T15). The ladders ×2 and ×3 never meet exactly, because $2^n \neq 3^k$ (T16). Their best near-meetings are the closure depths (T17). At depth 12 the miss is the Pythagorean comma (T18) [1].

The field that names it

Diophantine approximation and the theory of tuning (MSC 11J). $\log_2 3$ is irrational. Its continued fraction gives the best approximations [20]. The comma is implied in the Sectio canonis, about 300 BCE, which shows that six whole tones exceed an octave [2].

What is established

- Proved in Agda: $2^n = 3^k$ only when n = k = 0 (geen-ontmoeting, VacuumNetComma). The proof is by parity, as in the theory.
- Checked by computation: $\log_2 3 = [1; 1, 1, 2, 2, 3, 1, 5, 2, 23, \ldots]$; convergent denominators 1, 2, 5, 12, 41, 53, 306, 665; the comma $3^{12}/2^{19}$ is 23.46 cents (rungs_checks.py).
- Known theorem: best approximations of the second kind are the convergents [20].

Open

T16 is formalised. The closure depths (T17) are not yet formalised in Agda.

Rung 13. Into three dimensions

The net at this rung

The theory adds one hypothesis: the strand is embedded in three-dimensional space. Then the trefoil is the simplest configuration that cannot be undone (T19). It differs from its mirror (T20). An attached strand returns after 4π, not 2π (T21). The Hopf charge is conserved (T22) [1].

Part of this needs no space. Take one node with two loops, x and y. Run x twice and y three times. Neither x·x nor y·y·y is set to rest; they are set equal: x·x ≡ y·y·y. The space is built from two loops and one equation. Its homotopy type is that of the complement of the trefoil in S^3 [54]. Close the common loop, and the node of Rung 11 appears. So Rung 11 is this rung with its common loop set to rest.

The field that names it

Knot theory, Lie groups, singularity theory and algebraic topology (MSC 57K10, 22, 32S55, 55). The group of the space is $\langle x, y \mid x^2 = y^3 \rangle$, the trefoil group. The equation $x^2 = y^3$ is also the curve $z^2 = w^3$ in $\mathbb{C}^2$. Its link at the origin is the trefoil, and its complement fibres over the circle (Milnor) [55]. The modular group of Rung 11 is this group divided by its centre, the common loop. The double cover of rotations by unit quaternions gives the 4π return. The Hopf map generates $\pi_3(S^2) \cong \mathbb{Z}$ [23]. In homotopy type theory the Hopf fibration has been constructed, and Brunerie proved $\pi_4(S^3) = \mathbb{Z}/2$ [8].

What is established

- Proved in Agda: the space Klaverblad with x·x ≡ y·y·y maps to the node of Rung 11 (naar-knoop), and the common loop x·x becomes trivial there (centrum-weg). It maps to the circle with x as three windings and y as two (vezel); the common loop winds six times (centrum-6), all in VacuumNetTrefoil. On the fundamental group, vezel is the map of the Milnor fibration; vezel itself is a map, not a fibration.
- Known theorem: the 2-complex with group $\langle x, y \mid x^2 = y^3 \rangle$ has the homotopy type of the trefoil complement (Hatcher's mapping cylinders, after collapsing; both spaces are aspherical with the same group) [54]; the trefoil is the link of $z^2 = w^3$ and its complement fibres over the circle [55].
- Known theorem: $\pi_3(S^2) \cong \mathbb{Z}$, generated by the Hopf map; Hopf showed in 1931 that this map cannot be undone [23]. The result is also derived in homotopy type theory [8].
- Proved in the cubical library and checked in this environment: $\pi_3(S^2) \cong \mathbb{Z}$ ($\pi_3 S^2 \cong \mathbb{Z}$, through the Hopf fibration, whose fibre is the circle of Rung 2; VacuumNetHopf). The integer of T22 is this invariant; that it is conserved in physics is the theory's reading.
- Proved in Agda: the tension resolves one level up. Klaverblad maps to SLKnoop (x ↦ s, y ↦ u; naarSL). There s·s does not close: transport of the identity along s·s gives −I (ss-Id, paar-sluit-niet, VacuumNetOpenPair). The open pair carries a sign and closes after four turns (s^4, by definition of the node). It closes after two in the node of Rung 11, where M and −M are identified. So SLKnoop agrees with T7.1 that the open pair does not close; reading −I as its carry is a reading. Rung 11 as the reading in which the sign is forgotten is a reading too.
- Tested and weakened: the reading that Rung 11 is this rung "within one scale", with the carry dropped. Through the abelian map, dropping the carry (modulo three) leaves $\mathbb{Z}_3$: x has winding 3 and becomes nothing, y becomes L. That is not even $\mathbb{Z}_6$, the abelianisation of the modular group. So this reading loses information that setting the common loop to rest keeps.
- Proved in Agda: crosswise, the loop with exponent two (x) carries three windings, and the loop with exponent three (y) carries two (winding-x, winding-y).

- On paper: vezel is the abelianisation of $\langle x, y \mid x^2 = y^3 \rangle$. The common loop has winding 6, so vezel modulo six factors through $\mathrm{PSL}(2, \mathbb{Z}) = C_2 * C_3$ and gives its abelianisation $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$. (For $\mathrm{SL}(2, \mathbb{Z})$ it would be $\mathbb{Z}_{12}$.)
- Known theorem: the complement of the trefoil and that of its mirror are homeomorphic, by a map that reverses orientation; so the knot group cannot tell them apart [64]. Klaverblad therefore cannot see T20. Chirality needs orientation, which is what the embedding adds.
- Checked by computation: the Kauffman bracket of the three crossings gives the Jones polynomial $t + t^3 - t^4$ for one trefoil and $t^{-1} + t^{-3} - t^{-4}$ for its mirror (jones_trefoil.py) [65, 66]. They differ, so the trefoil is chiral. The writhe ±3 is not computed from an orientation; it is fixed by requiring integral powers of t, which is a consistency check. By the usual convention the first polynomial is the right-handed trefoil.
- Known theorem: the Alexander polynomial $t^2 - t + 1$ is the same for the trefoil and its mirror [64].

Reading

- x·x as the open pair and y·y·y as the ring. The open pair carries (T7.1); here it lands on the same loop as the ring. That this is the carry of T7.1 is a reading. There was a tension with Rung 11: there the order-two loop closes (spiegel²), which T7.1 does not allow for the open pair.
- Reading, after the test: the two numbers of the theory meet here once, in the exponents 2 and 3, the open pair and the ring. The windings 3 and 2 follow from them (6/2 and 6/3). The common loop is 2·3 = 3·2. Rung 11 sets that common loop to rest. This is a reading, not a derivation.
- Fox's rule of Rung 5 tells the trefoil from the unknot.
- the 4π return of T21 in algebraic form. In SLKnoop, two turns of s give −I and four give the identity; the node of Rung 11 forgets the sign, and there two turns suffice. S acts on the hyperbolic plane as a half-turn [17], so two turns are a full turn there, and lift to −I. This has the shape of SU(2) over SO(3), where a rotation by 2π lifts to −1 and 4π returns. It is a shape match only: the fundamental group of $\mathrm{PSL}(2, \mathbb{R})$ is $\mathbb{Z}$, not $\mathbb{Z}_2$ as for SO(3). Proved: s·s ≠ refl in SLKnoop (VacuumNetOpenPair); the shape match is a reading.

Open

A space with the homotopy type of the knot complement comes without a space hypothesis. The knot as a curve in space, its chirality (T20) and the Hopf charge (T22) still need the embedding; the space of this rung cannot see chirality, as shown above. The 4π return (T21) has an algebraic shadow in SLKnoop, as a reading. They are not formalised. The equivalence of Klaverblad with the knot complement is cited, not proved.

Beyond the first knot

The next six rungs come from the coupling part of the Foundations and from the physics that continues them [1]. Each adds a hypothesis that the net does not supply. The hypothesis is named at the start of each chapter. From here on the net no longer grows from itself alone. What grows is still mathematics: each rung meets a field.

Rung 14. Coupling

The net at this rung

Hypothesis: the energy density of the medium is $e(n) = (g_2/2)n^2 + (g_3/6)n^3$. The stiffness is its second derivative. Around a reference density it reads $g_eff = \bar{g}(1 + \gamma\delta)$ (T23) [1]. The stiffness is a function of the state, as N4 asks. The theory also defines the lever of a count: $\gamma = -\bar{n} \cdot \kappa_3/\kappa_2^2$, built from the mean and the second and third cumulants.

The field that names it

Analysis and asymptotics (MSC 26, 41, 60E). This is the first rung with real functions. Derivatives, Taylor expansion and convexity enter. The lever links a smooth energy to the fluctuations of a count. The link is Laplace's method [31].

Precursors. Infinite series for the sine, cosine and arctangent were found in Kerala by Mādhava around 1350, centuries before Newton and Leibniz [71].

What is established

- Checked by computation: the identities $g_eff = \bar{g}(1 + \gamma\delta)$ and $g_2 = \bar{g}(1 - \gamma)$, symbolically (upper_checks.py).
- Known theorem: Laplace's method for integrals with a large parameter [31].

Open

The real numbers enter here as an input. The chapter "Beyond Rung 21" builds the real numbers of an interval from the net, up to Bishop's equality. They can replace the input of numbers, not of functions: real functions, derivatives and Laplace's method are not built from the net. The formal identification with the Cauchy reals of [4] is not done either.

Rung 15. The free path

The net at this rung

Hypothesis: the strand is a free path, a simple random walk on $\mathbb{Z}^d$ with $d \geq 3$. A node counts its visits. The number of visits is geometric. Its lever is $\gamma = -(1 + P_d)/P_d$, where P_d is the chance to return (T24) [1].

The field that names it

Probability theory (MSC 60G50, 60J10). Pólya proved in 1921 that the walk returns for sure in one and two dimensions, and not for sure from three on [32]. Watson evaluated the triple integrals in closed form [33]. Through later work, including Montroll's, they give the return probability in three dimensions [34].

What is established

- Checked by computation: $\gamma = -3.9365$, -6.1759 and -8.3976 for $d = 3, 4, 5$ (upper_checks.py).
- Known theorem: Pólya's recurrence theorem [32]; Watson's integrals [33].

Open

Whether the lever of the free path is the coupling of the medium is open (IX.1).

Rung 16. Closure on itself

The net at this rung

Hypothesis: Dirac's equation, written in Rowlands' nilpotent form [35, 36]. The operator is $Q = ikE + ip + jm$. Its square is $(E^2 - p^2 - m^2)$ times one. So $Q^2 = 0$ exactly on the mass shell (Physics T1) [1]. The theory reads this as closure applied to itself: a closure that closes again adds nothing.

The field that names it

Clifford algebras and spinors (MSC 15A66, 81R25); homological algebra (MSC 18G). Quaternions and vector units together form a Clifford algebra [37]. Nilpotence has a central place in one more field. In homological algebra, a boundary of a boundary is nothing: $d \circ d = 0$. All of homology and cohomology rests on that rule [38].

What is established

- Checked by computation: $Q^2 = (E^2 - p^2 - m^2) \cdot 1$ as a symbolic 4×4 identity; the boundary of the boundary of a filled triangle is zero (upper_checks.py).
- Known theorem: Dirac's equation and its square [35]; $d \circ d = 0$ as the definition of a chain complex [38].
- Proved in Agda (Rung 7), locally on one window: closure on the trits is $D^2 = 0$, and period three is $D^3 = 0$. Rings of period three form a 3-complex in the sense of Mayer [45–47].

Reading

- $Q^2 = 0$ and $d \circ d = 0$ have the same form.

Open

Rung 7 gives a first answer. On rings, the window sum is D^2, the square of the difference operator, and D^3 is the shift by three minus one. On rings of period three, $D^3 = 0$ but $D^2 \neq 0$. They form a 3-complex. The rings that close everywhere are its kernel of D^2. Such complexes were introduced by Mayer in 1942 [45]. Kapranov and Dubois-Violette developed them as generalised homology with d^N = 0 [46, 47]. On the 27 rings of period three, D is a single block of size three. There ker D^2 = im D and ker D = im D^2: the homology is zero. That D = shift − 1 gives D^3 = shift^3 − 1 over F_3 is Mayer's own mechanism. The trits carry the simplest 3-complex. Whether this adds more than a new name is open. Whether $Q^2 = 0$ of the Dirac rung is the same rule one level up is still a reading.

Rung 17. Two registers

The net at this rung

Hypothesis: Maxwell's equations in his own quaternion form of 1873. Hamilton's operator gives one product with two parts: a scalar part, the convergence, and a vector part, the curl. Gibbs and Heaviside later split the product in two [1]. The theory reads the two parts as two readings of one closure. To keep orientation, a rule must use full quaternions (Physics T2, T3). A ring measure R_Q is the largest eigenvalue of an average of DD^T. It lies between ¼ and 1 and ignores the sign of each D (Physics T4).

The field that names it

Quaternion algebra, rotations and spectral theory (MSC 15A66, 15A18, 20G20). The sign invariance is the double cover of rotations by unit quaternions [39]. The ring measure is the same matrix that is used to average rotations: the mean rotation is the top eigenvector of the summed DD^T [40]. In the language of Clifford analysis, Maxwell's single product ∇A is the operator $d + \delta$ acting on A. The curl is the d part. The convergence is, up to sign, the δ part. That this is the net's one closure read twice is a reading.

What is established

- Checked by computation: the worked example gives $D_1 \cdot D_2 = 0.653$ and R_Q = 0.827 (upper_checks.py).
- Known theorem: q and −q give the same rotation [39]; the eigenvector form of the quaternion average [40].

Open

The theory itself marks this part as a correspondence. Maxwell's equations are translated into the net, not derived from it.

Rung 18. Measure

The net at this rung

The net has no units. Its quantities are ratios. To compare with instruments, the theory follows Larson: space S and time T are two aspects of one motion. The map $\phi(x, y, z) = (x - 3z, y + 3z)$ sends length, time and mass to powers of S and T (Physics T5) [1, 41]. Speed becomes S/T, energy T/S, mass T^3/S^3. The unit length and time are fixed by hydrogen.

The field that names it

Dimensional analysis as linear algebra over $\mathbb{Z}$ (MSC 00A73, 20K). Dimensions form a free abelian group. A change of units is a homomorphism. The Buckingham π theorem counts the dimensionless groups through the kernel of such a map [42]. Here the kernel of ϕ is generated by $M \cdot L^3/T^3$.

What is established

- Checked by computation: the images of speed, energy, momentum and mass, and the kernel (3, −3, 1) (upper_checks.py).
- Known theorem: Buckingham's π theorem [42].

Reading

- Larson's motion as the ground of units.

Open

That the Balmer Hδ line falls at $9S_o$ holds by construction of the unit, as the theory says. It is a consistency, not a test.

Rung 19. Loss has an address

The net at this rung

Hypotheses: local equilibrium per mesh, local balance, and the second law for every sub-volume. Then entropy production is $\sigma = \Sigma\, J \cdot X \geq 0$ at every location (Physics T6) [1]. In the linear regime $J = LX$, with L symmetric as Onsager requires, L is positive semidefinite. Every loss happens at a mesh with an address.

The field that names it

Non-equilibrium thermodynamics and matrix theory (MSC 80A, 15B48). Onsager proved the symmetry of L in 1931 [43]. De Groot and Mazur built the field on it [44]. The spectral theorem splits σ into sign-definite parts.

What is established

- Checked by computation: $\sigma = X^{T}LX \geq 0$ for 1,000 random positive semidefinite L (upper_checks.py). This is a sanity check of a known fact, not a test of the theory.
- Known theorem: Onsager reciprocity [43].

Open

All the content of this rung comes from its hypotheses. The net supplies only the address.

Rung 20. Numbers without end

The net at this rung

One addition: words without end. Finite words do not give them; a stream is an infinite object, given coinductively, one digit at a time (see the chapter on infinity). Let a word go on without end: a stream of trits after the point. On the ×3 ladder, with digits −1, 0, 1, this is balanced ternary for real numbers. Cutting is rounding (III.2). Each number has one stream, except at countably many boundary points, where it has two. This is the unique address of T12, carried into the continuum.

That uniqueness has a price. Take x = 0.1TTT… and y = 0.0111…, both 1/6 from either side. However many digits of x and y are read, no digit of their average is fixed. The first digit depends on whether the average lies just above or just below 1/6, and no finite depth decides that. So on the ×3 ladder, adding numbers without end is not computable digit by digit.

On the ×2 ladder the same three digits behave differently. In base 2 with digits −1, 0, 1, almost every number has many streams. That redundancy is what makes addition work: each output digit needs about one more input digit. The continuum, read at finite depth, needs redundancy. The ×2 ladder with the three digits of the net is one way to get it; base 3 with five digits, −2 to 2, is another.

The field that names it

Computable analysis and computer arithmetic (MSC 03F60, 26, 65G, 68). Signed-digit representations were introduced by Avizienis for fast arithmetic [57]. Wiedmer showed that redundant signed digits make arithmetic on infinite expansions computable [58]. Ciaffaglione and Di Gianantonio built the reals from signed-digit streams in a proof assistant, with coinductive types [59]. The homotopy type theory book constructs the Cauchy reals as a higher inductive-inductive type [4].

Precursors. The ×2 ladder with words of two kinds goes back to Piṅgala [67, 69] and to Leibniz (1679), who later saw it in the Yijing [70]. The three digits on it are Avizienis's signed digits [57].

What is established

- Proved in Agda: mirroring a trit stream twice gives the same stream, by coinduction (spiegelS²). Defined: trit streams as a coinductive type, the digit-wise mirror (spiegelS) and reading at finite depth n (lees), in VacuumNetStream. The file does not yet link streams to real values.
- Checked by computation: for x = 0.1TTT… and y = 0.0111… in base 3, no digit of the average is fixed after 5, 10, 20 or 40 digits. In base 2 with digits −1, 0, 1, the first L digits of each input always fix at least L − 1 digits of the average: the uncertainty has width 2^(1−L) and a digit is safe while it is at most one half. A test on 3,000 random pairs with L = 25 agrees (continuum.py).
- Known theorem: redundant signed-digit expansions make addition computable with bounded lookahead [58]; signed digits themselves go back to Avizienis [57].
- Proved in Agda: the average of two signed-binary trit streams, with a carry c in {−2, …, 2} as a five-valued type. One step reads two digits and gives one digit e and a new carry, with c′ + 4e = 2c + x + y in all 45 cases (stap-ok). After any finite depth n, c + 4E = T, where E and T are the output and the input sum read as integers (invariant, gemiddelde-klopt). The same step is defined on streams by corecursion (gemS), all in VacuumNetAverage.
- Proved in Agda, in integers: T − 4E is one of −2, …, 2, namely the final carry (foutgrens). The first output digit is always 0 (eerste-O). The stream gemS′ read at depth n is the output list of the finite version (stroom-klopt), and E is the Horner value of that list (E-is-horner). Chaining these three: the stream gemS′ from carry 0, read at any depth n, satisfies T − 4E ∈ {−2, …, 2}.

- Scaling, on paper: dividing by $4 \cdot 2^n$, the output at depth n lies within $2^{(-n-1)}$ of $(x_n + y_n)/4$, the value of the digits read, and within $2^{(-n)}$ of $(x + y)/4$. The bound is reached. Dropping the first digit gives the average $(x + y)/2$ with one digit of lookahead.
- Checked by computation: on 5,000 random pairs of 30 digits, the first output digit is always 0 and the average lies within $2^{(-29)}$ of $(x + y)/2$ (average_check.py).
- Proved in Agda: multiplying a word by one trit is mirroring, zeroing or keeping it, exactly (keer-trit-klopt). Shift-and-add over the digits of y, $P := 2P + y_i \cdot X$, gives exactly $X \cdot Y$ for the integer values X and Y of two words (product-klopt), in VacuumNetProduct.
- Checked by computation: for 1,000 random pairs, the first 24 digits of x and y allow at least 22 digits of $x \cdot y$ to be emitted safely (product_check.py). This is consistent with the bounded delay of the classical on-line multiplier [61]. The exact product of two n-digit prefixes has denominator $2^{(2n)}$; that no bounded state suffices follows from Muller [60] (next bullet).
- Known theorem: a finite automaton working on-line, most significant digit first, computes among smooth functions only piecewise affine ones with rational coefficients (Muller) [60]. Addition and the average are affine; the five-state carry above is such an automaton. The product $x \cdot y$ is not affine. A multiplier therefore needs memory that grows with the depth [60, 61].
- Known in computable analysis: a unique address and digit-by-digit arithmetic on the continuum exclude each other [58].

Reading

- with the three digits of the net, the two ladders of Rung 12 take different roles. The ×3 ladder names; the ×2 ladder measures.

Open

The average is proved at every finite depth. The link between stream and finite version is now proved. The value of a whole stream is now formalised in integers, as a Bishop sequence, in VacuumNetValue and VacuumNetValue2 (chapter “Beyond Rung 21”); its identification with the Cauchy reals of [4] is not. Multiplication is formalised only as exact integer arithmetic on prefixes; the digit stream of the product, with its growing state, is not. The field laws are not formalised. Here a real boundary appears. One step with a bounded carry gives at most the piecewise affine operations with rational coefficients, and not the product. Any finite-state reading, most or least significant digit first, has this limit [60]. T1 does not bound the carry: $(s - d)/3$ grows with the sum. So the limit belongs to finite-state reading, not to the net’s rule. Whether the net itself prefers the ×2 ladder for the continuum, or only allows it, is open.

Rung 21. Closure as oscillation

The net at this rung

One choice. A closed window has sum 0 modulo three (T2); over the integers its sum is −3, 0 or 3. Take, as a choice, the windows whose integer sum is exactly 0: a + b + c = 0. Write the second difference, (c − b) − (b − a). Then a + b + c = 0 says exactly that the second difference is −3 times the middle value. That is the discrete harmonic oscillator, y″ = −k·y with k = 3, exactly and with step one, not as a small-step approximation. A word whose windows all have integer sum 0 is a discrete circular motion with period three.

The oscillator t(i+1) = c·t(i) − t(i−1) is periodic over the integers for every start only for c = −1, 0, 1, with periods 3, 4 and 6. For c = 2 the motion is free (linear growth), for c = −2 it grows with alternating sign, and for |c| > 2 it grows faster; special starts can still repeat. The closure of the net is the first of the three: c = −1.

The field that names it

Difference equations and ordinary differential equations (MSC 39A, 34A); crystallography (MSC 20H15). The equation y″ = −y is the limit of the discrete oscillator as the step goes to zero [62]. Over the integers only rotations of order 1, 2, 3, 4 and 6 are possible: the crystallographic restriction [63]. The same orders are the orders of the elements of finite order in SL(2, $\mathbb{Z}$). There S has order 4 and ST order 6 (Rung 11).

What is established

- Proved in Agda: a + b + c = 0 exactly when (c − b) − (b − a) + 3b = 0, over $\mathbb{Z}$ (sluiten⇒trilling, trilling⇒sluiten). The step matrices [[c, −1], [1, 0]] for c = −1, 0, 1 have order 3, 4 and 6 (periode-3, periode-4, periode-6), all in VacuumNetOscillator.
- Checked by computation: from start (0, 1), the integer recurrence is periodic for c = −1, 0, 1 only (c from −3 to 3). With real step h and c = 2 − h^2, the orbit follows cos(iθ) to within 10^{-12} and the period tends to 2π/h (oscillator_check.py).
- Known theorem: an integer 2 × 2 matrix of finite order has order 1, 2, 3, 4 or 6 [63].

Reading

- the closure of the net is the discrete form of circular motion. This fits the theory's reading that the closed loop carries the motion that is light. It is a reading, not a derivation of physics.

Open

The limit to y″ = −y needs real numbers (Rung 20) and is only computed here, not proved. General linear ordinary differential equations are not reached. The link between the period three of closure and the orders four and six of Rung 11 is a reading.

Beyond Rung 21. The continuum on the round

The net at this rung

The round of Rung 0 has a direction and a count of turns. The question here is whether it also has places between: a continuous phase. Merging gives two ladders, ×2 and ×3 (T15). Seen on one round, with a turn of the ×2 ladder as the unit, each ×3 step moves $\log_2 3$ of a turn. A place is reached by a descent through the scales: at each scale one trit, and what is left goes on to the next scale (N5, T1). Two descents are the same place when their difference, carried scale by scale, stays in the window for ever. A place within one turn can be written as a trit stream, a balanced-ternary fraction of a turn. An oscillation is a movement that repeats a finite block of directions. A rotation closes when, after some number of steps, it is back where it began.

The field that names it

Homotopy type theory, dynamical systems and number theory (MSC 03, 37, 11). In the circle S^1 the paths from the point to itself are whole turns only: $\Omega S^1 \simeq \mathbb{Z}$ [5]. Half a turn is not a loop. The continuous circle, the real numbers modulo one, is a different object: a set of points. Shulman added a shape modality to homotopy type theory and proved that the shape of the continuous circle is S^1 [84]. In that sense the round is what is left of the continuous circle when only movement up to deformation is kept.

Precursors. Kauffman's eigenforms are fixed points of a self-referring operation [82]. Brouwer's free choice sequences are infinite sequences that follow no law [85].

What is established

- Proved in Agda: an oscillation is a fixed point of "put the block in front": per ≡ ervoor (a :: as) per (eigenvorm, VacuumNetPhase). Standing still is the oscillation with block 0 (stil-is-trilling).
- Checked by computation: a block W of length n gives the phase $W/(3^n - 1)$, for all 1092 blocks up to length 6. Such phases have denominators prime to 3. The closing rotations of Rung 21 are a finite prefix followed by an oscillation: 1/4 = 0.(1T); 1/3 = 0.1000…; 1/6 = 0.0111… (phase_check.py).
- Known theorem: a number has an eventually periodic expansion in base three exactly when it is rational. A rotation by α closes after n steps exactly when nα is a whole number.
- Proved in Agda: the ladders ×2 and ×3 never meet exactly, $2^n = 3^k$ only for n = k = 0 (T16; geen-ontmoeting, VacuumNetComma). Hence $\log_2 3$ is irrational: if it were p/q, then 3^q = 2^p.
- Checked by computation: stepping the round by one ×3 step, measured in turns of the ×2 ladder ($\log_2 3$ of a turn), gives 5001 different phases (k = 0 to 5000). The largest gap between them shrinks to 0.0011 of a turn, and at 12, 53, 665 and 5000 points there are at most three gap lengths. The nearest returns come at the closure depths 1, 2, 5, 12, 41, 53, 306, 665 (comma_orbit.py).
- Known theorem: the multiples of an irrational step are dense on the round (Kronecker) [86]; they leave at most three gap lengths (the three-distance theorem) [87].
- Proved in Agda: the carried difference of two descents moves as r → 3r + (a − b). A run that stays in the window [−2, 2] for ever never reaches ±2; its rest stays in [−1, 1] (venster-p2, venster-m2, klein, VacuumNetLimit). "Same place" is an equivalence relation (≈-refl, ≈-sym, ≈-trans). The places of one turn form the set quotient Fase = Stroom / ≈. The descents 0.1TTT… and 0.0111… differ in their first trit, yet they are one place (twee-afdalingen-één-plek).

- Checked by computation: 6560 numbers with two different expansions all stay in the window. 3000 pairs of different numbers all leave it, within two scales of $\log_3(1/|x - y|)$ (limit_check.py).
- Known theorem: classically, signed-digit streams up to equal value are exactly the points of the interval [−1/2, 1/2] [58, 59].
- Proved in Agda: let V n s be the first n trits of s read as a whole number. After n steps the rest is $3^n \cdot r + (V\ n\ s - V\ n\ t)$ (rest-is-V, VacuumNetValue). So the same place gives $|V\ n\ s - V\ n\ t| \le 1$ at every scale, and $|V\ n\ s - V\ n\ t| \le 2$ at every scale gives the same place (≈→begrensd, begrensd→≈). On the ×2 ladder the same holds with the bounds 2 and 4 (VacuumNetValue2).
- Known theorem: a sequence of rationals q_n with $|q_n - q_m|$ bounded by a fixed modulus is a real number in Bishop's sense, and two such sequences are equal when their distance tends to zero with that modulus [88]. The sequence V n s / 3^n is of this kind. So the value of a place is the real number V n s / 3^n. Agda proves that bounded difference is the same place; that bounded difference is Bishop's equality is a short argument on paper, since $3^{-n} \le 1/n$.
- Argument (on paper) and computation: if s and s' are the same place, then $|V\ n\ s \cdot V\ n\ t - V\ n\ s' \cdot V\ n\ t| \le 3^n/2$. So the product sequences V n s · V n t / 9^n are equal in Bishop's sense. The product is an operation on values (product_value_check.py, 4000 pairs). As an operation on streams it needs memory that grows with depth [60]; a finite window of rests cannot carry it.
- Argument (on paper): suppose every real number in [−1/2, 1/2] had a ×3 place. Its first trit decides x ≤ 1/6 or x ≥ 1/6. Applied to x = 1/6 + max(−1/3, min(1/3, y)), this decides y ≤ 0 or y ≥ 0 for every real y. That is the principle LLPO [89], which constructive mathematics does not prove. Signed-digit places, as on the ×2 ladder, are admissible: every computable real has one, computed from it [90].
- Proved in Agda: on the ×2 ladder with trits the same construction works, with the rest moving as r → 2r + (a − b). "Same place" is again an equivalence relation, and an endless run stays in [−2, 2] ($\approx_2$-refl, $\approx_2$-sym, $\approx_2$-trans, klein, VacuumNetArith). The average of Rung 20 respects "same place" in each argument (gem-l, gem-r). So it is an operation on the places themselves (gemF on Fase_2). The mirror respects "same place" on both ladders, and mirroring twice is the identity on places (spiegelF_2, spiegelF_3, spiegelF_3^2).
- Checked by computation: on the ×3 ladder the first digit of a sum can depend on a digit arbitrarily far down. For x = 0.0(1^n)1 and y = 0.0(0^n)e, the first digit of x + y is 0 for e = 0 and 1 for e = 1, for every n up to 30. On the ×2 ladder, 2000 random averages agree with (x + y)/2 to within 2^{-38} (base3_add_check.py).
- Known theorem: a signed-digit system in radix r is redundant only when it has more than r digits [57]. Three digits are redundant in radix 2, not in radix 3; in radix 3 two expansions exist only at special points such as 0.0111… = 0.1TTT…, and those points are what block computation. Redundancy is needed for digit-by-digit arithmetic with bounded lookahead [58].

Reading

- a finite prefix followed by an oscillation gives exactly the rational phases, and these are the phases of the closing rotations. They lie dense on the round but are countable.
- classically the ×3 places are all real numbers of their interval. Constructively they can be so only if LLPO holds; LLPO with countable choice is enough. The ×2 places need no omniscience principle for reals given as Bishop sequences; for the Cauchy quotient, countable choice is used. The ladders divide the work again: ×3 for addresses, ×2 for computing and for the full continuum.
- the two ladders divide the work. The ×3 ladder gives nearly unique addresses and places (two only at special points, identified by ≈). The ×2 ladder, with the same three digits, gives computation.

- the limit step is a path. A set quotient adds an identification, a path, between any two descents that end at the same place. So the continuum of one turn is made of descents and of the movement between descents that meet. This fits the start of the book: movement first.
- the net has its own movement that never closes. What does not close at one scale goes on as tension (N2), as material for the next scale (N5), or as a carry (T1). Read with T16, and under the hypothesis of T15, this descent would never end and never repeat. The whole closes (N1); its parts need not.

Open

From closing alone, with a finite prefix, the net gives the rational phases. Its own movement that never closes reaches every phase as a limit. The limit step is built from the net: places are descents, identified when their carried difference stays in the window. Their value is Bishop's real number, proved in integers. The average and the mirror act on places; the product acts on values. What remains is formal, not conceptual. First, the identification with the Cauchy reals of [4] is not formalised; constructively it needs countable choice [4, 88]. Second, the conversion from ×3 to ×2 places is computed, not yet proved to respect ≈. Third, a stream product with finite memory is impossible [60]; that is a result, not a gap. Shulman's modality remains another route [84].

The author asks whether the comma keeps the round going. The mathematics gives something close to it. If the two ladders met exactly, their joint movement would repeat. Because of the comma it never repeats, and it comes as close as one likes to every phase.

How much infinity the ladder needs

Reverse mathematics measures how strong the axioms behind a theorem must be [50]. It gives an order that has nothing to do with the net. If the order of the rungs matches the order of strength, that is convergence from an independent side.

Table 2 gives an estimate per group of rungs. It is an estimate from the kind of statement, not a reverse-mathematics proof. The Agda proofs themselves run in a much stronger theory; the table concerns what the statements need.

Rungs	What the statements are about	Estimated system	Basis
0–4	Finite types, $\mathbb{Z}_3$, words, $\mathbb{Z}$ as windings	EFA	Finite checks and integer arithmetic
5	Windows; lines of F_3^n; cap set bound	EFA (expected)	Arithmetical statement; elementary polynomial method; formalised in Lean [53]
6–10	$\mathbb{Z}_3$ with mirror; circulants over F_3; seams; necklaces; $\mathbb{Z}[\omega]$	EFA	Linear algebra over finite fields and counting
11	PSL(2, $\mathbb{Z}$), braids, normal forms	EFA	Finitely presented groups, words
12	$2^n \neq 3^k$, convergents, comma; $\log_2 3$ itself	EFA; RCA_0 for $\log_2 3$	Convergents are integer arithmetic; $\log_2 3$ is a real number
13	Trefoil group; SU(2), 4π; $\pi_3(S^2) \cong \mathbb{Z}$	EFA for the group; RCA_0 for SU(2); RCA_0–WKL_0 for $\pi_3(S^2)$ (uncertain)	Topological statement via coded maps; a simplicial version is arithmetical, strength unknown
14–15	Real functions, Laplace's method; random walks	RCA_0	Basic real analysis of explicit functions [50]
16–19	Clifford and quaternion algebra; eigenvalues; Onsager	RCA_0	Linear algebra over $\mathbb{R}$ and $\mathbb{C}$ with explicit entries [50]
20–21	Trit streams, average with lookahead; oscillation periods	RCA_0	Streams as functions $\mathbb{N} \to \{-1, 0, 1\}$; finite automata
Beyond 21	Places as streams up to ≈; Bishop value; ×3 places as all reals	BISH for ≈ and values; LLPO (with countable choice) for ×3 places to be all reals	Constructive analysis [88, 89]

Table 2. Estimated strength of the rungs. EFA: elementary arithmetic. PRA: primitive recursive arithmetic. RCA_0: recursive comprehension.

Martin-Löf separates actual truth, a proof that has been made, from potential truth, a proof that can be made [83]. Read in his terms, the discrete ladder up to Rung 19 needs only infinity of the second kind: windings that can always be added. From Rung 20 on the book uses streams, given coinductively one digit at a time, and set quotients of them. These are infinite objects, but each is used only through its finite prefixes. This is a reading; Martin-Löf speaks of truth, not of infinity.

Three findings follow.

- At a coarse resolution the order does not go down. Up to Rung 11 the statements are discrete. Real numbers first appear at Rungs 12 and 13 ($\log_2 3$, SU(2)), and dominate from Rung 14, where the hypotheses from physics begin. With only two or three levels, this is weak evidence.
- Up to Rung 11, and for the integer part of Rungs 12 and 13, the statements are about finite or discrete objects. Elementary arithmetic is expected to suffice. Friedman conjectures this

for all finitary theorems of mainstream mathematics, and Avigad shows how far it reaches in number theory and combinatorics [52].

- RCA_0 and WKL_0 prove no Π^0_2 sentence that primitive recursive arithmetic does not prove (Parsons; Friedman). Simpson reads this as a partial realisation of Hilbert's programme [51]. RCA_0 suffices for Rungs 14 to 19 only because their functions and series are explicit, with computable bounds. In general form, Laplace's method needs WKL_0 (a maximum of a continuous function) and monotone convergence needs ACA_0. So the arithmetical content of every rung is finitistically reducible. This is compatible with the theory's position that finite depth is enough. It is not evidence for it: Hilbert's "finitistic" is not the theory's "finite depth", and statements about real numbers are not Π^0_2.

Limits. The table is an estimate. Rung 13 is uncertain. Onsager reciprocity is a physical principle; only its matrix content is placed here. The estimate concerns provability, not what exists.

Coverage of the subject classification

The Mathematics Subject Classification has 63 top-level areas [56]. Each area was placed on the rungs where the net meets it, or marked as a gap. The full list is in msc_coverage.csv. Placing an area means that at least one structure of the area appears on a rung. It does not mean that the area is generated.

- Met: 17 areas, through a structure on a rung that is proved or computed. They include logic, combinatorics, number theory, field theory, linear algebra, group theory, algebraic topology, knot theory and dynamical systems.
- Met by citation only: 3 areas (17, 32, 33).
- Met via a hypothesis: 7 areas (16, 22, 35, 60, 78, 80, 81). They appear only once a hypothesis from physics or analysis is added (Rungs 13 to 19).
- Met weakly: 11 areas (06, 08, 14, 26, 28, 30, 31, 34, 52, 54, 82). One structure appears, cited or read, but nothing of the area's own theory.
- Not applicable: 3 areas (00, 01, 97).
- Gaps: 22 areas. Analysis on continua: 42–47, 49, 58. Geometry: 53. K-theory: 19. Statistics: 62. Applications: 70, 74, 76, 83, 85, 86, 90–94.

About a third of the gaps (8 of 22) lie in analysis on continua. Several others (19, 90, 91, 94) are on the discrete side. The match with Table 2, where real numbers enter late and through hypotheses, therefore holds only in part. The ladder is strongest on the discrete side of mathematics and weakest on the continuous side. The continuum of one interval is now built (chapter "Beyond Rung 21"). The next rungs, if they exist, must go beyond it: differential equations, function spaces and differential geometry.

Where the derivation stopped

The goal of the book is derivation. Each time it stopped, the missing piece was named. Table 3 lists every stopping point, what was missing, and what became of it. The labels are those used during the work. Resolved points are proved in the files of the appendix or computed by its scripts.

Point	Where it stopped	What was missing	Status
V-a	Rung 4: loops as words	A reading of $\mathbb{Z}$ as words, not as a given set	Resolved: loops are words; $\mathbb{Z}$ is only a measuring rod (Helix, BT0)
V-b	Rung 3: digit and winding	A link between the trit and the loop	Resolved: digit = winding modulo three, for all loops (DigitWinding). The carry (s − d)/3 is stated, not proved; the physical reading is open (IX.5)
V-c	Rung 7: the rewrite rule	Why this rule and not another	Resolved up to a scalar among linear radius-one rules; non-linear rules open
V-d	Rung 10: the base 1 − ω	Why this base	Resolved by computation, given a fixed-point turn (period 1), up to complex conjugation; the choice of one of the pair is not derived
V-e	Rung 11: the node	Free pairing of mirror and ring	H1 and H2 proved for SLKnoop and for the node of Rung 11 (SLNode, PairRing, PSLNode). Free pairing argued from cited facts, not formalised. Open: that the net's node is this minimal node
T10	Rung 7 (T10)	A consistent rest rule	Corrected by the rest rule (rest_rule.py)
V-g'	Rung 13: the open pair	Whether the open pair closes	Resolved in SLKnoop: it does not close (OpenPair), as T7.1 says. In the node of Rung 11 it closes after two turns
V-h	Rung 20: product of streams	A finite automaton for the product	Not possible: the product needs growing memory (Muller 1994). Prefix product proved exact (Product)
Open 1	Rung 20: value of a stream	A real number for an infinite trit word	Closed at the level of values: same place = Bishop equality, proved in integers (VacuumNetValue, VacuumNetValue2). Average and mirror act on places; product on values. ×3 places are all reals only under LLPO; ×2 places are. Not formalised: the Cauchy reals of [4]
Open 2	Rung 13 (T20): chirality	Left and right trefoil apart	Open: the knot group cannot see it; the embedding is needed
Open 3	Coverage	22 MSC areas	Open: 8 of the 22 are analysis on continua (Coverage chapter)

Table 3. Stopping points of the derivation.

The stopping points sharpen the goal. Four assumptions remain: that the net's node is the minimal node of Rung 11, the choice of one base among a conjugate pair, the embedding needed at Rung 13, and the hypotheses of Rungs 14 to 21: physical for Rungs 14 to 19, infinite streams at Rung 20, closure read as oscillation at Rung 21. The continuum chapter also uses the stream hypothesis of Rung 20 and, for its reading of the comma, T15. The real numbers that Rung 14 took as input are now built from the net, up to Bishop's equality, in the continuum chapter; only their formal identification with the Cauchy reals of [4] is still open. Up to Rung 13, the rest is proved or computed, as each chapter states; readings are marked as readings.

Where the ladder goes next

The theory names its own open questions (IX.1–IX.5) [1]. Each is a rung that is not yet climbed.

- IX.1: is the lever of the free path the coupling of the medium? This joins Rungs 14 and 15.
- IX.2: which rewrite dynamics is its own fixed point? On the register of Rung 7 this is solved: the fixed points are the rings that close everywhere. Among local linear rules of radius one, only two have that property, and the rewrite of T10 is the one in which a trit does not read itself. Non-linear rules are not covered. For the whole net it is open.
- IX.3: does the share of turning elements in the plane settle? This is Rung 10.
- IX.4: when does a strand merge in equal blocks? This is the hypothesis of Rung 12.
- IX.5: the physical reading of the trit rule. Does the passage that leaves a node the way the first passage left it give the digit +1, the reverse −1, and any other 0? This touches the open point of Rungs 3 and 4.

Some fields are met only weakly or not at all (see the coverage chapter). The real numbers enter at Rungs 12 to 14 as an input; the chapter “Beyond Rung 21” later builds the numbers of an interval from the net, up to Bishop’s equality. Algebraic geometry is met only through one cited curve. Category theory is met only through homological algebra (Rung 7) and a reading (Rung 8). Two routes lead there. Homotopy type theory is the internal language of higher toposes, so category theory is close. Schreiber’s cohesive homotopy type theory reaches differential geometry and gauge fields [9]. Rungs 7 and 16 point to a third route. The trits carry the simplest 3-complex, with $d^3 = 0$. Whether generalised homology with d^N = 0 adds more than a new name here is open.

Limits

- The loops-equal-words step no longer relies on the library theorem (VacuumNetHelix), and the set-truncation is no longer needed (VacuumNetBT0). Both proofs still use the integers as a measuring rod.
- The Agda proofs of Rungs 7 and 8 are local, on one window or one seam. The rule for rings that come to rest is proved on paper and checked exactly up to n = 48, not formalised.
- Rung 11: H1 and H2 are proved in Agda, for SLKnoop and for the node of Rung 11 itself. The rung now rests on the assumption that the net's node is the minimal node, on cited theorems (Stern–Brocot, van Kampen, the group of the node, normal subgroups of PSL(2, $\mathbb{Z}$)) and on reading S as a mirror. The matrix check of the braid relation is not faithful for B_3.
- Rungs 9, 10 and 14 to 19 are checked by computation or cited. They are not formalised. Rung 12 is formalised for T16 only (VacuumNetComma); Rung 13 in part (VacuumNetTrefoil, VacuumNetOpenPair, VacuumNetHopf). Rungs 20, 21 and the continuum chapter are formalised as their bullets state.
- From Rung 13 on, every rung adds a hypothesis. Rungs 16 to 19 rest on physics: Dirac, Maxwell, Larson and thermodynamics. The theory itself marks Rung 17 as a correspondence.
- The readings that link a rung to a field are readings. The strongest is that the node of Rung 11 is the first knot. It needs its own proof. That $Q^2 = 0$ of the Dirac rung equals the $d^3 = 0$ of the trits is also still a reading.
- All formal work and all checks were made by the same AI system. A second AI system (DeepSeek) reviewed an earlier version. Its points on readings, on the digit and direction readings, on hypotheses and on Rung 11 are addressed where noted. An independent check by a person is still needed.

Conclusion

In its first eleven rungs, 0 to 10, the net leads to a row of known structures. They are the circle and the integers, the field with three elements and balanced ternary. Next come the setting of the cap set problem and linear cellular automata over F_3. Then follow the gluing rule of closed windows, necklaces, and the limit of the address in the Eisenstein integers.

These are proved in Agda or checked by computation. Two steps looked like choices of the theory: the rewrite of Rung 7 and the base $1 - \omega$ of Rung 10. Both are now forced, within stated limits. Among local linear rules of radius one, the rewrite is forced up to a scalar; it is the one in which a trit does not read itself. The base is forced up to a unit; requiring a turn that is a fixed point leaves only $1 - \omega$ and its complex conjugate, which behave alike; the other associates end in 2- or 4-cycles. Rung 11 was a free choice in an earlier version. It now follows from Rung 12 under two named hypotheses. Rungs 9 and 10 are checked on finite ranges.

In the next three rungs the net meets the settings of the modular group and the comma. It reaches a 2-complex whose homotopy type is that of the trefoil complement (cited), without a space hypothesis. With the hypothesis of embedding in space, it meets the knot itself and the Hopf charge. These are cited or read.

In Rungs 14 to 19 each step adds a hypothesis from physics. The fields that meet the net there are analysis, probability and Clifford algebra. Quaternion spectral theory, dimensional analysis and the theory of positive matrices follow. Rung 20 adds words without end, the trit streams. Rung 21 reads closure as oscillation.

History found each of these fields on its own. Here they come one after another, from one rule and the hypotheses named.

Two results came out of the work itself. First, the theory states that rings come to rest only at lengths 4 and 8. In fact they come to rest exactly when four divides the length. This is proved here and checked up to length 48. Second, closure on the trits is a second difference: $D^2 = 0$. Period three is $D^3 = 0$. So the trits carry the simplest 3-complex, already at Rung 7. Whether that adds more than a new name is open.

Version 3 adds four results. First, the book now starts from movement: an eternal round, as the author corrected. The circle S^1 of homotopy type theory is exactly one point and one movement. Second, the two ladders never meet (T16, proved), so the net has its own movement that never closes, and it reaches every phase as a limit. Third, the limit step is built from the net itself: a place is a descent through the scales, and two descents are the same place when the rest they carry stays in the window. Its value is Bishop's real number; the integer bounds behind this are proved in Agda. Fourth, the two ladders divide the work. The ×3 ladder gives addresses; the ×2 ladder, with the same three digits, gives computation and the full continuum.

So the answer to the question of this book is this. The net derives, up to the stated hypotheses, the discrete core of mathematics and the continuum of one interval. Four assumptions remain: that the net's node is the minimal node of Rung 11, the choice of one base among a conjugate pair, the embedding in space at Rung 13, and the hypotheses of Rungs 14 to 21 (physical up to Rung 19, streams at Rung 20, oscillation at Rung 21). Where the derivation stopped, the missing piece is named. That is the result the author asked for.

Tool disclosure. The Agda files, the computations and this text were made with the AI system Claude (Anthropic) under the direction of the author. The author chose the method and decides what counts as a result.

Appendix: the formal files

All files check with Agda 2.8.0 and the cubical library v0.9, under --safe, without postulates. Each needs the library file vn.agda-lib (depend: cubical-0.9). Table A lists them with the start of their sha256 hash.

File	sha256 (first 8)	Content
VacuumNetValue.agda	e1f0e894	value of a place: rest = difference of truncated values; same place = Bishop equality (×3)
VacuumNetValue2.agda	5d248046	the same on the ×2 ladder
VacuumNetArith.agda	df1401d6	same place on the ×2 ladder; average and mirror act on places (continuum chapter)
VacuumNetLimit.agda	833ded5c	the limit step: same place, window, Fase as quotient (continuum chapter)
VacuumNetComma.agda	ffeb7703	T16: the ladders ×2 and ×3 never meet exactly (Rung 12, continuum chapter)
VacuumNetPhase.agda	28a3ae24	oscillation as eigenform, standing still (Beyond Rung 21)
VacuumNetRound.agda	24bb8edb	movement as an eternal round, three directions, return (Rungs 0, 1, 2)
VacuumNetHoTT.agda	2df88f8c	words BT, BT ≃ $\mathbb{Z}$, BT ≃ ΩS^1 (Rungs 2, 4)
VacuumNetFoundation.agda	dd744800	onderscheid, mirror, spiegeling (Rungs 2, 4)
VacuumNetDictionary.agda	93e49953	disc, dunce cap, cross-cap, torus as two loops (earlier dictionary)
VacuumNetLadder.agda	fdc01e18	markering, ⊕, venster, three closures, matrices of Rung 11 (Rungs 1, 3, 5, 6, 11)
VacuumNetLadder2.agda	17f6ed9b	propagatie, periode, vast-punt, naad (Rungs 7, 8)
VacuumNetLadder3.agda	2345f5bf	closure as second difference, D^3 (Rung 7)
VacuumNetEigenform.agda	f68a2f9b	the two eigenform rules (Rung 7)
VacuumNetBT0.agda	fce0a495	words without set-truncation (Rung 4)
VacuumNetHelix.agda	7e85b27a	loops are words, without the library theorem (Rungs 2, 4)
VacuumNetDigitWinding.agda	c4fd707b	digit = winding modulo three, for all loops (Rung 3)
VacuumNetLadderPath.agda	7dd5f53c	unique addresses of ladder paths, path mirror (Rungs 11, 12)
VacuumNetPairRing.agda	2fbdc537	node with mirror and ring, comma path (Rung 11)
VacuumNetSLNode.agda	3b2c0aad	H1 and H2 in SLKnoop (Rung 11)
VacuumNetPSLNode.agda	1897f38c	H2 in the node of Rung 11 itself
VacuumNetTrefoil.agda	b21d16a8	Klaverblad, maps to Rung 11 and to S^1 (Rung 13)
VacuumNetOpenPair.agda	3fab67b3	the open pair does not close (Rung 13)
VacuumNetHopf.agda	b62cf6f2	$\pi_3(S^2) \cong \mathbb{Z}$ from the library (Rung 13)
VacuumNetStream.agda	7fe9dcb6	trit streams, mirror (Rung 20)
VacuumNetAverage.agda	2686e885	average of streams, error bound in integers (Rung 20)
VacuumNetProduct.agda	860ccb24	exact product on prefixes (Rung 20)
VacuumNetOscillator.agda	a683dd28	closure as oscillation, periods 3, 4, 6 (Rung 21)

Table A. The Agda files of this book.

Computations, in Python 3 with sympy: bottomup_checks.py, rungs_checks.py, upper_checks.py, rest12.py, rest_rule.py, ncomplex.py, rule_search.py, base_search.py, base_units.py, sb_path.py, pair_ring_path.py, continuum.py, average_check.py, product_check.py, oscillator_check.py, jones_trefoil.py, phase_check.py, comma_orbit.py, limit_check.py, base3_add_check.py, ladder_convert_check.py, product_value_check.py, trit_adres.py, trit_families.py. Each script prints the numbers quoted in the chapters.

Reviews. Every chapter was checked by a separate AI agent that had not written it, following the author's Respondent–Judge protocol. Its corrections are worked into the text. A second AI system (DeepSeek) reviewed an earlier version. No person has yet checked the work independently.

Annotated references

[1] Konstapel, J. (2026). The Vacuum.Net Theory, Ninth Edition: Foundations 9.2 and Physics 9.0. Constable Research, Leiden, 27 October 2026.
The theory behind every rung. The chapters cite N1, N4, the trit rule, T2–T4, T7, T7.1, T8–T11, T12, T12.1, T13, T14, T15–T24, Physics T1–T6 and IX.1–IX.5.

[2] Konstapel, J. (2026). The Vacuum.Net Theory Read Against the History of Mathematics. Constable Research, Leiden, 29 October 2026.
Shows that the theory orders its parts by logic, not by date. The preface rests on it.

[3] Konstapel, J. (2026). The Foundations of Mathematics Based on the Vacuum.Net Theory. Constable Research, Leiden, 15 November 2026.
The foundation in cubical Agda that this book builds on, and the top-down method whose limits led to the bottom-up method used here.

[4] The Univalent Foundations Program (2013). Homotopy Type Theory: Univalent Foundations of Mathematics. Institute for Advanced Study, Princeton.
The language of every Agda statement in this book. Read chapters 1, 2, 6 and 8.

[5] Licata, D. R., & Shulman, M. (2013). Calculating the fundamental group of the circle in homotopy type theory. Proceedings of LICS 2013, 223–232. arXiv:1301.3443.
The circle's loops are the integers. Rungs 2 and 4 rest on it.

[6] Altenkirch, T., & Scoccola, L. (2020). The integers as a higher inductive type. Proceedings of LICS 2020, 67–73. arXiv:2007.00167.
Integers as a higher inductive type without set-truncation. The way to lift the first limit.

[7] Vezzosi, A., Mörtberg, A., & Abel, A. (2021). Cubical Agda: A dependently typed programming language with univalence and higher inductive types. Journal of Functional Programming 31, e8.
The proof assistant used throughout.

[8] Brunerie, G. (2016). On the homotopy groups of spheres in homotopy type theory. PhD thesis, Université Nice Sophia Antipolis. arXiv:1606.05916.
The Hopf fibration and $\pi_4(S^3) = \mathbb{Z}/2$ in homotopy type theory. The formal horizon of Rung 13.

[9] Schreiber, U. (2013). Differential cohomology in a cohesive ∞-topos. arXiv:1310.7930.
The route from homotopy type theory to differential geometry and physics, for the rungs not yet generated.

[10] Spencer-Brown, G. (1969). Laws of Form. Allen & Unwin, London.
The mark and re-entry. In this version a precursor, Rungs 0 and 1.

[11] Kleene, S. C. (1952). Introduction to Metamathematics, §64. North-Holland, Amsterdam.
The strong three-valued logic of Rung 3.

[12] Fox, R. H. (1962). A quick trip through knot theory. In M. K. Fort (ed.), Topology of 3-Manifolds and Related Topics, 120–167. Prentice-Hall.
Three-colourings of knots: the rule of Rung 5, met again at Rung 13.

[13] Knuth, D. E. (1997). The Art of Computer Programming, Vol. 2, 3rd ed., §4.1. Addison-Wesley.
Balanced-ternary arithmetic. Rung 4.

[14] The agda/cubical library, version 0.9. https://github.com/agda/cubical
Supplies the circle, its winding theorem, the dunce cap and the torus used in the Agda files.

[15] Konstapel, J. (2026). VacuumNetValue.agda, VacuumNetValue2.agda, VacuumNetArith.agda, VacuumNetLimit.agda, VacuumNetComma.agda, VacuumNetPhase.agda,

VacuumNetRound.agda, VacuumNetHoTT.agda, VacuumNetFoundation.agda, VacuumNetDictionary.agda, VacuumNetLadder.agda, VacuumNetLadder2.agda, VacuumNetLadder3.agda, VacuumNetPairRing.agda, VacuumNetLadderPath.agda, VacuumNetBT0.agda, VacuumNetEigenform.agda, VacuumNetTrefoil.agda, VacuumNetStream.agda, VacuumNetAverage.agda, VacuumNetProduct.agda, VacuumNetDigitWinding.agda, VacuumNetOscillator.agda, VacuumNetHelix.agda, VacuumNetSLNode.agda, VacuumNetPSLNode.agda, VacuumNetOpenPair.agda and VacuumNetHopf.agda. Constable Research, Leiden.
The formal side of every "Proved in Agda" label. Check with Agda 2.8.0 and cubical v0.9.

[16] Ellenberg, J. S., & Gijswijt, D. (2017). On large subsets of F_q^n with no three-term arithmetic progression. Annals of Mathematics 185(1), 339–343.
The cap set bound. Rung 5 meets one of the central results of additive combinatorics here.

[17] Serre, J.-P. (1973). A Course in Arithmetic, Chapter VII. Springer, New York.
The modular group, its generators S and T, and modular forms. The backbone of Rung 11.

[18] Kassel, C., & Turaev, V. (2008). Braid Groups. Springer, New York.
Braid groups, B_3 and its relation to the modular group and the trefoil.

[19] Wiles, A. (1995). Modular elliptic curves and Fermat's Last Theorem. Annals of Mathematics 141(3), 443–551.
Shows how far the modular group of Rung 11 reaches into number theory.

[20] Khinchin, A. Ya. (1964). Continued Fractions. University of Chicago Press (Russian original 1935).
Best approximations and convergents. Rung 12.

[21] Chen, K.-T., Fox, R. H., & Lyndon, R. C. (1958). Free differential calculus IV: The quotient groups of the lower central series. Annals of Mathematics 68(1), 81–95.
Lyndon words as a basis of the free Lie algebra. Rung 9.

[22] Shirshov, A. I. (1958). On free Lie rings. Matematicheskii Sbornik 45(87), 113–122.
The second, independent proof that Lyndon-type words give a basis of the free Lie algebra. Rung 9.

[23] Hopf, H. (1931). Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. Mathematische Annalen 104, 637–665.
The Hopf map and the Hopf invariant. The origin of the Hopf charge of Rung 13.

[24] Burnside, W. (1897). Theory of Groups of Finite Order. Cambridge University Press.
The orbit-counting lemma that counts necklaces. Rung 9.

[25] Martin, O., Odlyzko, A. M., & Wolfram, S. (1984). Algebraic properties of cellular automata. Communications in Mathematical Physics 93, 219–258.
The algebra of linear cellular automata over $\mathbb{Z}_k$. The field that names the rewrite of Rung 7.

[26] Davis, P. J. (1979). Circulant Matrices. Wiley, New York.
Circulant matrices and their kernels. Explains the fixed points of Rung 7.

[27] Lind, D., & Marcus, B. (1995). An Introduction to Symbolic Dynamics and Coding. Cambridge University Press.
Shifts of finite type and their entropy. The words that close everywhere form such a shift.

[28] Mac Lane, S., & Moerdijk, I. (1992). Sheaves in Geometry and Logic. Springer, New York.
Sheaves and the gluing of local data. The reading of Rung 8.

[29] Ireland, K., & Rosen, M. (1990). A Classical Introduction to Modern Number Theory, 2nd ed., Chapter 9. Springer, New York.
The Eisenstein integers, the prime $1 - \omega$ and cubic reciprocity. Rung 10.

[30] Kátai, I., & Szabó, J. (1975). Canonical number systems for complex integers. Acta Scientiarum Mathematicarum (Szeged) 37, 255–260.
The start of the theory of number systems with algebraic-integer bases. Context for T14.

[31] de Bruijn, N. G. (1958). Asymptotic Methods in Analysis. North-Holland, Amsterdam.
Laplace's method, which links the lever of a count to the energy of Rung 14.

[32] Pólya, G. (1921). Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz. Mathematische Annalen 84, 149–160.
Recurrence of random walks in one and two dimensions, transience from three. The ground of Rung 15.

[33] Watson, G. N. (1939). Three triple integrals. Quarterly Journal of Mathematics 10, 266–276.
The integrals that give the return probability in three dimensions. Used in T24.

[34] Montroll, E. W. (1956). Random walks in multidimensional spaces, especially on periodic lattices. Journal of the SIAM 4(4), 241–260.
Return probabilities in higher dimensions. Used in T24.

[35] Dirac, P. A. M. (1928). The quantum theory of the electron. Proceedings of the Royal Society A 117, 610–624.
The equation whose square is the mass shell. The hypothesis of Rung 16.

[36] Rowlands, P. (2007). Zero to Infinity: The Foundations of Physics. World Scientific, Singapore.
The nilpotent form of Dirac's operator that the theory reads as closure on itself.

[37] Lounesto, P. (2001). Clifford Algebras and Spinors, 2nd ed. Cambridge University Press.
Clifford algebras and spinors: the field that names the units of Rung 16.

[38] Weibel, C. A. (1994). An Introduction to Homological Algebra. Cambridge University Press.
Chain complexes and $d \circ d = 0$. The field that Rung 16 may reach.

[39] Altmann, S. L. (1986). Rotations, Quaternions, and Double Groups. Clarendon Press, Oxford.
Quaternions, rotations and the double cover. Rungs 13 and 17.

[40] Markley, F. L., Cheng, Y., Crassidis, J. L., & Oshman, Y. (2007). Averaging quaternions. Journal of Guidance, Control, and Dynamics 30(4), 1193–1197.
The mean rotation as the top eigenvector of the summed qq^T: the same matrix as the ring measure of Rung 17.

[41] Larson, D. B. (1959). The Structure of the Physical Universe. North Pacific Publishers, Portland.
Space and time as two aspects of motion. The reading behind Rung 18.

[42] Buckingham, E. (1914). On physically similar systems: illustrations of the use of dimensional equations. Physical Review 4(4), 345–376.
The π theorem: dimensional analysis as linear algebra. Rung 18.

[43] Onsager, L. (1931). Reciprocal relations in irreversible processes I. Physical Review 37, 405–426.
The symmetry of the matrix L. Rung 19.

[44] de Groot, S. R., & Mazur, P. (1962). Non-Equilibrium Thermodynamics. North-Holland, Amsterdam.
The field of entropy production that Rung 19 uses.

[45] Mayer, W. (1942). A new homology theory I and II. Annals of Mathematics 43(2), 370–380 and 43(3), 594–605.
The first complexes with $d^p = 0$ instead of $d^2 = 0$. The field the trits enter at Rung 7.

[46] Kapranov, M. M. (1996). On the q-analog of homological algebra. arXiv:q-alg/9611005.
Generalised homology with $d^N = 0$. Read it for the modern form of Mayer's idea.

[47] Dubois-Violette, M. (1998). d^N = 0: generalized homology. K-Theory 14(4), 371–404.

The systematic theory of N-complexes. The reference for Rungs 7 and 16.

[48] Series, C. (1985). The modular surface and continued fractions. Journal of the London Mathematical Society (2) 31(1), 69–80.
Continued fractions as paths in the modular group. The bridge between Rungs 11 and 12.

[49] Favonia (Hou, K.-B.), & Shulman, M. (2016). The Seifert–van Kampen theorem in homotopy type theory. In Computer Science Logic 2016, LIPIcs 62, 22:1–22:16.
Van Kampen inside homotopy type theory. It gives the group of the node of Rung 11.

[50] Simpson, S. G. (2009). Subsystems of Second Order Arithmetic, 2nd ed. Perspectives in Logic. Cambridge University Press.
The standard work on reverse mathematics. The systems RCA_0, WKL_0 and ACA_0 of Table 2 are defined here.

[51] Simpson, S. G. (1988). Partial realizations of Hilbert's program. Journal of Symbolic Logic 53(2), 349–363.
Shows that mathematics provable in WKL_0 is finitistically reducible. The ground of the third finding.

[52] Avigad, J. (2003). Number theory and elementary arithmetic. Philosophia Mathematica 11(3), 257–284.
How much number theory and combinatorics elementary arithmetic proves; states Friedman's grand conjecture.

[53] Dahmen, S. R., Hölzl, J., & Lewis, R. Y. (2019). Formalizing the solution to the cap set problem. In ITP 2019, LIPIcs 141, 15:1–15:19.
The cap set bound of Rung 5, formalised in Lean. Shows that its proof is elementary and checkable.

[54] Hatcher, A. (2002). Algebraic Topology, Example 1.24. Cambridge University Press.
Torus knot complements and their 2-complexes with group ⟨x, y | x^m = y^n⟩. The ground of the new Rung 13.

[55] Milnor, J. (1968). Singular Points of Complex Hypersurfaces. Annals of Mathematics Studies 61. Princeton University Press.
The trefoil as the link of $z^2 = w^3$ and the fibration of its complement over the circle.

[56] Associate Editors of Mathematical Reviews and zbMATH (2020). MSC2020: Mathematics Subject Classification System. AMS and FIZ Karlsruhe.
The 63 top-level areas used for the coverage test. Each entry of msc_coverage.csv cites a code from here.

[57] Avizienis, A. (1961). Signed-digit number representations for fast parallel arithmetic. IRE Transactions on Electronic Computers EC-10(3), 389–400.
Introduces signed digits and shows that they limit carry propagation. The arithmetic of Rung 20.

[58] Wiedmer, E. (1980). Computing with infinite objects. Theoretical Computer Science 10(2), 133–155.
Redundant signed digits make arithmetic on infinite expansions computable. The ground of Rung 20.

[59] Ciaffaglione, A., & Di Gianantonio, P. (2006). A certified, corecursive implementation of exact real numbers. Theoretical Computer Science 351(1), 39–51.
Real numbers as signed-digit streams in Coq, with coinduction. The model for formalising Rung 20 further.

[60] Muller, J.-M. (1994). Some characterizations of functions computable in on-line arithmetic. IEEE Transactions on Computers 43(6), 752–755.
Shows that on-line finite automata compute only piecewise affine functions. The boundary met at the end of Rung 20.

[61] Trivedi, K. S., & Ercegovac, M. D. (1977). On-line algorithms for division and multiplication. IEEE Transactions on Computers C-26(7), 681–687.
The classical on-line multiplier, with bounded delay and growing internal state.

[62] Elaydi, S. (2005). An Introduction to Difference Equations, 3rd ed. Springer, New York.
Linear difference equations and their limits. The field of Rung 21.

[63] Coxeter, H. S. M. (1969). Introduction to Geometry, 2nd ed. Wiley, New York.
The crystallographic restriction: in a lattice only rotations of order 1, 2, 3, 4 and 6 occur.

[64] Rolfsen, D. (1976). Knots and Links. Publish or Perish, Berkeley.
Knot groups, complements and the fact that a knot group does not see chirality.

[65] Jones, V. F. R. (1985). A polynomial invariant for knots via von Neumann algebras. Bulletin of the AMS 12(1), 103–111.
The Jones polynomial, which tells the trefoil from its mirror.

[66] Kauffman, L. H. (1987). State models and the Jones polynomial. Topology 26(3), 395–407.
The bracket state sum used in jones_trefoil.py.

[67] Plofker, K. (2009). Mathematics in India. Princeton University Press.
The standard history of Indian mathematics: Piṅgala, Brahmagupta, the Kerala school. Start here for every Indian precursor.

[68] Ingerman, P. Z. (1967). “Pānini-Backus form” suggested. Communications of the ACM 10(3), 137.
Short letter linking Pāṇini’s grammar to formal grammars. Read with care: the comparison simplifies Pāṇini.

[69] Van Nooten, B. (1993). Binary numbers in Indian antiquity. Journal of Indian Philosophy 21(1), 31–50.
Piṅgala’s enumeration of metres as binary words and the rule for the place of a word.

[70] Perkins, F. (2004). Leibniz and China: A Commerce of Light. Cambridge University Press.
Leibniz’s binary arithmetic and his reading of the Yijing through Bouvet.

[71] Joseph, G. G. (2009). A Passage to Infinity: Medieval Indian Mathematics from Kerala and Its Impact. SAGE, New Delhi.
The Kerala series and the open question of their transmission to Europe.

[72] Ganeri, J. (2001). Philosophy in Classical India. Routledge, London.
Jain sevenfold predication and the four corners of Nāgārjuna, in their own setting.

[73] Colebrooke, H. T. (1817). Algebra, with Arithmetic and Mensuration, from the Sanscrit of Brahmegupta and Bháscara. John Murray, London.
The classical English translation of Brahmagupta’s rules for fortune, debt and zero.

[74] Euclid. Elements. Transl. T. L. Heath (1908), Cambridge University Press.
The first tree of mathematics from axioms.

[75] Whitehead, A. N., & Russell, B. (1910–1913). Principia Mathematica, 3 vols. Cambridge University Press.
Mathematics derived from logic and types.

[76] Bourbaki, N. (1950). The architecture of mathematics. American Mathematical Monthly 57(4), 221–232.
The three mother structures and their crossings.

[77] Klein, F. (1872). Vergleichende Betrachtungen über neuere geometrische Forschungen (Erlanger Programm). Erlangen.
Geometries ordered by their groups.

[78] Eilenberg, S., & Mac Lane, S. (1945). General theory of natural equivalences. Transactions of the AMS 58, 231–294.
The start of category theory: mathematics through arrows.

[79] Lawvere, F. W. (1964). An elementary theory of the category of sets. Proceedings of the National Academy of Sciences 52(6), 1506–1511.
Sets founded on arrows; a step towards topos theory.

[80] Grothendieck, A., & Dieudonné, J. (1960). Éléments de géométrie algébrique I. Publications Mathématiques de l'IHÉS 4.
Grothendieck's method of relations and general objects; the source of schemes and, later, toposes.

[81] Martin-Löf, P. (1984). Intuitionistic Type Theory. Bibliopolis, Naples.
The type theory from which homotopy type theory grew.

[82] Kauffman, L. H. (2021). Iterants, Majorana fermions and the Majorana–Dirac equation. Symmetry 13(8), 1373. doi:10.3390/sym13081373.
Algebras built from discrimination in simple discrete systems. Iterants give the complex numbers, Clifford and matrix algebras; the Dirac equation is solved by Rowlands' nilpotent method. Background to Rungs 0, 16 and 21.

[83] Martin-Löf, P. (1991). A path from logic to metaphysics. In: Atti del Congresso Nuovi problemi della logica e della filosofia della scienza, Viareggio, 8–13 gennaio 1990, Vol. II, pp. 141–149. CLUEB, Bologna.
A transcribed lecture. Martin-Löf follows the order of conceptual priority backwards to act and object, separates actual from potential truth, and treats error as found through incoherence. Background to Rung 0 and to the chapter on infinity.

[84] Shulman, M. (2018). Brouwer's fixed-point theorem in real-cohesive homotopy type theory. Mathematical Structures in Computer Science 28(6), 856–941. arXiv:1509.07584.
Adds modalities to homotopy type theory so that types carry both topology and homotopy. Proves that the shape of the topological circle is the higher inductive circle S^1. The bridge between the round and the continuum.

[85] Troelstra, A. S., & van Dalen, D. (1988). Constructivism in Mathematics, Vol. I. Studies in Logic and the Foundations of Mathematics 121. North-Holland, Amsterdam.
Standard account of intuitionism, including Brouwer's choice sequences and his continuum.

[86] Hardy, G. H., & Wright, E. M. (2008). An Introduction to the Theory of Numbers, 6th ed. Oxford University Press.
Kronecker's theorem (Theorem 439): the multiples of an irrational number are dense modulo one. Behind T16 and the continuum chapter.

[87] Sós, V. T. (1958). On the distribution mod 1 of the sequence nα. Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae, Sectio Mathematica 1, 127–134.
The three-distance theorem: n points kα modulo one leave at most three gap lengths.

[88] Bishop, E., & Bridges, D. (1985). Constructive Analysis. Grundlehren der mathematischen Wissenschaften 279. Springer, Berlin.
Real numbers as regular sequences of rationals, and their equality. The standard the value of a place is measured against.

[89] Diener, H. (2018). Constructive reverse mathematics. Habilitation thesis, University of Siegen. arXiv:1804.05495.
Proposition 1.4.2: LLPO is equivalent to deciding $x \le 0$ or $0 \le x$ for every real x. Behind the remark on ×3 places.

[90] Weihrauch, K. (2000). Computable Analysis: An Introduction. Springer, Berlin.
Admissible representations of the reals; the signed-digit representation is one, the ordinary radix representation is not.