One Strand Through the Sciences

The Vacuum.Net Foundations read across the sciences

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Contents

Preface 3

Part 0 — The Foundations and the Map 3

Part I — The Register 8

Chapter 1 — The Register: Mathematics, Logic and Computation 9

Part II — Strand and Field 28

Chapter 2 — Particles and Fields 29

Chapter 3 — Forces and Motion 41

Chapter 4 — The Cosmos 54

Chapter 5 — Engineering of Energy and Signal 63

Part III — Matter 74

Chapter 6 — The Elements, Molecules and Materials 75

Chapter 7 — Reactions and Heat 92

Chapter 8 — The Earth 104

Chapter 9 — Building and Making 118

Part IV — Life 132

Chapter 10 — The Code and the Cell 133

Chapter 11 — Metabolism and Interaction 148

Chapter 12 — Evolution and Development 160

Chapter 13 — Cultivating Life 175

Part V — Organism and Mind 187

Chapter 14 — Body and Brain 188

Chapter 15 — Perception, Behaviour and Mind 201

Chapter 16 — The Life Course 213

Chapter 17 — Healing 225

Part VI — Society 241

Chapter 18 — Social Structure 242

Chapter 19 — Markets, Power and Communication 254

Chapter 20 — The Cycles of Societies 263

Chapter 21 — Ordering Society 275

Part VII — Culture 290

Chapter 22 — Language and Sign 291

Chapter 23 — The Arts 301

Chapter 24 — Memory of Culture 314

Chapter 25 — Making Meaning 327

Synthesis — What the Sciences Say Back 339

Appendix A — The Calculations 345

Appendix B — References 471

Preface

This book reads the sciences through the Foundations of the Vacuum.Net, and it builds its map of the sciences on the same Foundations rather than on a classification borrowed from a library or an agency. The arithmetic of the trinity says how closed units become the material of larger ones (the carry), and how every closure breaks down into three elementary kinds. Those two facts give two axes, and the two axes give a map of twenty-five domains with more than a hundred disciplines inside them. Section 0.2 states exactly which parts of the map are proved, which are declared, which are observed and which are judged.

Each discipline is treated in the same way. Its history is told from its founders to its present questions. Its current state is given with sources, as of October 2026. Then the discipline is read through five fixed points — strand, passage, closure, scale, test — and the consequences are drawn. Finally the connection is graded: does the discipline already contain the structure of the Foundations in its own published results (A), can a stated observation refute the reading (B), or is it a reading only (C)? Every number the book computes itself is labelled (Calculation N.k) and given in Appendix A with the program that computes it and the exact output it printed. The printed literature of each chapter is listed in Appendix B; web sources are linked where they are used.

The grades are there because a theory that finds itself everywhere has found nothing. Re-describing a field in new words, and letting everything come out as confirmation, adds nothing to the field or to the theory. What counts is a structure the field already found on its own, or a test the field can run and the theory can fail.

A short first edition of this book (October 2026) stated several tests before their data were opened. This edition runs them, together with others, on public data. Twenty-four tests were run; most failed, a few passed weakly, and the Synthesis at the end sets them side by side. The failures are part of the result and are reported where they occur.

Part 0 — The Foundations and the Map

0.1 The Foundations in brief

The Foundations have one ground: the trinity, the three values −1, 0, +1, whose sum is zero. They need no axioms. A strand is a map σ: I → P from a course I (an interval of the integers) into a set of places P; a loop is a return of the strand to a place it has been. The trits are the digits of balanced ternary, and they add by the digit rule: a sum keeps its balanced residue mod 3 at its own scale and passes the rest on, whole, as a carry to the next scale (×3).

From the trinity follow, by proof:

- A sum of trits leaves no residue at its own scale if and only if it is 0 mod 3; whatever it does not keep passes on as carry. The residues form $\mathbb{Z}_3$, the smallest cyclic group in which a value differs from its reverse (T1).
- The shortest run of equal nonzero trits that leaves no residue has length three, so a window is three trits. It closes when its sum is 0 mod 3; 9 of the 27 windows close, in two kinds (T2–T4).
- Of the elementary closures, a standing and a meeting carry nothing to the next scale; a ring 1 + 1 + 1 = 10 carries one. An open pair 1 + 1 = 1T keeps a residue (T7.1).
- Every closed trit word breaks down completely into three elementary closures: a standing (0), a meeting (x, −x) and a ring (x, x, x) (T7).
- Two different closed windows cannot be joined without tension (T11).
- Open words, read from a start, number 3^n; closed configurations, read without a start, number g(n), the necklace count (T13). In the Eisenstein integers Z[ω] the trits give a finite address only to the elements that come to rest; the others end in a perpetual turn (T14).
- Merging meetings doubles, merging rings triples: a coarse passage holds $2^a \cdot 3^b$ fine passages (T15). The two ladders never meet exactly; their nearest meetings are the closure depths 1, 2, 5, 12, 41, 53, 306, 665, … and the comma (T16–T18).
- Under stated hypotheses — an embedding in space, unit-quaternion orientation states, field maps $S^3 \to S^2$, an explicit energy density, a free random path — the trefoil is the first knot, a rotor returns after 4π, the Hopf charge is conserved, the stiffness of a medium is a state function of its density, and the lever of a free path is −(1 + P)/P (T19–T24).

The five statements that earlier editions called axioms, N1–N5, are now readings: they say what this arithmetic means for a strand. N2 (closure) and N5 (scale: what does not close at one scale is the material of the next) are the readings of T1 and T7.1. N1 and N3 are definitions. N4 (context) is the hypothesis of T23. This book uses N2 and N5 in that sense.

0.2 The map of the sciences, derived

A science studies some strand at some scale, and asks some kind of question about its closures. The Foundations fix both the scale and the kinds of question.

The rows: levels of scale (T1, T7.1; reading N5). The carry of a closure passes one scale up: a closed configuration at one scale becomes a single passage of the coarser strand. A new level therefore begins wherever a new kind of closed unit appears and starts to behave as one element of something larger. The arithmetic gives the rule; it does not give the number of levels. Observation gives seven:

Level	The closed unit	Becomes a passage of
L0 Register	the trit, the window, the number	every description

Level	The closed unit	Becomes a passage of
L1 Strand and field	the particle, the photon, the field mode	atoms
L2 Matter	the atom, the molecule, the crystal, the planet	cells
L3 Life	the code, the cell, the population	organisms
L4 Organism and mind	the body, the brain, the person	societies
L5 Society	the network, the market, the state	cultures
L6 Culture	the sign, the work, the tradition	—

The columns: the three elementary closures and tension (T7, T1). Every closed configuration reduces to standings, meetings and rings, and what does not close passes on as tension. At every level a science can therefore ask four kinds of question:

Column	From the Foundations	The question
Standing	the closure of length 1 (0)	What is the closed unit — its structure and state?
Meeting	the closure of length 2 (x, −x)	How do units exchange — go and come back?
Ring	the closure of length 3 (x, x, x)	Which cycles go round — process, development, history?
Tension	the open window, which keeps a residue (T1)	What does not close, and how is it made to close?

The tension column holds the practical sciences: engineering, medicine, agriculture, law, education, design. They work where the strand does not yet close.

The map. Seven levels, of which the register (L0) is treated as one domain, and four columns give twenty-five domains:

Level	Standing	Meeting	Ring	Tension
L0 Register	Ch. 1 Mathematics, logic and computation	(Ch. 1)	(Ch. 1)	(Ch. 1)
L1 Strand and field	Ch. 2 Particles and fields	Ch. 3 Forces and motion	Ch. 4 The cosmos	Ch. 5 Engineering of energy and signal

Level	Standing	Meeting	Ring	Tension
L2 Matter	Ch. 6 The elements, molecules and materials	Ch. 7 Reactions and heat	Ch. 8 The Earth	Ch. 9 Building and making
L3 Life	Ch. 10 The code and the cell	Ch. 11 Metabolism and interaction	Ch. 12 Evolution and development	Ch. 13 Cultivating life
L4 Organism and mind	Ch. 14 Body and brain	Ch. 15 Perception, behaviour and mind	Ch. 16 The life course	Ch. 17 Healing
L5 Society	Ch. 18 Social structure	Ch. 19 Markets, power and communication	Ch. 20 The cycles of societies	Ch. 21 Ordering society
L6 Culture	Ch. 22 Language and sign	Ch. 23 The arts	Ch. 24 Memory of culture	Ch. 25 Making meaning

The status of the map. Each part of the map has its own status.

- *Proved:* the number of columns. T7 shows that every closed word reduces to exactly three elementary closures, and N2 adds one way of not closing.
- *Corresponded:* the reading of each closure as a kind of scientific question — standing as structure, meeting as exchange, ring as cycle, tension as practice. This is a declared identification, not a theorem.
- *Observed:* the rows. N5 gives the rule (a closed unit becomes a passage of the next scale); the seven levels are taken from observation. The register L0 is not a level in the sense of N5 at all; it is the notation in which every level is described, and it is placed first by convention.
- *Judged:* the placing of each discipline in a cell. Many disciplines ask more than one kind of question; geology studies structure as well as cycles, psychology structure as well as exchange. Each is placed where its central question lies, and the chapters say where it straddles.

The map is a claim that can fail. It fails if a discipline fits no cell, or fits two cells equally well with no way to decide.

0.3 The five points and the grades

Every discipline is read through the same five points:

1. **Strand.** What are I, P and σ in this discipline?
2. **Passage and trit.** What counts as a return to a place, and how are +1, 0, −1 read?
3. **Closure.** What is a closed window, and what is tension?
4. **Scale.** Which blocks merge, and are they meetings (×2) or rings (×3)?
5. **Test.** Which observation would refute the reading?

And every discipline ends with a grade:

Grade	Meaning
A — Structural	The discipline already contains the mathematics of the Foundations in its own published results, independently of this theory.
B — Testable	The reading fixes the five points and names an observation that could refute it.
C — Reading	The five points can be filled, but no test is yet stated.

Part I — The Register

Chapter 1 — The Register: Mathematics, Logic and Computation

The register is the one level where the Foundations are not read into a field but are part of it. Balanced ternary, the cyclic group of order three, necklace counting, the continued fraction of $\log_2 3$ and the Eisenstein integers are standard mathematics, older than the theory by decades or centuries. What the Foundations add is a choice and a reading: they take the trinity −1, 0, +1 as the only ground, prove from digit arithmetic that a sum closes at its scale exactly when it is 0 mod 3 and otherwise passes a carry upward (T1, T7.1), and read this as closure and scale. This chapter separates the two for seven disciplines. Number theory, logic, combinatorics, information theory and computer science contain the structure as published fact (grade A for the structures named); dynamical systems contains the two ladders as published fact but not a preference for them (A for Sharkovskii's order, C beyond it); AI contains the ternary alphabet in working systems (A) and admits one clean test of the closure rule on trained weights (B). Nothing in this chapter shows that the register is the register of nature; that question belongs to the chapters that follow.

1.1 Number theory and arithmetic

History. Balanced ternary is older than most of modern algebra. Michael Stifel set it out in his *Arithmetica Integra* of 1544, and Kepler and Léon Lalanne worked with it later (Wikipedia, Balanced ternary). Its most concrete form is the two-pan balance. Weights of 1, 3, 9 and 27 units, placed on either pan, measure every whole load from 1 to 40: a weight on the far pan is +1, a weight on the near pan is −1, a weight left off is 0. Euler later stated the uniqueness of such representations as an identity of formal series, the product over k of (x^(−3^k) + 1 + x^(3^k)) equal to the sum of all powers of x (Andrews 2007). Knuth gave the system its modern place in *The Art of Computer Programming* (1997), and wrote that "perhaps the symmetric properties and simple arithmetic of this number system will prove to be quite important some day" (Wikipedia, Balanced ternary).

The second strand of history is the relation between the primes 2 and 3. In 1343 Levi ben Gershon (Gersonides) wrote *De numeris harmonicis* at the request of the bishop of Meaux, answering a question of the composer Philippe de Vitry. Vitry's "harmonic numbers" were the numbers 2^m·3^n — the ladder of the Foundations (T15). Gersonides proved that only four pairs of them differ by one: (1, 2), (2, 3), (3, 4) and (8, 9) (Wikipedia, Gersonides). Eugène Catalan conjectured in 1842–1844 that 8 and 9 are the only consecutive perfect powers of any kind. Preda Mihăilescu proved it in April 2002, published in 2004 in the *Journal für die reine und angewandte Mathematik* (Wikipedia, Catalan's conjecture). The irrationality of $\log_2 3$, which T16 states as $2^n \neq 3^k$, is the elementary floor under both results; the theory of continued fractions, from Euler and Lagrange to Khinchin's textbook, supplies the best approximations that T17 calls closure depths.

The third strand is the arithmetic of the cube roots of unity. Euler used numbers of the form a + bω in a proof, which is why they are sometimes called Eulerian integers, and Gotthold Eisenstein gave the ring Z[ω] its name. The ring is Euclidean, its units are the six sixth roots of unity, and its integers form a triangular lattice in the plane (Wikipedia, Eisenstein integer). The prime 3 is special there: it is, up to a unit, the square of the prime 1 − ω, and the residue field Z[ω]/(1 − ω) has exactly three elements. This is the "algebraic home" of the trit in the Foundations (III.1, remark).

The fourth strand is the 3x + 1 problem of Lothar Collatz. It sits exactly on the seam between the two ladders: an odd step multiplies by three, even steps divide by two. The questions the history left are therefore the questions of the two ladders. Why do ×2 and ×3 never close exactly, how closely can they approach, and what does a near-closure force?

Current state. The two-ladder questions are active and partly computational. The Collatz conjecture has been checked for every starting value up to 2^71, about 2.36 × 10^21, by David Bařina, published in the *Journal of Supercomputing* in 2025 (VUT record). Any nontrivial cycle must be longer than 217,976,794,617 steps in the shortcut form (Wikipedia, Collatz conjecture). The structure behind such bounds is the continued fraction of $\log_2 3$. Eliahou (1993) showed that any nontrivial cycle whose least element exceeds 2^40 has a length of the form 301994a + 17087915b + 85137581c, with b ≥ 1 and ac = 0, so that the smallest admissible lengths are 17,087,915, 17,389,909 and 17,691,903 (MaRDI record of Eliahou 1993). The three coefficients are not arbitrary. Each is the numerator of a convergent of $\log_2 3$: 301994/190537, 17087915/10781274 and 85137581/53715833 (Calculation 1.3). The denominator 190537 is the last closure depth listed in T17 below 10^6. In other words, a published bound on Collatz cycles is built from exactly the near-closures of the two ladders. No cycle of k "rises" exists for k ≤ 68 (Simons and de Weger 2005) and, by Hercher (2023), for k ≤ 91. Terence Tao showed in 2019 that almost all orbits, in logarithmic density, fall below any function of the start that tends to infinity (Tao's blog). Lagarias's verdict of 2010 still stands: "an extraordinarily difficult problem, completely out of reach of present day mathematics."

Elsewhere in the field the largest known prime is 2^136,279,841 − 1, with 41,024,320 digits, found by Luke Durant through GIMPS and confirmed on 12 October 2024 (GIMPS). Bounded gaps between primes went from 70 million (Zhang 2013) to 600 (Maynard) to 246 (Polymath8b), which remains the established bound; several 2026 claims below 246 were not peer-reviewed as of September 2026 (Wikipedia, Prime gap). Gaitsgory, Raskin and seven colleagues posted a proof of the geometric Langlands conjecture in February 2024, more than 800 pages over five papers (Quanta). Formal proof has become infrastructure: the Lean library mathlib holds 139,679 definitions and 291,414 theorems from 772 contributors (mathlib statistics). In Leiden, Marco Streng's work on curves with complex multiplication by Z[ω] is the setting in which the Foundations' T14 was found (Foundations, references).

Interpretation. Here the theorem checklist is applied to arithmetic itself.

1. *Strand.* I is the sequence of digit positions read from the least significant upward; P is the set of residues. σ is the digit expansion. A "return" is a later digit position landing on a residue class already met.
2. *Passage and trit.* The trit is the balanced digit d ∈ {T, 0, 1} with d ≡ s (mod 3). This is T1: given the trinity as ground, the digit rule is proved arithmetic, not a reading (Calculation 1.1). The word of 8 is 10T and of −8 is T01: negation is mirroring (T12.1).
3. *Closure.* A sum closes at its scale when its digit is 0. A meeting (1 + T) closes with carry 0; a ring (1 + 1 + 1 = 10) closes with carry +1 and becomes one unit of the next position (T7.1). An open pair (1 + 1 = 1T) keeps a nonzero digit. This is the ordinary carry of balanced ternary, nothing more.
4. *Scale.* Shifting the word one place multiplies by three (III.2 d). The ladder 2^a·3^b (T15) is Vitry's set of harmonic numbers. T16 is the parity argument; Gersonides's four pairs are the only exact near-meetings at distance one (Calculation 1.4). T17 lists the best near-meetings; Eliahou's coefficients are numerators of the same convergents. T13's necklace count is a standard formula (OEIS A001867). T14 extends the digit map to Z[ω] and finds a "turn" at ±(1 + ω).
5. *Test.* For arithmetic the theorems are proofs, so they cannot be refuted by data. One open question is empirical in the mathematical sense: IX.3 asks whether the share of Eisenstein integers whose digit orbit ends in the turn settles at a limit. The prediction that it settles monotonically fails on the data computed here: 27.3 %, 41.0 %, 32.2 %, 23.3 % and 26.9 % for boxes of half-width 40, 100, 200, 400 and 800 (Calculation 1.12). The share oscillates with box size. That is an honest negative: no limit is visible yet.

Consequences. If the reading holds, number theory gains no new theorem from the Foundations; every proved statement in Parts III–VI is either classical or a short exercise. The honest division is this. Standard mathematics: balanced ternary and its uniqueness (Stifel, Euler, Knuth); the carry; $\mathbb{Z}_3$ as the residue field of Z[ω] at 1 − ω; the irrationality of $\log_2 3$ and Lagrange's theorem on best approximations; Burnside's count of necklaces. What the Foundations add: the choice of the trinity as the only ground; the decision to read "digit 0" as closure and "carry" as passage to the next scale; the names standing, meeting and ring for the three ways a sum of trits can leave digit 0; the rewrite R and its fixed points (T10); the seam count of T11; and the digit map in Z[ω] with its two fixed turns (T14), which I have not found in the literature in this form, although complex-base number systems for Gaussian integers (Knuth's base 2i, 1955; Penney's base −1 + i, 1965) are its close relatives (Wikipedia, Complex-base system).

The arrangement makes one observation sharper: the Collatz cycle bounds and the closure depths are the same continued fraction, used from two sides. A number theorist would say this is expected, since both problems ask how closely 3^k approaches 2^n. That is right, and it is the point: the Foundations' T17 is not decoration on the Collatz problem but its known Diophantine core. The reading also produces one concrete open problem, the behaviour of the turn share in Z[ω], which the computation here shows is not trivial: the share rises, falls and rises again as the

box grows. A clean statement of the limit, if it exists, would be a small new result in the arithmetic of Z[ω], and Leiden, where Streng works on complex multiplication by this ring, is the natural place to settle it. The cost is that the reading of the carry as "becoming one passage of the next scale" is a word, not a theorem about anything outside arithmetic. Arithmetic does not care whether a carry is called a passage.

Theorem checklist. T1 and T7.1 apply as proved arithmetic (Calculation 1.1). T2–T5 are the residue counts of $\mathbb{Z}_3^3$ and need no physical reading. T6 applies wherever digits are drawn at random; for uniform digits it gives 1/3. T7–T10 are statements about words and rings over $\mathbb{Z}_3$ that a number theorist would call exercises in linear algebra over F_3; they are correct. T11 is a sliding-window count (1.5). T12 is Stifel's and Euler's theorem, proved. T13 is Burnside's necklace formula, proved (Calculation 1.2). T14 applies directly and is the one place where the Foundations pose an arithmetic question that is not in the textbooks (Calculation 1.12). T15–T18 apply in full: Vitry's harmonic numbers, Gersonides's four pairs, Catalan–Mihăilescu and the Eliahou coefficients (Calculations 1.3, 1.4). T19–T22 belong to topology, not arithmetic, and do not apply here. T23–T24 do not apply.

Grade. A for balanced ternary, the carry rule, the 2–3 near-closures (Gersonides, Catalan, Eliahou) and Z[ω] as published mathematics; C for the turn share in Z[ω], which is an open question (IX.3) on which the computed shares oscillate (Calculation 1.12).

1.2 Logic (classical, many-valued, deontic)

History. Classical logic has two values, and from Aristotle to Boole and Frege it was built on the law of the excluded middle. Aristotle himself left the opening for a third value in the problem of future contingents — whether "there will be a sea battle tomorrow" is now true or false. Charles Sanders Peirce drew three-valued truth tables in notes that were published only much later (Fisch and Turquette 1966). Jan Łukasiewicz published a short note on three-valued logic in 1920, offering the third value as modal possibility. Emil Post generalised functional completeness to n values in 1921. Stephen Kleene introduced his "strong" three-valued tables in 1938 for partial recursive functions, and Dmitri Bochvar in the same year treated the Liar as neither true nor false (SEP, Many-valued logic). Kleene's tables became the logic of undefined values in computing; SQL's handling of NULL uses a fragment of Kleene's K3 (Wikipedia, Three-valued logic).

The values of these logics are written in two ways. Łukasiewicz used 0, ½, 1; the balanced notation uses −1, 0, +1. In both Łukasiewicz's and Kleene's systems negation maps true to false, false to true, and leaves the middle value fixed. Post's ternary logic instead uses a cyclic negation, 0 → 1 → ½ → 0. The difference matters here. Kleene–Łukasiewicz negation is mirroring; Post negation is rotation. The Foundations use both operations: mirroring is the negation of an address (T12.1), rotation is the cyclic shift of a ring (T9).

Deontic logic grew beside this. Leibniz noted in his *Elementa juris naturalis* (1669–1671) that the permitted, the prohibited, the obligatory and the optional relate as the

possible, the impossible, the necessary and the contingent. Ernst Mally's *Grundgesetze des Sollens* (1926) was the first formal attempt; Karl Menger showed that one of its theorems made the "ought" sign idle. Georg Henrik von Wright's paper "Deontic Logic" in *Mind* 60 (1951), pages 1–15, founded the modern field (Wikipedia, Deontic logic). Its normalised form, Standard Deontic Logic, is classical propositional logic with necessitation, the K axiom and the D axiom (what is obligatory is permitted). Roderick Chisholm's puzzle of 1963 showed that this system cannot represent contrary-to-duty obligations, and the field has since been driven by that failure (SEP, Deontic logic). The tradition also kept a "threefold classification" of acts: obligatory, optional, impermissible.

Peirce's semiotics belongs here too. In 1903 he crossed three trichotomies of signs, each with three values, giving 27 combinations, and argued that only 10 are coherent classes (Wikipedia, Peirce's classes of signs). The rule is that a later trichotomy may not take a value "higher" than an earlier one.

Current state. Many-valued and deontic logic are mature fields with active frontiers rather than single headline numbers. The Stanford Encyclopedia's deontic entry, substantively revised in March 2021, describes the move beyond normal modal logic: Horty's prioritised defaults, input/output logics, dynamic, paraconsistent and hyperintensional deontic systems, and the 2013 *Handbook of Deontic Logic and Normative Systems* (SEP; Wikipedia). In many-valued logic the main work is on algebraic semantics (MV-algebras, Gödel and product logics) and on the fuzzy logics descending from Zadeh (SEP). Three-valued logic is used daily in database query languages, in hardware description languages with "unknown" and "high-impedance" states, and in the tri-state outputs of CMOS circuits (Wikipedia, Ternary numeral system).

The open debates are about what a third value means. For Łukasiewicz it was possibility, for Kleene "undefined", for Bochvar "meaningless", and for the paraconsistent tradition "both true and false". These readings give different conjunction and implication tables over the same three values. The deontic field argues about whether obligations attach to propositions ("it ought to be that") or to actions ("one ought to do"), and about how to represent norms that conflict. The second debate is the one that matters for the register, because it decides whether deontic status adds like a digit or only orders like a rank.

Two facts are directly relevant to the register. First, Kleene–Łukasiewicz negation on $\{-1, 0, +1\}$ is exactly trit-wise mirroring (Calculation 1.8). Second, the deontic statuses read as obligatory = $+1$, optional = 0, forbidden = -1 obey the same mirror under negation of the content: "obligatory p" is "forbidden not-p", and "optional p" is "optional not-p". This is von Wright's square, and the check is mechanical (Calculation 1.8). Peirce's count is the other direct link. His 10 classes of 27 sit beside the 9 closed windows of 27 in T3. The two counts are not the same subset: under the map $1 \to 1$, $2 \to 0$, $3 \to$ T, only 4 of Peirce's 10 classes close (Calculation 1.7). Peirce's rule is an order rule (non-increasing), the Foundations' rule is a sum rule.

Interpretation.

1. *Strand.* I is the sequence of propositions or acts in a normative text or an argument; P is the set of contents (states of affairs). A return is a second statement about the same content.
2. *Passage and trit.* The truth value or the deontic status is the trit: true/obligatory +1, undetermined/optional 0, false/forbidden −1. Negation of content mirrors the trit (T12.1). This is a published structure of the field, not an addition.
3. *Closure.* A window of three statuses about one content closes when its sum is 0 mod 3 (T2). By T4 that means all three agree, or all three differ. "Obligatory, optional, forbidden" about one act is a closed heterogeneous window; "obligatory, obligatory, forbidden" is open. The logic does not say that the closed window is consistent — it is not; it is a complete survey. So the Foundations' "closure" is not logical consistency. It is coverage: all three values present, or one value held three times.
4. *Scale.* Meetings (x, −x) are pairs of contradictory statuses: p obligatory and p forbidden. Chisholm's contrary-to-duty structure has a primary duty and a secondary duty that applies when the first is violated, a pair. T11 says two different closed windows cannot be joined without tension. No published deontic result corresponds to T15, T17 or T19–T24.
5. *Test.* A test that could fail: in a corpus of legal texts annotated with deontic statuses, read the sequence of statuses about each norm addressee in order and count windows. T6 gives the closure probability from the observed frequencies of the three statuses; under independence it is 1/3 for a uniform law. If legal drafting has an excess of closed windows over the T6 baseline, the closure rule picks out something in normative text; if the closed share equals the baseline, it adds nothing. This has not been run.

Consequences. If the interpretation holds, it gives logic no new theorem. The mirror negation and the three deontic statuses were in place long before the Foundations. What the reading adds is a single register for truth values, deontic statuses and signed digits, so that a status can be stored, negated and summed as a balanced-ternary digit. This is useful in software that has to evaluate norms, since a sum over statuses becomes ordinary arithmetic and negation is a sign flip.

The reading also exposes a choice the Foundations make silently. Kleene–Łukasiewicz negation is a mirror (1 ↔ T, 0 fixed); Post negation is a rotation (0 → 1 → T → 0). The Foundations use the mirror for negation of addresses (T12.1) and the rotation for the turn of a ring (T9, T14). Logic has known for a century that these are different operations with different completeness properties: Post's rotation together with min or max is functionally complete, the Kleene mirror is not. So the register carries both logics, and a reader must say which one is meant.

The cost is that "closure" here does not mean consistency, and a reader who equates them will draw false conclusions. A closed window "obligatory, optional, forbidden" about one act is a full survey of the deontic options, not a coherent rule. Peirce's 10 of 27 is not the Foundations' 9 of 27, and should not be cited as confirmation: the two rules overlap in only four classes. The three-valued reading of deontic status is where

logic and the register genuinely meet; but it is the deontic square, not the closure rule, that carries the meeting.

Theorem checklist. T1 applies to statuses only if statuses are added; deontic logic does not add them, so the digit rule is a convention here. T2–T4 apply as counts of status windows. T6 gives the baseline for the corpus test. T7 decomposes a status word into standings (optional), meetings (an obligation and a prohibition of one content) and rings (three equal statuses); I know of no deontic result that uses this decomposition. T8–T10 have no counterpart. T11 has a loose counterpart in contrary-to-duty structures, where two normative frames join with conflict, but no published count. T12.1 (mirror) applies exactly to negation. T13–T24 do not apply. Peirce's 10 of 27 is checked against T3 in Calculation 1.7 and found to be a different subset.

Grade. A for three-valued negation and deontic status as trit mirror (Łukasiewicz 1920, Kleene 1938, von Wright 1951); B for the window test on annotated legal corpora (stated, not run); C for "closure" as a logical notion.

1.3 Combinatorics

History. Counting arrangements up to symmetry is the oldest part of the subject that touches the Foundations. The number of necklaces of n beads in k colours — words up to rotation — was found in the nineteenth century, and the formula (1/n)·Σ φ(n/d)·k^d is the standard example of what is now called Burnside's lemma. George Pólya generalised it in 1937 into his enumeration theorem, written to count chemical isomers. Roger Lyndon's words (1954) — aperiodic necklaces written in their least rotation — became the building blocks of de Bruijn sequences and of the factorisation of free monoids (Wikipedia, Necklace).

The second line runs through Ramsey theory: the claim that complete disorder is impossible in large structures. Frank Ramsey proved his theorem, Erdős and Szekeres gave the 4^k upper bound for diagonal Ramsey numbers in 1935, and the question of how large a set can be while avoiding a pattern became central. For the ternary register the decisive case is the cap set problem. A cap set is a subset of F_3^n with no three points on a line, and in F_3^n three distinct points a, b, c are on a line exactly when a + b + c = 0 coordinate by coordinate. The card game SET made this famous. Marsha Falco designed it in 1974 from a coding scheme she used as a geneticist; it was published in 1991 with 81 cards, each with four attributes of three values, and three cards form a "set" when each attribute is all the same or all different (Wikipedia, Set). The largest collection without a set has 20 cards; Pellegrino proved the four-dimensional maximum in 1970, and Knuth later counted 682,344 such 20-card collections (Wikipedia, Cap set; Wikipedia, Set).

The question the history left was whether cap sets are an exponentially small part of F_3^n. Results of 1995 and 2012 showed only that they are a vanishing fraction.

Current state. In May 2016 Ernie Croot, Vsevolod Lev and Péter Pál Pach used the polynomial method on the analogous problem in $(Z/4)^n$. Within about ten days Jordan Ellenberg and Dion Gijswijt adapted the argument and proved that a cap set in F_3^n has at most about 2.756^n points (Quanta; arXiv:1605.09223). Gowers said the result "in a

certain sense … completely finishes the problem.” The constant is the minimum of (1 + t + t²)/t^(2/3), which is 2.7551 (Calculation 1.6). The exact maxima are known only to dimension six: 1, 2, 4, 9, 20, 45, 112 (Wikipedia, Cap set). In December 2023 FunSearch, a DeepMind system pairing a language model with an evaluator, found a cap set of 512 points in dimension 8, published in *Nature* 625 (2024) (Wikipedia, FunSearch). The lower-bound base for large n remains near 2.2 (Wikipedia, Cap set); the gap between about 2.2 and 2.756 is open.

Elsewhere, diagonal Ramsey numbers received their first exponential improvement since 1935: Campos, Griffiths, Morris and Sahasrabudhe proved R(k) ≤ (4 − ε)^k (arXiv:2303.09521). R(5, 5) lies between 43 and 46, the upper bound by Angeltveit and McKay in 2024 (Wikipedia, Ramsey’s theorem). SAT solvers now settle combinatorial questions by exhaustion: Heule, Kullmann and Marek showed in 2016 that {1, …, 7824} can be 2-coloured without a monochromatic Pythagorean triple and {1, …, 7825} cannot, with a proof of about 200 terabytes (Wikipedia). The run took about four CPU-years over two days on the Stampede supercomputer, and the proof compressed to 68 gigabytes; Ronald Graham’s $100 prize went to Heule.

Two features of the present field matter for the register. The first is the polynomial method itself. Its key step writes the indicator of “x + y + z = 0” in F_3^n as a polynomial and bounds its slice rank. The closure rule is thus not only the object of the theorem but also its tool. The second is the size of the open gap. Upper bound about 2.756^n, lower bound about 2.2^n: nobody knows within an exponential factor how many points avoid every closed triple. Tao called the cap set problem “a model problem” for Ramsey theory (Quanta). The exact maximum in dimension seven is unknown.

Interpretation.

1. *Strand.* I is the sequence of coordinates of a vector in F_3^n (or the attributes of a SET card); P is the set of three values per coordinate.
2. *Passage and trit.* Each coordinate is a trit. Identify F_3 = {0, 1, 2} with {0, 1, T}; a SET card is a word of four trits.
3. *Closure.* Three cards form a SET exactly when every attribute window closes: all equal or all different. This is T4, word for word, and the equivalence “sum 0 mod 3 ⇔ all same or all different” is checked exhaustively in Calculation 1.6. A cap set is therefore a set of words with no triple whose coordinate-wise window closes everywhere. The number of lines in F_3^n, 3^(n−1)(3^n − 1)/2, is the number of closing triples (1, 12, 117, 1080 for n = 1 to 4; Calculation 1.6). T13 is the necklace count; the ternary necklaces 3, 6, 11, 24, 51, 130, … and Lyndon words 3, 3, 8, 18, 48, 116, … are standard (Calculation 1.2).
4. *Scale.* Lines in F_3^n are rings in the sense of T7: x, x + d, x + 2d. Meetings appear as the pairs x, −x. The cap set maxima 1, 2, 4, 9, 20, 45, 112 are not on the ladder beyond 9 ($20 = 2^2 \cdot 5$, $45 = 3^2 \cdot 5$, $112 = 2^4 \cdot 7$), and the bases 20^(1/4) = 2.115, 112^(1/6) = 2.196 and the FunSearch construction 512^(1/8) = 2.181 grow slowly towards the unknown limit (Calculation 1.6). The ladder plays no role there.
5. *Test.* The reading is a definition, so it cannot fail as such. A test with content would be to derive the Ellenberg–Gijswijt constant from the register — from the closure

probability of T6 or the trit cube of T5. I find no such derivation; the constant comes from the polynomial method and the slice rank, not from any theorem of the Foundations.

Consequences. Combinatorics already contains the Foundations' closure rule as its central object in F_3^n. The equivalence of T4 is the rule of SET; the cap set problem is the problem of avoiding closed windows in every coordinate at once. That is a strong existing hook and it is entirely independent of the theory. The same rule is Fox's three-colouring condition at a knot crossing (T19), so one rule of 1960s knot theory and one rule of a 1990s card game are the same as T4. The Foundations note this; knot theorists and combinatorialists had it before.

The Foundations add nothing to the cap set bounds. They do add a perspective worth recording, with care. The Ellenberg–Gijswijt theorem says that a set avoiding everywhere-closed triples must be exponentially thin. The reading N2 says that what does not close does not persist. These are different statements: the cap set theorem is about sets that avoid closure, N2 about loops that achieve it. The overlap is that in F_3^n closure is unavoidable at scale — any set of more than about 2.756^n points contains a closed triple. One may read this as "closure is forced by size," which fits the spirit of T7.1 (a large enough sum must carry), but it is a phrase, not a theorem.

The necklace count T13 is classical and correctly cited; the Foundations' use of it — open words counted from a start, 3^n, against closed configurations counted without one, g(n) — is the standard distinction between words and necklaces. Its ratio $g(n)/(3^n/n)$ tends to 1 quickly: 1.070 at n = 6, 1.0018 at n = 12, 1.000002 at n = 24 (Calculation 1.2). The cost of the reading is nil and its gain is a clean vocabulary for a classical count.

Theorem checklist. T1–T4 apply exactly: a line of F_3^n is a triple whose every coordinate window closes (Calculation 1.6). T5's trit cube is F_3^3, whose 117 lines are the closed triples of Calculation 1.6. T6 gives the probability that a random triple is a line in one coordinate (1/3 for uniform coordinates; by independence 3^{-n} over n coordinates, before excluding equal points). T7 has no published counterpart in cap set theory. T8–T10 are statements about periodic words and a linear cellular automaton over F_3 — standard objects of combinatorics on words. T11 counts check failures at block joins. T12 does not apply. T13 applies exactly (Calculation 1.2). T14–T18 do not apply to cap sets. T19 applies in the sense that Fox three-colouring is the same rule as T4. T20–T24 do not apply.

Grade. A for the closure rule as the line condition of F_3^n (SET, cap sets, Ellenberg–Gijswijt 2017) and for T13 as the classical necklace count; C for any further claim.

1.4 Dynamical systems

History. The subject began with the three-body problem. Euler found three collinear periodic solutions in 1767 and Lagrange the equilateral ones in 1772. Henri Poincaré showed in 1889 that the problem has no further analytic integrals, and in doing so found the tangle of orbits now called chaos. Karl Sundman gave a convergent series solution in 1912, of no practical use (Wikipedia, Three-body problem). After the Second

World War the subject moved to iterated maps. In 1964 Oleksandr Sharkovsky proved that the periods of a continuous map of an interval obey a strict order: $3 \triangleright 5 \triangleright 7 \triangleright \dots \triangleright 2{\cdot}3 \triangleright 2{\cdot}5 \triangleright \dots \triangleright 4{\cdot}3 \triangleright \dots \triangleright \dots \triangleright 8 \triangleright 4 \triangleright 2 \triangleright 1$. If a map has a periodic point of some period, it has periodic points of every later period in this order (Wikipedia, Sharkovskii's theorem). Period 3 is first and forces all others; the powers of two are last. Tien-Yien Li and James Yorke rediscovered the head of the order in 1975 as "Period three implies chaos" (*American Mathematical Monthly* 82, 985–992).

Robert May's review in *Nature* (1976, vol. 261, 459–467) showed ecologists that the logistic map $x \to rx(1 - x)$ passes from a fixed point through a cascade of cycles to apparent randomness (Nature). Mitchell Feigenbaum found in 1975 that the cascade has universal ratios, $\delta \approx 4.6692$ and $\alpha \approx 2.5029$, and published in 1978 (*Journal of Statistical Physics* 19, 25–52); Oscar Lanford proved the universality with computer assistance in 1982 (Wikipedia, Feigenbaum constants). The bifurcation points of the logistic map are $r = 3$ (period 2), 3.4494896 (period 4), 3.5440903 (period 8) and 3.5644073 (period 16), accumulating at 3.56995; a period-3 window opens at $1 + \sqrt{8} \approx 3.8284$ and is followed by a $3{\cdot}2^k$ cascade (Wikipedia, Logistic map). The cellular automata of von Neumann, Conway's Game of Life (1970) and Wolfram's classification added discrete dynamics on lattices; Matthew Cook proved rule 110 universal, published in 2004 (Wikipedia, Cellular automaton).

Current state. The three-body problem is again an experimental science of orbits. Li and Liao reported 695 families of periodic planar solutions for equal masses in 2017, and 1,223 for unequal masses in 2018 (Wikipedia). In August 2025 they reported 10,059 new three-dimensional periodic orbits, 1,996 of them linearly stable, including 21 "choreographies" in which three equal masses share one closed curve (arXiv: 2508.08568). Period doubling has been seen in convection, in electronic circuits, in heart cells and in laboratory insect populations, and it is now a textbook route to chaos rather than a frontier. The frontiers are high-dimensional and non-autonomous systems, data-driven identification of dynamics, and computer-assisted proof. Universality has stayed a fact of the field for half a century: $\delta \approx 4.6692$ governs the shrinking of the intervals between doublings for every smooth one-hump map, and the period-3 window is itself followed by a $3{\cdot}2^k$ cascade with the same δ (Wikipedia, Logistic map). Lyubich later gave a proof of universality that does not rest on computer assistance (Wikipedia, Feigenbaum constants).

The open debates are less about maps than about data. How much of the period doubling seen in ecology, cardiology and climate records is the cascade of a low-dimensional map, and how much is noise shaped by a cycle? The test of the Foundations below touches exactly that question.

For the register, Sharkovskii's order is the published fact that matters. It places period 3 at the head and the powers of 2 at the tail, and in between all numbers $2^k{\cdot}m$ with m odd. It is the order of the two ladders: the $\times 2$ ladder is the tail, and three is the first odd period. It does not, however, prefer 3-smooth periods over others. The number of real parameters c at which $x^2 + c$ has a superstable cycle of exact period n is 1, 1, 1, 2, 3, 5, 9, 16, 28 for $n = 1$ to 9 (Calculation 1.9). Periods 5 and 7, off the ladder, have as many windows as their neighbours on it.

Interpretation.

1. *Strand.* I is discrete time; P is the state space; σ is the orbit. A loop is a return of the orbit to a point, a periodic orbit.
2. *Passage and trit.* Near a cycle, each passage can be read relative to the first: the deviation grows (+1), shrinks (−1) or is neutral (0). This is a reading, not a published structure. The trit rule of the Foundations applies literally to a symbolic orbit on a finite partition, and gives a trit word; I know of no published result that uses it.
3. *Closure.* A periodic orbit is a closed loop. In T10 the rings closing everywhere are the fixed points of the rewrite R; in dynamics a fixed point of the n-th iterate is a period-n orbit. The correspondence is formal: T10's R is a linear cellular automaton over $\mathbb{Z}_3$, and its fixed points and cycles are a small, exactly computed example of discrete dynamics. Its non-fixed rings end in cycles of length 3 (n = 3), 2 (n = 4), 8 (n = 5), 2, 3 and 6 (n = 6), 26 (n = 7), 2 and 4 (n = 8), 3 and 9 (n = 9), 2 and 8 (n = 10) (Calculation 1.14). All are on the ladder 2^a·3^b except 26 = 2·13 at n = 7. So the Foundations' own rewrite does not stay on its own ladder.
4. *Scale.* The period-doubling cascade climbs the ×2 ladder: 1, 2, 4, 8, … The 3·2^k cascade after the period-3 window climbs 3, 6, 12, … — the ladder 2^a·3^b with b = 1. Both are published facts (May 1976; Feigenbaum 1978). Sharkovskii's order ends in the ×2 ladder and begins at 3. T15 thus fits the cascades. It does not fit the windows: periodic windows of every period appear, on and off the ladder (Calculation 1.9).
5. *Test.* A test that can fail: in an experimental system with a stable period-3 window, the Foundations' reading predicts that the cascade from it climbs ×2 (3, 6, 12) and not ×3 (3, 9, 27). Standard theory predicts the same for generic one-hump maps. So the test does not separate the Foundations from standard theory. A separating test would need a system in which a ×3 cascade (period tripling) is generic; T15 allows it, unimodal theory forbids it, and two-dimensional maps do show it in some cases. I state it as open, not as evidence.

Consequences. The two ladders are in dynamical systems as published fact. Sharkovskii's order has period 3 at its head and the ×2 ladder at its tail; the cascades climb ×2 from 1 and ×2 from 3. This is a genuine structural echo, and it is older and independent of the theory. Li and Yorke's title, "Period three implies chaos," is as close as mathematics comes to the Foundations' claim that three is the first period that goes round (T2, T7).

But the field also shows where the Foundations' ladder stops being special. Windows of period 5, 7 and every other integer exist in the same maps, in comparable numbers: 3 windows of period 5 and 9 of period 7 for the quadratic family, against 2 of period 4 and 5 of period 6 (Calculation 1.9). The rewrite R of T10, the Foundations' own dynamics, reaches a 26-cycle on rings of 7 trits. So dynamics does not select the ladder; it orders periods, and the ladder sits at both ends of the order.

The Foundations add a vocabulary (meeting = period doubling, ring = period 3) but no result the field lacks. The vocabulary has one use: it predicts that wherever a system's

states are counted by sums of trits, the carry of T7.1 forces structure at the next scale. In logistic dynamics nothing is counted by trits, so the prediction is silent. The Foundations' IX.2 — which rewrite dynamics is its own fixed point — is a real dynamical question, and the register case is solved in T9–T10. For the whole net it is open, and dynamical systems is the discipline that would answer it. That is the strongest consequence: the open problem of the theory is a problem of this field.

Theorem checklist. T1–T6 apply only if a symbolic dynamics on three symbols is chosen; none is canonical for the logistic map. T7's meeting and ring match period doubling and period three in name. T8–T10 apply to the Foundations' own rewrite R, which is a dynamical system and is computed in Calculation 1.14. T11 does not apply. T12–T14 do not apply. T15 is checked against superstable windows (Calculation 1.9) and against R's cycle lengths (Calculation 1.14); it holds for cascades and fails for windows and for one cycle of R. T16–T18 apply to circle maps through the irrational rotation number $\log_2 3$, but not to the logistic family. T19–T22 apply to three-dimensional flows with knotted periodic orbits (a published subject since the 1980s), which this section does not cover. T24 is a statement about random walks, the stochastic counterpart of dynamics; Pólya's recurrence (Chapter 3.1) is its source.

Grade. A for Sharkovskii's order and the ×2 and 3·2^k cascades as published structure; C for the trit reading of stability and for period tripling (open, and not separating from standard theory).

1.5 Information theory

History. Claude Shannon's "A Mathematical Theory of Communication" (*Bell System Technical Journal* 27, 379–423 and 623–656, 1948) made information a measurable quantity in bits and showed that a source cannot be compressed below its entropy per symbol without loss. A trit carries $\log_2 3 \approx 1.585$ bits (Calculation 1.5). Error-correcting codes followed at once. Marcel Golay published in 1949 a perfect ternary code of length 11 with 729 codewords and minimum distance 5; a Finnish football-pool enthusiast, Juhani Virtakallio, had found it in 1947 and printed it in the magazine *Veikkaaja* (Wikipedia, Ternary Golay code). Its automorphism group is the Mathieu group M11, the smallest sporadic simple group. The extended code, of length 12, adds one check digit that makes the coordinates of every codeword sum to zero mod 3.

Rolf Landauer argued in 1961 that erasing a bit costs at least kT ln 2 of heat, about 0.018 eV at room temperature; Bérut and colleagues measured the heat of a single erasure in 2012 (Wikipedia, Landauer's principle). László Lovász bounded the zero-error capacity of the pentagon by √5 in 1979 with his theta function (*IEEE Transactions on Information Theory* IT-25, 1–7), a number that later reappeared as the quantum bound in the simplest contextuality inequality (Wikipedia, Lovász number). Erdal Arıkan's polar codes (2009, *IEEE Transactions on Information Theory* 55, 3051–3073) were the first explicit codes proved to reach capacity on symmetric binary-input channels; 3GPP adopted them for 5G control channels in November 2016 (Wikipedia, Polar code). The data channel of 5G uses LDPC codes instead.

The history of the ternary Golay code shows how early the zero-sum rule entered coding. The extended code is built by adding to each word of length 11 one symbol that makes the total sum 0 mod 3. Its words of weight 6 have supports that form the Steiner system S(5, 6, 12), and its automorphism group is 2.M12 (Wikipedia, Ternary Golay code); there are 264 such words on 132 supports, the 132 blocks of the Steiner system (Calculation 1.10). The question the history left for the register is simple: why are perfect codes so rare? The ternary Golay code is one of very few nontrivial perfect codes, which is why it is still a reference object.

Current state. Information theory is a large, mature field whose frontiers are quantum information, network and distributed coding, and the information theory of learning. The IEEE Information Theory Society's Shannon Award went to Peter Shor in 2025 and to Frans Willems in 2026 (Wikipedia, Shannon Award). Quantum error correction now runs in hardware: Google's Willow chip (105 qubits) showed in December 2024 that each increase of a surface code's distance by two suppressed the logical error by a factor of 2.14, reaching 0.143 % per cycle at distance 7 on 101 qubits (arXiv: 2408.13687; Google). The Landauer bound has been tested in nanomagnets at 300 K (2016, about 44 % above the minimum) and at 1 K in molecular nanomagnets (2018) (Wikipedia). Computers in 2012 still spent about a billion times the Landauer minimum per operation, which is why the bound is a horizon rather than a constraint on present hardware.

Quantum information has also moved beyond two levels. Qutrits — three-level quantum systems — have been run on superconducting circuits with randomized benchmarking (Morvan et al. 2021) and high-fidelity entangling gates (Goss et al. 2022), and quantum error correction of qudits beyond break-even was reported in *Nature* in 2025 by Brock and colleagues (Wikipedia, Qutrit). The smallest system in which quantum contextuality appears is three-dimensional: Gleason's theorem holds from dimension 3 and has counterexamples in dimension 2 (Wikipedia, Gleason's theorem), and the Kochen–Specker theorem likewise needs dimension at least three (Wikipedia). This is a published fact in which three is the least number that carries a new structure; Chapter 3 returns to it.

For the register two facts are structural. The ternary Golay code is perfect: the 729 codewords and the Hamming balls of radius 2 around them fill the 3^11 = 177,147 words exactly, since 3^6·(1 + 11·2 + 55·4) = 177,147 (Calculation 1.10). And its extension is a zero-sum code: every extended codeword closes mod 3, the parity check is the closure rule (Calculation 1.10). The radix economy of III.1 gives 2.731 for base 3 against 2.885 for bases 2 and 4 (Calculation 1.5); this is standard and the Foundations say themselves that it does not prove T1.

Interpretation.

1. *Strand.* I is the sequence of transmitted symbols; P is the alphabet. A return is a repeated symbol.
2. *Passage and trit.* With a ternary alphabet the symbols are trits. With a binary alphabet they are not; T1 then applies only to sums of symbols written in balanced ternary.

3. *Closure.* A word closes when its digit sum is 0 mod 3 (T1, T2). The extended Golay code is a set of 729 words all of which close; the check digit is the carry-free completion of the sum. More generally, every ternary code with an all-ones vector in its dual is a code of closed words. The 9 closed windows of T3 form a linear [3, 2] code over F_3, the zero-sum code of length 3. This is standard coding theory. T6 is the probability that a window of independent symbols passes this check: 1/3 for uniform symbols.
4. *Scale.* The Golay code's length 11 and dimension 6 are not on the ladder, but its extended length $12 = 2^2 \cdot 3$ and its $729 = 3^6$ codewords are; the information-theoretic content of the observation is nil. Lovász's √5 for the pentagon is off the ladder and connects to the five-cycle, not to the trinity.
5. *Test.* Information theory offers a test of III.1's remark but not of T1: if, at equal energy per operation, ternary symbols carried more information per joule than binary ones, the radix-economy remark would have physical weight. Landauer's bound is kT ln 3 per trit and kT ln 2 per bit, equal per bit of information; the test reduces to engineering (see 1.6).

Consequences. Information theory contains the closure rule as the parity check of ternary zero-sum codes, and the perfect ternary Golay code is the most famous of them. That is independent of the Foundations and grade A as structure. The reading adds nothing to coding theory; coding theorists have used sum-zero checks over F_3 since 1949.

One observation is worth keeping. In T11 the Foundations prove that a closed window is fixed by its first and last trit, because the middle is −(first + last). In coding language, a closed window is a codeword of the [3, 2] zero-sum code, and the middle trit is its check symbol. The Foundations' "seam" then counts how many check failures appear when two codewords are concatenated and read through a sliding window: 0 in 9 of 81 pairs, 1 in 36, 2 in 36. That is a small sliding-window code analysis, correct and elementary.

The more useful consequence is negative. Information per symbol is $\log_2 3$ bits, and the thermodynamic cost per bit is the same in any radix: kT ln 3 per trit is exactly kT ln 2 per bit times $\log_2 3$. The Foundations therefore cannot claim an informational or thermodynamic advantage for three beyond radix economy, which is a statement about counting digits, not about physics. The Foundations say as much in III.1. Lovász's √5 is a reminder that the simplest non-classical structure in zero-error information theory is a five-cycle, not a three-cycle; the trinity is not the whole of the register's interesting mathematics.

Theorem checklist. T1–T3 apply exactly: the closed windows are the [3, 2] zero-sum code. T4 gives its two kinds of codeword: constant (3) and all-different (6). T5 does not add to coding theory. T6 is the probability that a random ternary window passes the zero-sum check. T7 does not apply to codes. T8–T9 describe the 9 words (and 3 or 9 rings) that pass every sliding check — a sliding-window code of rate 1/3 per symbol beyond the first window. T10 does not apply. T11 counts check failures at codeword joins. T12 gives the length of a balanced-ternary integer code. T13 counts cyclic codes' orbits. T14–T24 do not apply. Radix economy (III.1) is checked in Calculation 1.5.

Grade. A for the zero-sum check of the extended ternary Golay code and the [3, 2] zero-sum code as the 9 closed windows; C for any informational priority of three.

1.6 Computer science

History. Alan Turing defined computation in 1936 with a machine of finitely many states reading symbols on a tape. John von Neumann's EDVAC report (1945) fixed the stored-program design, and binary became the standard. One serious machine did not follow: Setun, designed at Moscow State University under Sergei Sobolev with Nikolay Brusentsov as chief designer, computed in balanced ternary from 1958. Fifty were built from 1959 to 1965 at the Kazan Mathematical plant; a word had 18 trits and the fast memory held 81 words (Wikipedia, Setun). Its successor Setun-70 extended the design. The history left the balanced machine as a working counterexample: binary was a choice, not a necessity.

Complexity theory gave the register a second, subtler place. Stephen Cook showed in 1971, and Leonid Levin independently in 1973, that Boolean satisfiability is NP-complete; Richard Karp listed 3-SAT among 21 NP-complete problems in 1972; and Thomas Schaefer proved in 1978 that every Boolean constraint language is either in P or NP-complete (Wikipedia, Boolean satisfiability). In 1987 Alexander Razborov and Roman Smolensky proved the strongest circuit lower bounds known: for primes p, the function MOD_m cannot be computed by constant-depth circuits with MOD_p gates unless m is a power of p (Wikipedia, ACC0). In particular, counting mod 3 and counting mod 2 are mutually unreachable in that model — the ladders ×2 and ×3 of T16 in circuit form.

Current state. Ternary hardware has a modern line. Jeong and colleagues reported in *Nature Electronics* (2019, vol. 2, 307–312) a wafer-scale ternary CMOS technology using a single threshold voltage, with the third state from a band-to-band tunnelling current that can be held below a picoampere at 0.5 V (KNU record). Computing-in-memory chips run ternary-weighted networks: a 65 nm ReRAM macro reached 16.95 TOPS/W and 98.8 % MNIST accuracy with a "split binary-input ternary-weighted" model (Nature Electronics). The fifth Busy Beaver value, BB(5) = 47,176,870, was settled in 2024 by an online collaboration with a proof formalised in Rocq (preprint arXiv:2509.12337, September 2025); for two states and three symbols, Σ(2,3) = 9 and S(2,3) = 38 (Wikipedia, Busy beaver); for six states, a machine running more than $2\uparrow\uparrow\uparrow 5$ steps was found in June 2025 (Quanta). Quantum computing's error correction is now below threshold on superconducting chips (1.5 above). Data centres used about 415 TWh in 2024, 1.5 % of world electricity, projected at about 945 TWh in 2030 (IEA). That energy figure is why the question of radix is again practical.

Three-state logic is already everywhere in binary hardware, in a hidden form. CMOS and totem-pole TTL outputs can drive high, drive low or float in a high-impedance state, which lets many devices share a bus (Wikipedia, Ternary numeral system). Hardware description languages model unknown values as a third (and further) state. What has not happened is ternary arithmetic in mass-produced processors. The obstacles are not mathematical: they are the cost of a reliable middle voltage level, noise margins, and the inertia of a binary toolchain. The 2019 tunnelling device

addresses the first. Formal verification is another frontier where the register appears: Lean's mathlib (1.1) and the Rocq proof of BB(5) show that large mathematical results now live as machine-checked code.

Interpretation.

1. *Strand.* I is the sequence of machine steps; P is the set of machine states or memory cells. A loop is a revisit of a state; a program loop is literally one.
2. *Passage and trit.* In a balanced-ternary machine each cell holds a trit. The Foundations' register is then not an analogy but the machine's number system: Setun's arithmetic is T1 and T12 in hardware.
3. *Closure.* Facts (a)–(d) of III.2 — no sign bit, carry-free single-trit products, truncation equal to rounding, shift equal to scale step — are the engineering reasons given for balanced ternary since Knuth. T1's carry rule is the adder. A ring 1 + 1 + 1 gives 10: digit 0 and carry 1 (Calculation 1.1).
4. *Scale.* Words of k trits address (3^k − 1)/2 values on each side of zero (T12.3); 13 trits suffice for 10^6 states against 20 bits. Razborov–Smolensky separates the ×2 and ×3 ladders inside constant-depth circuits; this is the nearest published analogue of T16 in complexity theory, and it is a theorem, not a reading. Of the 64 binary words of length 6, 22 have a sum divisible by 3 and 32 have even parity (Calculation 1.13); deciding the first with parity gates, or the second with mod-3 gates, needs circuits that grow with the input.
5. *Test.* The radix question is testable. If ternary logic at equal technology cannot match binary energy per bit of information, III.1's remark gains nothing in hardware. The 2019 tunnelling CMOS shows that a third state can be cheap at 0.5 V; a full comparison per bit, on the same node, has not been published to my knowledge.

Consequences. Computer science is where the register already is engineering: Setun ran on it, ternary CMOS exists, and ternary weights run in memory chips. The Foundations add the closure rule as a filter over the register. In software it is a check: a window of three trits passes when its digit is zero. Whether any algorithm benefits from that filter is open; I know of none.

The Razborov–Smolensky separation gives the Foundations a strong formal ally for T16: the two ladders are not only numerically incommensurable, they are computationally separated in shallow circuits. This is a better citation for "the ladders never close onto each other" than any physical example, because it is a theorem about computation rather than a coincidence of numbers. It should be stated with its limits. The separation holds for constant-depth circuits with modular gates; for general polynomial-size circuits nothing comparable is known, and ACC^0 lower bounds remain one of the hardest problems in complexity theory.

Two further points are plain consequences. First, Turing machines with three symbols are a standard object (the busy beaver $\Sigma(2,3) = 9$), and nothing in computability theory singles out three; universality needs only two. The register is therefore a design choice for arithmetic, not a requirement of computation. Second, the energy figure for data centres makes radix a practical question again. If the closure filter has no algorithmic

value, the case for ternary hardware rests entirely on facts (a)–(d) of III.2 and on device physics, which is where the B test lies.

Theorem checklist. T1 and T7.1 are the balanced-ternary adder (Calculation 1.1). T2–T6 apply to any ternary register as checks. T7 has no published use in algorithms. T8–T10: T10's rewrite is a linear cellular automaton, a standard model of computation (Calculation 1.14). T11 does not apply. T12 and T12.3 are the address and capacity of a ternary word, applied in Setun's 18-trit words. T13 counts distinct contents of a circular register. T14 has a computational analogue in complex-base number systems. T15–T16 apply through the Razborov–Smolensky separation of mod-2 and mod-3 counting (Calculation 1.13). T17–T24 do not apply.

Grade. A for the balanced-ternary register in hardware (Setun; Jeong et al. 2019) and for the mod-2/mod-3 separation as published theorem (Razborov 1987; Smolensky 1987); B for the energy-per-bit comparison of ternary and binary logic on one node.

1.7 Artificial intelligence

History. Neural networks began with binary threshold neurons and grew, through associative memories and back-propagation, into trainable systems. John Hopfield and Geoffrey Hinton received the 2024 Nobel Prize in Physics "for foundational discoveries and inventions that enable machine learning with artificial neural networks" (Wikipedia, Nobel laureates in Physics). AlexNet (2012) and the Transformer (2017) began the present era. The symbolic side of AI ran in parallel. Expert systems encoded rules; and three-valued and deontic logics were used to represent unknown facts and norms in such systems. The two traditions — numbers and rules — met only late, in systems that use a language model to propose and a formal checker to verify.

Compression of networks followed their growth. In May 2016 Fengfu Li and colleagues proposed Ternary Weight Networks, with every weight in {−1, 0, +1} times a scale, reporting up to 16× compression and accuracy close to full precision on MNIST and CIFAR-10 (arXiv:1605.04711). The question the history left was whether such extreme quantisation could work for large language models trained from scratch.

Current state. In February 2024 Ma and colleagues at Microsoft reported BitNet b1.58: a language model whose every weight is −1, 0 or +1, obtained by dividing a weight matrix by its mean absolute value and rounding (arXiv:2402.17764). The authors report that it matches a full-precision LLaMA baseline in perplexity from 3 billion parameters, is 2.71 times faster and uses 3.55 times less GPU memory at that size, and is 4.1 times faster at 70 billion (arXiv HTML). In April 2025 the open BitNet b1.58 2B4T, 2 billion parameters trained on 4 trillion tokens, used 0.4 GB of non-embedding memory, 29 ms per token on CPU and 0.028 J per token, against 2.6 GB, 65 ms and 0.347 J for Qwen2.5 1.5B; its benchmark average was 54.19 against 55.23 (arXiv:2504.12285). These are the authors' figures.

AI has also become a tool in mathematics. FunSearch found the 512-point cap set in dimension 8 (1.3). AlphaEvolve, announced on 14 May 2025, found a way to multiply 4 × 4 complex matrices with 48 multiplications against Strassen's 49, a kissing configuration of 593 spheres in 11 dimensions, and matched the best known answers

on about 75 % of more than 50 open problems while improving about 20 % (DeepMind). An advanced Gemini Deep Think scored 35 of 42 at the 2025 International Mathematical Olympiad, gold standard; in 2024 AlphaProof with AlphaGeometry 2 had scored 28, silver (DeepMind). Frontier training compute has grown about 5× a year since 2020, doubling every 5.2 months, and training cost about 3.5× a year (Epoch AI).

Interpretation.

1. *Strand.* I is the sequence of weights along a row of a weight matrix (or the sequence of tokens in a context); P is the set of weight values. In a BitNet layer, P = {T, 0, 1}.
2. *Passage and trit.* Each weight is a trit. The Foundations' register is the weight alphabet of a working model; this is a fact, not a reading. A matrix-vector product with trit weights is a signed sum without multiplication, which is fact (b) of III.2.
3. *Closure.* A window of three consecutive weights closes when its digit is 0 (T1, T2). There is no published reason why trained weights should prefer closed windows. T6 gives the baseline. For Gaussian weights quantised by the absmean rule, the probabilities are $p_0 = 0.3101$ and $p_+ = p_- = 0.3450$, and a window closes with probability 0.33330 — almost exactly 1/3 (Calculation 1.11). Any excess over this baseline in trained weights would be a signal.
4. *Scale.* The scaling laws of language models (compute ×5 per year) are not on the 2^a·3^b ladder in any meaningful sense; I find no published ladder structure in network sizes or training curves. T13 suggests one structural observation: in a convolution or a circular buffer, a closed configuration counted without a start number $g(n) \approx 3^n/n$, which matters for weight-sharing; this is standard and adds nothing.
5. *Test.* Stated before data: take the released weights of BitNet b1.58 2B4T, read every row of every ternary layer as a trit word, and count the share of closed windows. The Foundations' closure rule predicts a share above the T6 baseline computed from that layer's own frequencies. If the share equals the baseline within sampling error across layers, the closure rule has no content for trained networks. The test has not yet been run.

Consequences. If the test showed an excess of closed windows, the closure rule would become a candidate regulariser or a compression prior for ternary networks: a window that closes can be stored as two trits, since the third is fixed (T11's proof uses this). A one-third saving on closed windows is concrete. A second use follows from T7: a closed row reduces completely to standings, meetings and rings, and a meeting x, −x in adjacent weights cancels on equal inputs. Whether trained models contain more meetings than chance is part of the same test.

If the test shows no excess, the Foundations' contribution to AI is the alphabet alone, which the field reached by its own route — from quantisation, not from the trinity. Either way the alphabet is not the theory's discovery, and the Foundations cite Ma et al. correctly as precedent rather than as confirmation.

The energy figures matter to the larger argument. A ternary model at 0.028 J per token is a working instance of III.1's "maximal richness from minimal means." But the saving comes from avoiding multiplication and from memory traffic, which binary-weight

networks also achieve; the third value 0 adds sparsity, which is the measurable benefit of ternary over binary weights. The original BitNet paper reports that its binary predecessor was weaker than the ternary model. That is an empirical point in favour of having a zero, not of closure mod 3. AI in mathematics (FunSearch, AlphaEvolve) is relevant in another way: it is the tool most likely to find, or fail to find, structure in the turn share of T14 and in other open questions of the register.

Theorem checklist. T1 applies to sums of ternary weights. T2–T6 apply to windows of weights; T6 gives the baseline 0.3333 for absmean-quantised Gaussian weights (Calculation 1.11). T7 applies to rows of weights and is the basis of the meeting count in the stated test. T8–T10 have no counterpart. T11 applies to concatenated closed windows if the test finds an excess. T12 gives the integer range of a sum of trit weights. T13 does not apply. T14–T24 do not apply to network weights; T15's ladder has no published structure in model sizes.

Grade. A for the ternary weight alphabet in working models (Li et al. 2016; Ma et al. 2024; BitNet 2B4T 2025); B for the closed-window test on trained ternary weights.

Part II — Strand and Field

Chapter 2 — Particles and Fields

This chapter reads the standing column of level L1: the closed units of physics itself, from hadrons to atoms. Its strongest result is old and independent of the theory. Only colour-neutral combinations of quarks are seen free, and a combination of quarks and antiquarks can be colour-neutral only if its triality — the number of quarks minus antiquarks, mod 3 — is zero. In the digit arithmetic of the Foundations (T1, T7.1) the carry of that sum is the baryon number, exactly, for every composition (Calculation 2.1). Particle physics is grade A for hadron composition; quantum field theory is A for the centre $\mathbb{Z}_3$ of SU(3) and for the 4π return of spinors (T21), with B for the companion document's tests; nuclear physics is A only for the three-alpha ring of carbon and C for shell structure, which falls off the ladder after 8; atomic physics carries the one sharp prediction of this chapter, a shift of the fine-structure constant on a white dwarf, which is grade B and currently untestable by a factor of about 700.

2.1 Particle physics

History. By the early 1960s accelerators had found dozens of short-lived hadrons, and the problem was to order them. Murray Gell-Mann and Yuval Ne'eman did so in 1961 with the "eightfold way", grouping hadrons into octets and decuplets of the symmetry group SU(3). In 1963–64 Gell-Mann and George Zweig proposed that hadrons are built from three kinds of smaller constituents, which Gell-Mann named quarks (Wikipedia, Quantum chromodynamics). The model had a defect. The Δ++ baryon is made of three up quarks with parallel spins, which seems to break Pauli's exclusion principle. Oscar Greenberg in 1964, and Moo-Young Han and Yoichiro Nambu in 1965, solved this by giving each quark a further three-valued property, later called colour; Han and Nambu proposed that an octet of gauge bosons carries the force (Wikipedia, Colour charge).

Deep inelastic scattering at SLAC in 1969 showed point-like constituents inside the proton. In 1973 Harald Fritzsch, Gell-Mann and Heinrich Leutwyler formulated quantum chromodynamics, colour as the source of a Yang–Mills field, and David Gross, Frank Wilczek and David Politzer found asymptotic freedom: the force weakens at short distances. The three shared the 2004 Nobel Prize. Three-jet events at the PETRA collider in 1979 gave the first direct evidence of gluons (Wikipedia, QCD). The electroweak theory of Glashow, Weinberg and Salam, the discovery of the W and Z bosons in 1983, the top quark in 1995 and the Higgs boson in 2012 completed the Standard Model. On 4 July 2012 ATLAS and CMS reported a new particle near 125 GeV; it was confirmed as the Higgs boson in March 2013 (Wikipedia, Standard Model).

The Standard Model has three generations of quarks and leptons, a gauge group SU(3) × SU(2) × U(1), and 19 free parameters fixed by experiment (Wikipedia, Standard Model). The number of light neutrino species was measured at LEP from the width of the Z boson. The questions the history left are many: why three generations, why these masses, what is dark matter, and why free quarks are never seen. The last is confinement, and it is the question closest to the Foundations.

Quark-model hadrons were mesons (quark–antiquark) and baryons (three quarks) for forty years. In 2003 the Belle experiment found X(3872), the first of a family of states that do not fit the simple pattern. Zc(3900) followed in 2013 at BESIII and Belle; in April 2014 LHCb confirmed Z(4430)⁻ with minimal content $c\bar{c}d\bar{u}$, and in July 2015 LHCb announced the pentaquarks Pc(4380) and Pc(4450) (Wikipedia, Exotic hadron). QCD itself sets no limit on the number of quarks in a hadron, only that the whole be colour-neutral (Wikipedia, Exotic hadron).

Current state. The LHC has found 72 new bound states as of August 2023; 64 of them came from LHCb, and 23 of the LHCb discoveries are exotic hadrons — tetraquarks and pentaquarks (LHCb outreach). Every one of them has a minimal quark content of zero triality. The Z-width measurement of light neutrino species now gives $N_\nu = 2.9963 \pm 0.0074$ after corrections to the luminosity and the Bhabha cross-section; the original LEP combination was 2.984 ± 0.008 (PDG review). The corrected value is 0.5 standard deviations from 3 (Calculation 2.3). The Higgs mass is 125.09 ± 0.24 GeV in the Standard Model parameter table (Wikipedia, Standard Model).

The muon's anomalous magnetic moment was the most-watched test of the decade. Fermilab's final result, released on 3 June 2025, is $a_\mu = 0.001\,165\,920\,705$ with a precision of 127 parts per billion, better than the design goal of 140 (Fermilab). The 2025 theory update, which uses lattice QCD for the hadronic vacuum polarisation, lies closer to the measurement than the earlier data-driven prediction; a 2026 *Nature* paper reports a hybrid calculation of the hadronic term to 0.48 % and agreement at 0.5σ (Wikipedia, Muon g−2). The anomaly that seemed to point beyond the Standard Model has largely closed.

Neutrino masses are bounded from cosmology. DESI's second data release (19 March 2025) gives a summed mass below 0.0642 eV under ΛCDM, in about 3σ tension with the 0.059 eV floor from oscillation experiments; allowing evolving dark energy removes the tension and raises the limit to 0.163 eV (DESI). Antimatter tests have reached gravity: on 27 September 2023 the ALPHA collaboration reported that antihydrogen falls like hydrogen, ruling out repulsive "antigravity" (Wikipedia, ALPHA experiment). The open debates are the origin of neutrino mass, the nature of dark matter, the strong CP problem and whether any new particle lies within reach of the LHC's high-luminosity run.

Interpretation.

1. *Strand.* I is the sequence of constituents in a bound state, quarks and antiquarks read in any order; P is the set of colour states. A "return" is a second constituent in a colour state related to the first.
2. *Passage and trit.* Each quark carries triality +1, each antiquark −1, each gluon 0. Triality is the charge of a representation of SU(3) under its centre $\mathbb{Z}_3$ (2.2 below). This is a published structure of the field, not an addition. The trits are then the digits of T1: a hadron's total triality is the balanced digit of (number of quarks − number of antiquarks).
3. *Closure.* A colour singlet can only be built from constituents of total triality 0. In the language of T1, the sum of trits leaves digit 0. Of the 20 compositions with one to

five constituents, exactly 6 close: $\bar{q}\bar{q}\bar{q}$, $q\bar{q}$, $q\bar{q}\bar{q}\bar{q}$, $qq\bar{q}\bar{q}$, qqq and $qqqq\bar{q}$ (Calculation 2.1). T7 decomposes each closed word into meetings and rings: a meson is a meeting (1, T); a baryon is a ring (1, 1, 1); a tetraquark is two meetings; a pentaquark is a ring and a meeting.

4. *Scale.* T7.1 adds something the closure rule alone does not. A meeting closes with carry 0; a ring closes with carry ±1. Apply the digit rule of T1 to the six closed compositions: the carry is 0 for mesons and tetraquarks, +1 for baryons and pentaquarks, −1 for antibaryons and antipentaquarks. In every case the carry equals the baryon number $B = (q - \bar{q})/3$ (Calculation 2.1). The carry of the Foundations is the conserved baryon number of particle physics. This is arithmetic: $B = (q - \bar{q})/3$ is the standard definition, and for a sum divisible by 3 the carry is exactly the quotient. But it means the reading “a ring becomes one passage of the next scale” has a precise physical meaning here: three quarks make one baryon, a unit of baryon number at the hadronic scale. The electric charges agree as well: for all 149 combinations of u, d, $\bar{u}$, $\bar{d}$ with one to six constituents, the total charge is an integer exactly when the triality is zero (Calculation 2.2).
5. *Test.* The rule is established physics and cannot refute the net; it fixes the reading. The prediction that adds risk is negative: no free hadron with nonzero triality and no free state whose constituents cannot be grouped into meetings and rings. QCD forbids both. A fractionally charged free particle would refute both QCD’s confinement and this reading at once.

Consequences. If the reading holds, particle physics already contains the Foundations’ closure rule, and has done since 1965. Nature builds every observed hadron, including the 23 exotic states of LHCb, from meetings and rings. The Foundations add two things. First, a single sentence that covers the whole spectrum of hadron compositions: closed words of trits, decomposed by T7. Second, the identification of the carry with baryon number (T7.1), which turns “a ring becomes one passage at the next scale” from a slogan into the arithmetic of B.

The costs are equally clear. The Foundations do not explain why there are three colours; they take the trinity as ground, and particle physics takes three colours from the Δ++ and from measured rates. Nor do the Foundations explain the 19 parameters, the three generations or $N_\nu = 3$. A reader may be tempted to read $N_\nu = 2.9963 \pm 0.0074$ as “three because of the trinity”. That is a coincidence of the integer 3, not a derivation, and the Foundations do not claim it. The three generations have no known link to triality: leptons carry triality 0 in every generation.

The muon $g - 2$ story is a reminder of the standard for this book. A 4–5σ anomaly lasted fifteen years and dissolved when the theory calculation was redone. Any claim of the Foundations about particle properties should be held to the same standard: a stated prediction, a measurement and an error budget.

Theorem checklist. T1 and T7.1 apply exactly to triality and baryon number (Calculation 2.1). T2 gives the baryon as the shortest closing run of equal trits. T3 counts closed colour windows: 9 of the 27 ordered words of three constituents drawn from quark, antiquark and gluon close. T4 splits them into 3 of one kind (qqq, $\bar{q}\bar{q}\bar{q}$ and three gluons) and 6 orderings of a quark, an antiquark and a gluon (Calculation 2.1).

T5–T6 add nothing. T7 decomposes every hadron (meson = meeting, baryon = ring). T8–T10 have no counterpart. T11 has a loose counterpart in hadron molecules (two closed hadrons bound together, as proposed for some exotic states), but no published count. T12.1 is charge conjugation: the mirror of a hadron's word is its antiparticle. T13 does not apply. T14 applies through the SU(3) weight lattice (2.2, Calculation 2.13). T15 is checked against the generation count and the Z-width (Calculation 2.3) and gives nothing. T16–T18 do not apply. T19–T22 belong to 2.2. T23–T24 do not apply.

Grade. A for hadron composition as closed trit words (triality, Greenberg 1964; Han and Nambu 1965), with the carry of T7.1 equal to baryon number for all compositions; C for three generations and $N_\nu = 3$.

2.2 Quantum field theory

History. Quantum field theory began with Paul Dirac's relativistic equation for the electron (*Proceedings of the Royal Society A* 117, 610–624, 1928), which required four-component spinors and predicted the positron. Spinors themselves were older: Élie Cartan found them in 1913 as representations of rotations that return to themselves only after 4π, not 2π (Wikipedia, Spinor). The 1930s brought the infinities of quantum electrodynamics, and the late 1940s their cure by renormalisation in the work of Tomonaga, Schwinger, Feynman and Dyson. Chen-Ning Yang and Robert Mills generalised gauge symmetry to non-commuting groups in 1954. With the Higgs mechanism (1964) and 't Hooft and Veltman's proof of renormalisability (1971) the Yang–Mills theories became the language of the Standard Model.

Topology entered field theory in parallel. Heinz Hopf showed in 1931 that maps from the three-sphere to the two-sphere fall into classes labelled by an integer, the Hopf invariant; the Hopf fibration that realises it is the fibre bundle of Dirac's magnetic monopole and the geometry of the Bloch sphere of a qubit (Wikipedia, Hopf fibration). Skyrme, Faddeev and others built particle-like solitons whose charge is such a topological integer; Faddeev and Niemi showed in 1997 that knotted solitons, now called hopfions, exist in a classical field model (Wikipedia, Hopfion). Edward Witten connected quantum field theory and knot invariants in 1989: Chern–Simons theory computes the Jones polynomial (*Communications in Mathematical Physics* 121, 351–399).

Confinement became a question about the centre of the gauge group. In SU(3), the matrices that commute with everything form the group $\mathbb{Z}_3$ of cube roots of unity times the identity. At finite temperature the Polyakov loop — a Wilson line wound round imaginary time — picks up a centre phase under centre transformations; its expectation vanishes in the confined phase, and a nonzero value signals deconfinement (Wikipedia, Polyakov loop). Kenneth Wilson's lattice gauge theory of 1974 made these statements computable. The question the history left is the mass gap: a proof that Yang–Mills theory in four dimensions has a lowest positive mass. It is one of the Clay Institute's Millennium Problems (Wikipedia, QCD).

Current state. Lattice QCD now delivers precision numbers. The HotQCD collaboration places the chiral crossover of QCD at zero density at $T_c = 156.5 \pm 1.5$

MeV (arXiv:1812.08235). With dynamical quarks the $\mathbb{Z}_3$ centre symmetry is explicitly broken, so confinement must be read from the spectrum of colour-neutral states rather than from the Polyakov loop alone (Wikipedia, Polyakov loop). Lattice computations of the hadronic vacuum polarisation resolved the muon g − 2 discrepancy (2.1).

The global form of the Standard Model gauge group is an open question that experiments cannot yet settle. David Tong showed in 2017 that the group is SU(3) × SU(2) × U(1)/Γ with Γ a subgroup of $\mathbb{Z}_6$, and that the choice of Γ changes the allowed line operators and the periodicity of the theta angles, but "cannot be determined by current experiments" (arXiv:1705.01853). The quotient is possible because every field's hypercharge is tied to its triality and its SU(2) charge: for all Standard Model fields, 6Y ≡ 3d − 2t (mod 6), where t is triality and d is 1 for doublets (Calculation 2.10). Reduced mod 3, this says 6Y ≡ t: hypercharge in units of 1/6 carries triality.

Topological solitons have become experimental objects. Hopfions have been observed in chiral liquid crystals and colloids (Ackerman and Smalyukh 2017), in Ir/Co/Pt magnetic multilayers (Kent et al., *Nature Communications*, March 2021), in the polarisation of free-space light (Sugic et al., November 2021) and as fractional hopfions in the helimagnet FeGe (Yu et al., *Advanced Materials*, May 2023) (Wikipedia, Hopfion). Trefoil vortex knots were created in water by Kleckner and Irvine and observed to untie by reconnection (*Nature Physics* 9, 253–258, 2013) (Nature Physics). The 4π return of spinors was shown directly with neutrons in 1975: Rauch and colleagues (*Physics Letters A* 54, 425–427) and Werner and colleagues (*Physical Review Letters* 35, 1053–1054) sent neutrons through a magnetic field on one arm of an interferometer and saw the interference return after a 4π precession; Werner's group measured a field period of 62 ± 2 G for that full return (Werner et al. 1975; Rauch et al. 1975, CrossRef).

Interpretation.

1. *Strand.* I is the course of a field line or a world line; P is the set of points of space-time or of the target space of the field. In the companion document, *The Vacuum.Net Theory: Physics*, the strand is given Dirac's equation in Rowlands' nilpotent form (Physics T1): the state is a nilpotent operator whose square vanishes, a closure condition written in quaternion algebra.
2. *Passage and trit.* For SU(3), the centre element e^(2πi/3) acts on a representation as a power ω^t; t mod 3 is the triality, a trit. This is also T14's setting, exactly. The weights of the three quarks, drawn in the plane, are 1, ω and ω^2 of the Eisenstein lattice Z[ω]; the gluons' roots are their differences, such as 1 − ω, of norm 3. The root lattice is the ideal (1 − ω), of index 3 in the weight lattice, and the triality of a weight a + bω is its residue a + b mod 3 — the trit of the Foundations' remark in III. 1. Every quark has residue +1, every antiquark −1, every gluon root 0, and every qqq and $q\bar{q}$ weight sum lies in the root lattice (Calculation 2.13). For spinors, the passage is a rotation, and T21 states that the return needs 4π (Calculation 2.6). For hopfions, the Hopf charge $Q \in \mathbb{Z}$ is an integer winding (T22), whose address is a balanced-ternary word with mirror −Q (T12.1).
3. *Closure.* The Polyakov loop is the closure test of a world line wound once round the thermal circle: it vanishes in the confined phase, where only centre-neutral (triality-

zero) objects exist. The Foundations' T1 states the same selection at the level of digits. Dirac–Rowlands nilpotency (Physics T1) is a second closure: a state that squares to zero.

4. *Scale.* The relation $6Y \equiv 3d - 2t \pmod 6$ is the only place in the Standard Model where ×2 (SU(2) doublets) and ×3 (SU(3) triality) are tied in one modulus, $6 = 2 \cdot 3$. In the language of T15 it is the first rung of the ladder 2^a·3^b with a = b = 1. The quotient $\Gamma \subseteq \mathbb{Z}_6$ that Tong discusses is the group of this rung. T16 says ×2 and ×3 never close onto each other; in the Standard Model they meet only modulo 6, not in any exact equality.
5. *Test.* Two tests are clean. First, T21 predicts that every attached fermion returns after 4π. The neutron experiments of 1975 confirm it, so this test has been passed by others. Second, T22 states that Hopf charge is conserved under continuous evolution without boundary crossing. Hopfion experiments that report creation or annihilation of hopfions in a bulk without a boundary or a singular core would refute T22's hypotheses as a physical description. Observed annihilations so far occur at surfaces or through singular points, which T22 excludes by hypothesis.

Consequences. Quantum field theory contains the Foundations' register in three published places: the $\mathbb{Z}_3$ centre of SU(3), the 4π return of spinors, and integer topological charges. All three are independent of the theory and older than it. The Foundations' contribution is to put them in one chain (T1, T21, T22) and to name the shared structure: closure as digit 0, return after 4π as the double cover of rotations, and charge as an integer winding.

The relation between hypercharge and triality deserves notice. It is the reason the Standard Model gauge group can be divided by $\mathbb{Z}_6$, and in the digit language it says that hypercharge carries the triality digit. The Foundations predict nothing about Γ. If future experiments (for example the observation of a magnetic monopole of particular charge) fixed Γ, that would be physics the Foundations could read but not have predicted.

The identification of SU(3) weights with Eisenstein integers is standard Lie theory (the A_2 root system is the hexagonal lattice), but it gives the Foundations' "algebraic home" of the trit a physical address: the residue modulo the prime $1 - \omega$ is colour triality. That is a link of the A kind, and it is the most direct one between T14's ring and physics.

The companion document's T1–T4 (Dirac–Rowlands, Maxwell quaternions) restate known equations in quaternion form; they are correspondences, not new results. The cost of the reading is the hypothesis load: T21 needs unit-quaternion orientation states, T22 needs field maps $S^3 \to S^2$, and both hypotheses carry the physics. The Foundations say so. The proof of T22 itself is the standard homotopy argument; the Foundations explicitly do not claim that it replaces Noether's theorem.

Theorem checklist. T1 applies to the centre $\mathbb{Z}_3$. T2–T7 apply as in 2.1. T8–T10: T10's fixed points have no field-theory counterpart I know of. T11 does not apply. T12.1 is charge conjugation on addresses of conserved charges. T13 does not apply. T14 applies exactly: the A_2 weight lattice is $\mathbb{Z}[\omega]$ and triality is the residue mod $1 - \omega$ (Calculation 2.13). T15 applies in the single rung $6 = 2 \cdot 3$ of the $\mathbb{Z}_6$ quotient (Calculation 2.10). T16–

T18 do not apply. T19–T20 apply through Witten's Chern–Simons theory of the Jones polynomial and through knotted vortices. T21 applies exactly (Calculation 2.6). T22 applies exactly to hopfions. T23 applies as a constitutive form for an effective coupling; field theory's running couplings are a different state dependence (on scale, not density). T24 does not apply.

Grade. A for the $\mathbb{Z}_3$ centre and triality as the residue of SU(3) weights in Z[ω] modulo $1 - \omega$, for $6Y \equiv 3d - 2t \pmod 6$ behind the $\mathbb{Z}_6$ quotient (Tong 2017), for the 4π return of spinors (Rauch et al. 1975; Werner et al. 1975) and for integer Hopf charge; B for T22 as a statement about bulk hopfion dynamics; C for the companion document's quaternion forms of Dirac's and Maxwell's equations as anything beyond rewriting.

2.3 Nuclear physics

History. Ernest Rutherford found the nucleus in 1911 and James Chadwick the neutron in 1932. Carl Friedrich von Weizsäcker's semi-empirical mass formula of 1935 treated the nucleus as a liquid drop. In 1948 Maria Goeppert Mayer published evidence for unusually stable nuclei with 50 or 82 protons and 50, 82 or 126 neutrons; she and Hans Jensen explained these "magic numbers" by strong spin–orbit coupling in a shell model and shared the 1963 Nobel Prize (Wikipedia, Magic number). The magic numbers are 2, 8, 20, 28, 50, 82 and 126. Without spin–orbit coupling a three-dimensional harmonic oscillator would close shells at 2, 8, 20, 40, 70, 112 and 168 (Calculation 2.4); the spin–orbit term moves the closures above 20.

The origin of carbon was a separate puzzle. Edwin Salpeter showed in 1952 how helium burning in red giants could fuse three alpha particles into carbon via unstable beryllium-8. In 1953 Fred Hoyle argued from the observed abundance of carbon and oxygen that carbon-12 must have a resonance near 7.68 MeV; Ward Whaling and colleagues in William Fowler's laboratory at Caltech found one near 7.65 MeV a few months later (Wikipedia, Triple-alpha process). Beryllium-8, the meeting of two alpha particles, has a half-life of 8.19×10^{-17} s, long enough that a third alpha can arrive before it falls apart.

The three-body problem returned in nuclear form. In 1970 Vitaly Efimov predicted that three identical bosons with a two-body interaction tuned to threshold have an infinite series of bound states whose sizes grow by a fixed factor. The states can be bound when no pair binds — the Borromean property, named after three rings linked as a whole though no two are linked (Wikipedia, Efimov state). Halo nuclei such as lithium-11, a core and two neutrons bound only as a triple, are the nuclear examples. Fission, discovered in 1938, gave the chain reaction; Leó Szilárd conceived it on 12 September 1933, and Fermi's Chicago Pile-1 sustained it on 2 December 1942 (Wikipedia, Nuclear chain reaction). A natural reactor ran at Oklo in Gabon about two billion years ago.

Current state. Rare-isotope facilities are mapping the edges of nuclear stability. The Facility for Rare Isotope Beams at Michigan State began experiments in June 2022, accelerated a 10.4 kW uranium beam in December 2023, and doubled that to 20.0 kW in March 2025; in its first eight hours of uranium running it found gallium-88,

arsenic-93 and selenium-96 (Wikipedia, FRIB). Oxygen-28, with 8 protons and the magic 20 neutrons, was first observed in 2023 and turned out not to be doubly magic: it is unbound and lacks a closed neutron shell (Wikipedia, Magic number). Magic numbers move far from stability.

Superheavy elements have reached oganesson, Z = 118, with up to 177 neutrons; ^{294}Og lives about 690 µs. Theory predicts the next neutron closure at N = 184 and proton closures at 114, 120 or 126; in 2024 Berkeley published a reliable route to element 116 using a titanium beam, a method that may reach beyond 118 (Wikipedia, Island of stability). Efimov states were first seen in an ultracold caesium gas by the Innsbruck groups in 2006, and in 2015 as an excited state of the helium trimer (Wikipedia, Efimov state). The universal scaling factor between successive Efimov states is e^(π/s_0) ≈ 22.69 in length, with s_0 = 1.00624 (Calculation 2.5).

The thorium-229 nucleus has an isomer only 8.355733554021(8) eV above its ground state, low enough to excite with a laser. In April 2024 two groups excited it in thorium-doped crystals; in June 2024 the transition frequency was measured to parts per trillion, about 2,020,407,384,335 kHz, and linked to a strontium clock. The first optical nuclear clocks were announced in June 2026 and are not yet competitive with atomic clocks (Wikipedia, Thorium-229). Fission still produces three fragments in 0.2–0.4 % of events, about 90 % of them with an alpha particle as the light third fragment; true ternary fission into three comparable pieces occurs about once in 100 million fissions (Wikipedia, Ternary fission).

Interpretation.

1. *Strand.* I is the sequence of nucleons, or of alpha clusters, in a nucleus; P is the set of single-particle orbits.
2. *Passage and trit.* For nucleons there is no published trit. Each nucleon has triality 0 as a colour singlet, and baryon number 1. A nucleus of A nucleons is a word of A carries from the hadronic scale, each worth one ring of quarks (T7.1). For alpha clustering, the passage is the arrival of another alpha.
3. *Closure.* The triple-alpha process is the cleanest case. Two alphas make beryllium-8, a meeting that does not hold: it decays in 8.19 × 10^{-17} s. Three alphas make carbon-12 through the Hoyle state, a ring that holds. In T7 and T7.1, a meeting closes with carry 0 and a ring closes with carry 1. The nuclear facts are a published instance of “the meeting does not persist, the ring does”, independent of the theory. Efimov’s Borromean trimers make the same point more strongly: three bind where two do not. The magic numbers are closed shells, but their closure is not a sum mod 3; it is the filling of degenerate levels.
4. *Scale.* The magic numbers are not on the ladder beyond 8: 2 = 2 and 8 = 2^3 are, but 20 = 2^2·5, 28 = 2^2·7, 50 = 2·5^2, 82 = 2·41 and 126 = 2·3^2·7 are not (Calculation 2.4). The oscillator closures 2, 8, 20, 40, 70, 112, 168 leave the ladder at the same place. T15 does not describe shell structure. The Efimov factor 22.69 is irrational and off the ladder. The ternary fission statistics have no ladder structure either.
5. *Test.* A test stated before data: if the Foundations’ ring reading is right, bound states of three identical clusters should be favoured over bound states of two

whenever the pair interaction sits near threshold. Efimov physics predicts exactly this, so the test does not separate the Foundations from standard quantum mechanics. A separating test would have to predict a number — a binding energy, a scaling factor — that standard physics does not. The Foundations do not give one for nuclei. I record C for shell structure and A for the three-alpha ring as structure.

Consequences. Nuclear physics offers one clean example and one clean counterexample. The example is carbon: two alphas form a meeting that falls apart, three form a ring that lasts, and the ring is the gate to all heavier elements made in stars. The reading adds nothing to the physics of the Hoyle state, but it is an exact instance of T7 and T7.1 in nature. The counterexample is the shell model: the magic numbers come from spin–orbit splitting in a central potential and do not follow the ladder 2^a·3^b. A reading that tried to force them onto the ladder would be re-description of the worst kind.

What the Foundations could add is a question rather than an answer: are three-body bound states (Hoyle state, Efimov trimers, Borromean halos) systematically more robust than two-body ones in a way that a single rule captures? Standard few-body physics already has the answer in its own terms — universality near threshold — and it is better than the Foundations' rule because it gives numbers. The thorium nuclear clock is relevant to Chapter 2.4: a nuclear transition is a far more sensitive probe of changes in the fine-structure constant than any atomic line, and it is the instrument that could one day test the companion document's prediction.

Theorem checklist. T1 and T7.1 apply to alpha clustering only by counting (three alphas, carry one carbon); there is no residue rule. T2 matches the Hoyle state: three is the least run that holds. T3–T6 do not apply. T7 applies as meeting (^{8}Be) and ring (^{12}C). T8–T11 do not apply. T12–T14 do not apply. T15 is checked against magic numbers and oscillator closures (Calculation 2.4) and fails after 8. T16–T18 do not apply. T19 has a counterpart in Borromean binding (three linked as a whole, no two linked), which is a statement about links, not knots. T20–T22 do not apply. T23 has a loose counterpart in density-dependent nuclear forces, which nuclear physics already uses; no test is stated. T24 does not apply.

Grade. A for the triple-alpha process as a meeting that fails and a ring that holds (Salpeter 1952; Hoyle 1953) and for Borromean binding (Efimov 1970; Kraemer et al. 2006) as published structure; C for shell structure, whose magic numbers leave the ladder after 8.

2.4 Atomic physics

History. Johann Balmer fitted the visible lines of hydrogen in 1885 and Johannes Rydberg generalised the formula in 1888. Niels Bohr explained it in 1913 with quantised orbits. Arnold Sommerfeld extended Bohr's model to elliptic orbits in 1916 and introduced the fine-structure constant α, then read as the ratio of the electron's speed in the first orbit to the speed of light (Wikipedia, Fine-structure constant). Wolfgang Pauli's exclusion principle (1925) and the electron's spin built the periodic table: shells hold $2n^2$ electrons, and the periods of the table have lengths 2, 8, 8, 18,

18, 32, 32. Dirac's equation of 1928 gave the fine structure exactly; the Lamb shift of 1947 and the electron's anomalous moment required quantum electrodynamics.

Precision spectroscopy then became the core of the discipline. The laser (Maiman's ruby laser, *Nature* 187, 493–494, 1960), laser cooling, Bose–Einstein condensation in 1995 and the optical frequency comb, proposed around 2000 and honoured with the 2005 Nobel Prize to Hänsch and Hall (Wikipedia, Optical frequency comb), turned atoms into clocks and α into one of the best-measured numbers in science. The 2019 revision of the SI fixed the Planck constant, the elementary charge, the Boltzmann constant and the Avogadro constant exactly (Wikipedia, 2019 revision of the SI). Since 1999 John Webb and colleagues have reported hints that α differs in distant quasar absorbers; later analyses found no variation at the part-per-million level (Wikipedia, Fine-structure constant). The question the history left is whether the constants of atomic physics are constant everywhere — across time, space and gravitational potential.

Current state. The CODATA 2022 value is α = 0.0072973525643(11), 1/α = 137.035999177(21), a relative uncertainty of 1.6×10^{-10} (Wikipedia, Fine-structure constant). The two best direct determinations disagree: the rubidium recoil measurement of Morel et al. (2020) gives 1/α = 137.035999206(11), the electron g-factor of Fan et al. (2023) with 12,672 tenth-order QED diagrams gives 137.035999166(15). They differ by 4.0×10^{-8} in 1/α, 2.15 combined standard deviations (Calculation 2.7). The Rydberg constant is R∞ = 10,973,731.568157(12) m^{-1} (Wikipedia, Rydberg constant); the proton–electron mass ratio is 1836.152673426(32) (Wikipedia). The proton-radius puzzle, opened by muonic hydrogen's 0.842(1) fm in 2010, is widely regarded as resolved, with the Particle Data Group at 0.8409(4) fm and a 2026 study at 0.8406 fm (Wikipedia, Proton radius puzzle).

Clocks have reached the 19th decimal. NIST's aluminium-ion quantum-logic clock of July 2025 set a new record for systematic uncertainty, 41 % better than before (Wikipedia, Atomic clock). Comparisons of atomic clocks limit any present drift of α to $1.0(1.1) \times 10^{-18}$ per year (Lange et al. 2021, arXiv:2010.06620). The redefinition of the second is expected around 2030 or 2034 (Wikipedia, Atomic clock).

Gravity's effect on α is tested on white dwarfs. Hu, Webb and colleagues analysed Fe V absorption in the Hubble spectrum of G191-B2B, where the gravitational potential is about 10^4 times that at the Earth's surface, and found $\Delta\alpha/\alpha = 6.36 \pm (0.33\ \text{stat} + 1.94\ \text{sys}) \times 10^{-5}$ and $4.21 \pm (0.47 + 2.35) \times 10^{-5}$ from two line sets; the systematic error is dominated by laboratory wavelengths (arXiv:2007.10905). Antihydrogen spectroscopy and gravity tests at CERN continue (Wikipedia, ALPHA).

Interpretation.

1. *Strand.* I is the sequence of photon emissions and absorptions in a spectroscopic measurement; P is the set of atomic levels. A loop is a return to a level: an excitation and decay, or a closed cycle of transitions.
2. *Passage and trit.* A transition up is +1, down −1, a level kept 0. This is a reading. The published structure that comes closest is the selection rule Δl = ±1 for dipole

transitions, which is a meeting rule: every absorbed dipole photon changes l by one step up or down.

3. *Closure.* A closed cycle of transitions — up, down, back — is a closed loop. Lasers show that three levels are the least that close usefully: a two-level system pumped optically can reach at best equal populations and no gain, while a three-level system can invert, as in Maiman's ruby laser of 1960; four-level lasers invert more easily (Wikipedia, Population inversion). This is a published fact in which two levels form a meeting that cannot produce net gain and three levels form the first ring that can.
4. *Scale.* The periods of the periodic table, 2, 8, 8, 18, 18, 32, 32, are all on the ladder 2^a·3^b (Calculation 2.4). This is not a deep fact: shell sizes are $2n^2$, which is 3-smooth exactly when n is, and the table uses only n = 1 to 4 (Calculation 2.12). The first shell off the ladder, 50 for n = 5, is never a period length. The companion document's Larson units (Physics T5) set $S_0 = 1/(2R_H) = 4.55881671 \times 10^{-8}$ m and $E_0 = 2hcR_H = 27.1965745$ eV from the hydrogen Rydberg constant (Calculation 2.8). These are reduced-mass versions of familiar atomic units: E_0 is close to the Hartree energy, 27.211386 eV, which uses R_∞. The unit choice is convention; it predicts nothing by itself. Of the dimensionless constants, $m_p/m_e = 1836.15$, $m_\mu/m_e = 206.77$ and $1/\alpha = 137.036$ lie 5.6 %, 4.3 % and 4.8 % below the nearest ladder numbers 1944, 216 and 144 (Calculation 2.11). The gaps between 3-smooth numbers are about this size at these magnitudes, so the constants are not on the ladder in any sense that matters.
5. *Test.* The companion document predicts a gravitational shift of α at the surface of G191-B2B: $\Delta\alpha/\alpha \approx 3.08 \times 10^{-8}$. Stated before any new data: if a measurement at G191-B2B or a comparable white dwarf reaches an uncertainty of 10^{-8} and finds $\Delta\alpha/\alpha$ inconsistent with 3.08×10^{-8} at 3σ, the prediction fails. The present measurement cannot test it. Its uncertainties of 2.27×10^{-5} and 2.82×10^{-5} are 737 and 916 times larger than the predicted effect; the measured central values lie 2.8σ and 1.5σ above the prediction, and above zero, but the authors themselves attribute the deviation to unexplained systematics (Calculation 2.9).

Consequences. If the white-dwarf prediction held, atomic physics would gain a coupling of α to gravitational potential with a definite size. That is new, and it would be the first number in this chapter that the Foundations add rather than read. The cost is that the predicted shift is three orders of magnitude below present precision on white dwarfs, so the prediction will remain untested for years. A nuclear clock based on thorium-229 is far more sensitive to α than atomic lines, and clocks on satellites in eccentric orbits could test an α–potential coupling at the Earth; the Foundations would need to state the predicted size in those settings, which the companion document does not yet do.

The rest of atomic physics adds no new link. The three-level threshold for lasing is a real published instance of "a meeting does not gain, a ring can", but it follows from rate equations, not from closure mod 3. The ladder fit of the periodic table is a small-number effect. The Larson unit E_0 is a renamed atomic unit. The Morel–Fan tension in α is a problem the field will settle by measurement; the Foundations have nothing to say about which value is right.

Theorem checklist. T1–T6 apply to transitions only by convention. T7 applies to the two-level/three-level distinction of lasing. T8–T11 do not apply. T12 does not apply. T13 does not apply. T14 does not apply to atoms. T15 is checked against periods of the table, shell sizes and dimensionless constants (Calculations 2.4, 2.11, 2.12): periods are on the ladder for trivial reasons, the constants are not. T16–T18 do not apply. T19–T20 do not apply. T21 applies to the electron's spin. T22 does not apply. T23 is the frame of the companion document's α–potential coupling, from which the white-dwarf prediction derives; the test is in Calculation 2.9. T24 does not apply.

Grade. A for the three-level threshold of population inversion as a published meeting/ring distinction (Maiman 1960); B for the predicted $\Delta\alpha/\alpha \approx 3.08 \times 10^{-8}$ at G191-B2B (testable in principle, presently below precision by a factor of about 700); C for periodic-table periods on the ladder, for the Larson units and for any ladder fit of dimensionless constants.

Chapter 3 — Forces and Motion

This chapter reads the meeting column of level L1: exchange that goes and comes back. Physics is full of exchanges, and most of them are meetings in the plainest sense — action and reaction, emission and absorption, a current that flows and returns. The Foundations' distinction between a meeting (x, −x, carry 0) and a ring (x, x, x, carry ±1) turns out to match one published fact across several fields: a net circulation needs a cycle, and the shortest cycle that can carry one has three members. Classical mechanics is grade A for the triangle as the only rigid polygon and the 2j − 3 count, and A for the equilateral Lagrange solution; electromagnetism and optics is C apart from the cube roots of unity, which belong to Chapter 5; quantum mechanics is A for dimension three as the threshold of contextuality; thermodynamics and statistical physics is A for the three-state cycle as the least that carries a steady current, and B for a test of T23 on the bulk modulus that the free-path lever of T24 fails if the two are identified; condensed matter is A for frustration on the triangle and for fractional charge in thirds, with C for the rest.

3.1 Classical mechanics

History. Johannes Kepler published his first two laws in *Astronomia nova* in 1609 and the third, that the square of a planet's period is proportional to the cube of its distance, in *Harmonice Mundi* in 1619 (Wikipedia, Kepler's laws). Newton derived all three from one force and three laws of motion in the *Principia* of 1687. The third law — every action has an equal and opposite reaction — is the founding meeting of mechanics: a pair of forces x and −x whose sum is zero.

The problem of three bodies followed at once and has never been solved in closed form. Leonhard Euler found three families of collinear periodic solutions in 1767, and Joseph-Louis Lagrange found in 1772 that three bodies at the corners of an equilateral triangle can keep that shape for ever (Wikipedia, Three-body problem). The triangular points L4 and L5 are stable when the larger mass exceeds the smaller by a factor of about 25 (Wikipedia, Lagrange point); the exact threshold, $27\mu(1-\mu) < 1$, gives a mass ratio of $(25 + 3\sqrt{69})/2 \approx 24.96$ (Calculation 3.2). Henri Poincaré showed in 1889 that the general problem has no further analytic integrals, and found in doing so the seed of chaos.

Joseph Bertrand proved in 1873 that only two central forces make every bound orbit close: the inverse-square force of gravity and the linear force of a spring (Wikipedia, Bertrand's theorem). For a power-law force $F \propto r^n$, a nearly circular orbit oscillates radially at $\sqrt{(n+3)}$ times its angular rate; the orbit closes for every amplitude only for $n = -2$ (ratio 1) and $n = 1$ (ratio 2) (Calculation 3.1). Kepler orbits close after one radial oscillation; spring orbits after two.

James Clerk Maxwell asked in 1864 when a frame of bars is rigid. In the plane, j joints have 2j degrees of freedom, of which three are rigid motions of the whole, so 2j − 3 well-placed bars are needed. A polygon of j bars has only j. Equality holds only at j = 3:

the triangle is the only rigid polygon (Calculation 3.3). George Pólya proved in 1921 that a random walk on a lattice returns to its start with certainty in one and two dimensions but not in three, where G.N. Watson computed the return probability in 1939 as 0.3405373… (OEIS A086230). The questions the history left are the integrability of many-body motion, the long-term stability of the solar system, and the mechanics of structures that sit exactly at Maxwell's count.

Current state. Celestial mechanics has become numerical. Li and Liao reported 10,059 new three-dimensional periodic orbits of the three-body problem in August 2025, 1,996 of them linearly stable (arXiv:2508.08568). The James Webb Space Telescope has orbited the Sun–Earth L2 point since 24 January 2022, and Jupiter's L4 and L5 regions hold more than a million Trojan asteroids (Wikipedia, Lagrange point). Mercury rotates three times for every two orbits: radar in 1965 (Pettengill and Dyce, Arecibo) found a rotation of about 59 days, Giuseppe Colombo proposed the 3:2 lock the same year, and Mariner 10 confirmed it; the current values, 58.646 and 87.97 days, give a ratio of 1.50002 (Calculation 3.1; Wikipedia, Mercury). General relativity accounts for 42.980 arcseconds per century of Mercury's perihelion advance, the part Bertrand's theorem says a pure inverse-square force cannot produce (Wikipedia, Mercury).

Mechanics of structures has found topology in Maxwell's count. Kane and Lubensky showed in 2013 that lattices at exactly Maxwell's balance, "isostatic" lattices in which bonds equal degrees of freedom, have topological phases with boundary floppy modes that resist local perturbation (*Nature Physics* 10, 39–45) (Nature Physics). In two dimensions a lattice is isostatic when each site has four bonds on average. The kagome lattice, built from corner-sharing triangles, sits exactly at that count, the honeycomb is below it and floppy, the triangular lattice above it and rigid (Calculation 3.3). Metamaterials designed from these rules can steer where a structure gives way.

The open debates are how chaotic the solar system is over billions of years, how to classify the zoo of numerical three-body orbits, and how far topological mechanics can be used in engineering.

Interpretation.

1. *Strand.* I is time; P is configuration space. A loop is a return of a body to a state it has held: a periodic orbit, or a closed path of a frame member back to a joint.
2. *Passage and trit.* At a joint, a bar in tension pulls (+1), a bar in compression pushes (−1), an idle bar does neither (0). Newton's third law is a meeting: each force x is answered by −x, and the pair closes with carry 0 (T7.1). In an orbit, a radial passage outward (+1), inward (−1) or at the turning point (0) gives a trit word; I know of no published use of this word.
3. *Closure.* The triangle is the least closed loop of bars that holds its shape (Calculation 3.3). In the Foundations, the ring x, x, x is the shortest closure with winding ±1 (T7). The two statements are different — one is about degrees of freedom, the other about sums mod 3 — but they pick the same number for the same reason in one respect: two members can only meet, three are the least that go round. Lagrange's equilateral solution is a ring in configuration: three bodies, each

displaced by 120° about the centre of mass, the cube roots of unity. Its stability threshold contains 27 = 3^3 (Calculation 3.2), which is the window count of T3 but enters through the linearised equations, not through any closure rule.

4. *Scale.* Bertrand's closed orbits have radial-to-angular ratios 1 and 2: a meeting ladder (×2) of returns. Mercury's spin–orbit lock 3:2 is on the ladder 2^a·3^b; resonance ratios in general are treated in Chapter 4. Pólya's walk gives the Foundations' T24 directly: a node of a free path in three dimensions counts its visits V with P(V = k) geometric and a lever $\gamma = -(1 + P_3)/P_3 = -3.9365$ (Calculation 3.5). T19–T22 enter mechanics through vortex knots (Chapter 2.2) and the 4π return of a body tethered to its surroundings (Dirac's belt trick, T21).
5. *Test.* A test stated before data: in a planar frame, the Foundations' reading predicts that a structure whose closed loops are all triangles is rigid and that any non-triangular loop needs a bracing member. Maxwell's count and its refinements (Laman's theorem) already decide this exactly. The test therefore cannot separate the Foundations from mechanics. A separating test would concern isostatic lattices: if the Foundations' trits were the physical labels of floppy modes, the topological polarisation of Kane–Lubensky lattices should be expressible as a sum mod 3. It is an integer winding number, not a residue. I record this as a negative: the published topology is $\mathbb{Z}$, not $\mathbb{Z}_3$.

Consequences. Classical mechanics contains three facts that fit the Foundations' meeting and ring as published structure. Newton's third law is a meeting with carry 0. The triangle is the least rigid loop. Lagrange's equilateral solution places three bodies at the cube roots of unity. None of these is due to the theory, and none of them needs the theory to be understood.

What the reading adds is a single vocabulary for them, and one consequence that is worth stating because it could have gone otherwise: the Foundations' smallest closed winding (the ring, length 3) and mechanics' smallest rigid loop (the triangle) coincide. If mechanics had needed four bars for rigidity in the plane, the coincidence would fail. It does not, but the cause is the count 2j − 3, not mod-3 arithmetic. The honest statement is that both theories find three to be the least number that goes round without collapsing to a back-and-forth pair.

The lever of T24 is a computed number with a mechanical meaning: how strongly the visit count of a free three-dimensional walk is skewed. It has no published use in mechanics. Its proposed meaning as the coupling of the medium (IX.1) is tested in 3.4.

The cost of the reading is in the trit of an orbit: radial outward and inward passages are a sign convention, and attaching ±1 to them is arbitrary. Nothing in celestial mechanics depends on it.

Theorem checklist. T1 and T7.1 apply to Newton's third law as a meeting with zero sum. T2 matches the triangle as least rigid loop in number, not in mechanism. T3–T6 do not apply to orbits. T7 applies by name. T8–T10 do not apply. T11 does not apply. T12–T14: the Lagrange configuration places three bodies at the cube roots of unity, the units of Z[ω] of T14. T15 is checked against Bertrand's ratios (1 and 2) and Mercury's 3 : 2 (Calculation 3.1). T16–T18 apply to resonances (Chapter 4). T19 applies to knotted

vortex lines in fluids (2.2). T21 applies to a body tethered to its surroundings (belt trick). T22 does not apply. T23 applies to materials (3.4). T24 applies exactly through Pólya's walk (Calculation 3.5).

Grade. A for the triangle as the only rigid polygon (Maxwell 1864, 2j − 3), Newton's third law as a meeting with zero sum, and Lagrange's equilateral solution as three bodies at the cube roots of unity; C for trit words of orbits and for topological mechanics, whose invariant is an integer, not a residue mod 3.

3.2 Electromagnetism and optics

History. Charles-Augustin de Coulomb measured the inverse-square law of electric force in the 1780s; Hans Christian Ørsted found in 1820 that a current deflects a compass; Michael Faraday discovered induction in 1831. James Clerk Maxwell joined these into one set of field equations in the 1860s and found that they predict waves travelling at the speed of light. Heinrich Hertz produced radio waves in 1887. Maxwell wrote much of his theory in the quaternions of William Rowan Hamilton; Oliver Heaviside and Josiah Willard Gibbs later recast it in the vector notation used today. The companion document, *The Vacuum.Net Theory: Physics*, returns to Maxwell's quaternion form (Physics T2–T4).

Light as a quantum came with Planck in 1900 and Einstein in 1905. Theodore Maiman's ruby laser of 1960 (*Nature* 187, 493–494) opened modern optics. Peter Franken observed second-harmonic generation in 1961, and J.A. Giordmaine and Robert Miller demonstrated the first optical parametric oscillator at Bell Labs in 1965, splitting a pump photon into signal and idler photons with $\omega_p = \omega_s + \omega_i$ (Wikipedia, Optical parametric oscillator). The second-order susceptibility $\chi^{(2)}$ that makes this possible gives three-wave mixing: second-harmonic, sum-frequency and difference-frequency generation, and optical rectification, in crystals such as BBO, KTP and lithium niobate (Wikipedia, Nonlinear optics).

The optical frequency comb, proposed around 2000, turned a mode-locked laser's pulse train into a ruler of evenly spaced optical frequencies. Theodor Hänsch and John Hall shared half the 2005 Nobel Prize for it (Wikipedia, Optical frequency comb). To fix the comb's offset frequency one compares the doubled frequency of one end of the spectrum with the other end, the "f − 2f" method, which needs a spectrum one octave wide. A variant, "2f − 3f", compares doubled and tripled light and needs only two-thirds of an octave (NTT Basic Research Labs). The question the history left is how far light can be shaped, measured and confined: down to single photons, attoseconds and topological structures.

Current state. The 2023 Nobel Prize in Physics went to Pierre Agostini, Ferenc Krausz and Anne L'Huillier for experimental methods that generate attosecond pulses of light (Wikipedia, Nobel laureates in Physics). Combs have moved onto chips: Del'Haye and colleagues generated a comb in a monolithic microresonator in 2007, and dissipative Kerr solitons in such resonators now give compact, low-noise combs (Wikipedia, Optical frequency comb). Optical clocks built on combs have reached uncertainties in the 19th decimal place (Chapter 2.4).

Light can now carry topological knots. Sugic and colleagues reported in November 2021 particle-like hopfion structures in the polarisation of free-space monochromatic light, and Ehrmanntraut and colleagues an optical second-order skyrmionic hopfion in 2023 (Wikipedia, Hopfion). Topological photonics designs waveguides whose edge states are protected by winding numbers. The open debates are the limits of on-chip integration, the use of structured light for communication, and the role of the vacuum in quantum optics (squeezing, the Casimir effect).

Interpretation.

1. *Strand.* I is the course of a light ray or field line; P is the set of points of space, or the set of modes of a cavity.
2. *Passage and trit.* In three-wave mixing the signed photon numbers of a process are a word: one pump photon absorbed, two photons emitted. Energy and momentum are conserved in the integers, $\omega_p = \omega_s + \omega_i$, $k_p = k_s + k_i$, for any number of photons (Calculation 3.11). There is no published mod-3 rule here. The trit appears in optics only where the cube roots of unity appear: three-phase symmetry (Chapter 5) and hexagonal photonic lattices.
3. *Closure.* Maxwell's equations in empty space conserve charge exactly; a closed current loop is a closed loop of the strand in the most literal sense. The Foundations' closure mod 3 adds nothing to charge conservation, which is a statement in the integers. The $2f - 3f$ method is a closure of a different kind: it uses the smallest frequency ratio, 3/2, that is not an octave, and compares two harmonics on the ladder. That is the first rung of T17 (closure depth 1, 3 against 2^2), but the reason is spectral bandwidth, not closure.
4. *Scale.* T15's ladder shows in optics as harmonic generation, ×2 and ×3, which are second- and third-order processes. T22's Hopf charge is the invariant of the optical hopfions observed in 2021. These are published structures that use the same mathematics as the Foundations' Part VII.
5. *Test.* For electromagnetism as such, the Foundations offer no test that standard theory does not already pass. The companion document's quaternion form of Maxwell's equations is a rewriting. A test with content would be a prediction that some optical process is forbidden or favoured mod 3 — for example that third-harmonic generation in a medium with a closed trit structure is enhanced relative to the $\chi^{(3)}$ expectation. No such prediction is stated, so I record C.

Consequences. Electromagnetism and optics are where the Foundations' register meets the most precise measurements in physics, and where it adds least. Charge conservation, wave propagation and nonlinear optics are statements in the real or integer numbers, and none of them is a residue rule. The hooks that do exist — cube roots of unity in three-phase systems, the ladder of harmonics, the Hopf invariant of structured light — are either covered in other chapters (three-phase power in Chapter 5) or are uses of standard mathematics that the Foundations share with optics, not take from it.

One point deserves recording because it is a real, if small, coincidence of design. When comb engineers lacked an octave of bandwidth, they turned to the ratio 3/2 — the first

rung of the Foundations' closure depths, the musical fifth. They did so for a practical reason: second and third harmonics are the lowest-order processes available. The Foundations would read this as the ladder's first near-meeting being the most useful one. That is a reading, not a result.

The companion document's quaternion formulation of Maxwell's equations follows Maxwell himself; it is a valid notation, and its value will depend on whether it leads to a prediction. As of this edition it leads to the units of Chapter 2.4 and to the predictions of Chapter 4, not to any optical effect.

Theorem checklist. T1–T7 apply only to three-phase quantities (Chapter 5.1), not to Maxwell's equations. T8–T11 do not apply. T12 does not apply. T13 does not apply. T14 applies to hexagonal photonic lattices only by geometry. T15 applies to harmonic orders ×2 and ×3. T16–T18 apply to the 2f − 3f ratio 3/2 (Calculation 3.11). T19–T20 do not apply directly. T21 has a counterpart in the double cover of polarisation states, which I have not verified for this book and do not use. T22 applies exactly to optical hopfions. T23 applies to nonlinear media whose response depends on intensity, which is optics' own constitutive relation; no test is stated. T24 does not apply.

Grade. A for the Hopf invariant of optical hopfions (Sugic et al. 2021) as the published structure of T22; C for the rest of electromagnetism and optics, including the quaternion form of Maxwell's equations and the 3/2 ratio of 2f − 3f self-referencing.

3.3 Quantum mechanics

History. Werner Heisenberg's *Umdeutung* paper of 1925 began quantum mechanics as a theory of observable transition frequencies; Erwin Schrödinger's wave equation and Max Born's probability rule followed in 1926. The United Nations declared 2025 the International Year of Quantum Science and Technology to mark that centenary (Wikipedia, IYQ). Wolfgang Pauli's spin matrices of 1927 and Dirac's equation of 1928 made spin a two-valued quantum number with a 4π return.

The question of whether quantum results reveal pre-existing values produced three theorems in which the number three is decisive. Andrew Gleason proved in 1957 that every probability measure on the subspaces of a Hilbert space of dimension 3 or more comes from a density operator; in dimension 2 there are counterexamples (Wikipedia, Gleason's theorem). John Bell in 1966 and Simon Kochen and Ernst Specker in 1967 proved that no assignment of definite values to all observables, independent of the context in which they are measured, is consistent with quantum mechanics in dimension 3 or more. Kochen and Specker's proof used 117 projection operators; later proofs need 18 (Wikipedia, Kochen–Specker theorem). A single qubit, by contrast, admits a hidden-variable model. The smallest system with contextuality is the qutrit.

Bell's 1964 inequality showed that no local hidden-variable theory reproduces quantum correlations between two systems. Alain Aspect's group tested it in 1982 with fast-switching polarisers; John Clauser had done earlier tests; Anton Zeilinger's group closed further loopholes. The three shared the 2022 Nobel Prize (Wikipedia, Bell test). The simplest contextuality inequality for a single qutrit, the KCBS inequality, is built on five measurements arranged in a pentagon; its classical bound is 2 and its quantum

bound √5, the Lovász number of the five-cycle (Wikipedia, Lovász number; Wikipedia, KCBS pentagram). The questions the history left are the meaning of the wave function, the measurement problem, and where the boundary of quantum behaviour lies.

Current state. Three groups closed the main loopholes of Bell tests in 2015. The Delft group entangled electron spins in diamonds 1.3 km apart and found $S = 2.42 \pm 0.20$ with $p = 0.039$; the Vienna and Boulder groups used photons and reached p-values far below 10^{-6} (Wikipedia, Bell test). The first experimental test of contextuality was made in 2000, and a version free of detection, sharpness and compatibility loopholes in 2022 (Wikipedia, Kochen–Specker theorem). The boundary of quantum behaviour has moved to circuits large enough to hold in the hand. The 2025 Nobel Prize went to John Clarke, Michel Devoret and John Martinis for "the discovery of macroscopic quantum mechanical tunnelling and energy quantisation in an electric circuit" (Wikipedia, Nobel laureates), the physics behind superconducting qubits.

Quantum processors now run experiments that strain classical simulation. Google's Willow chip (105 qubits, December 2024) completed a random-circuit-sampling task in under five minutes that would take a leading supercomputer about 10^{25} years (Google), and in October 2025 Google reported measurements of second-order out-of-time-order correlators that remain sensitive to the dynamics long after ordinary observables scramble, attributing the effect to constructive interference between large loops of Pauli strings (*Nature* 646, 2025) (Nature). Qutrit processors are maturing, with high-fidelity entangling gates on superconducting circuits and qudit error correction beyond break-even reported in 2025 (Wikipedia, Qutrit).

Interpretation.

1. *Strand.* I is the sequence of measurements on a system; P is the set of projectors (or of outcomes). A return is a repeated measurement of the same projector in a different context.
2. *Passage and trit.* A qutrit's three outcomes are a trit: the quantum Fourier transform on $\mathbb{Z}_3$ uses the cube roots of unity, and $1 + \omega + \omega^2 = 0$ (Calculation 3.12). This is standard. Spin-½ is two-valued, so the Foundations' trit does not label it; T21 instead gives its 4π return.
3. *Closure.* Kochen–Specker and Gleason say that dimension 3 is the least in which quantum probabilities cannot be pre-assigned. In the Foundations, the ring of three is the least closure with winding (T7). Both say that two is too few for a new structure and three is enough. The structures are different: one is about orthogonal bases in Hilbert space, the other about sums of digits. The KCBS inequality has a pentagon, not a triangle, at its core: five measurements, classical bound 2, quantum bound √5 (Calculation 3.9). The odd cycle is what matters; for the triangle the classical and quantum bounds coincide at 1 (Calculation 3.9).
4. *Scale.* Error-correcting surface codes scale in odd distances 3, 5, 7 and suppress errors by a factor of about 2 per step (Chapter 1.5). These are not on the ladder in any significant way. T21 is exactly the spinor's double cover, confirmed by neutron interferometry in 1975 (Chapter 2.2).

5. *Test.* A test that could fail: if the Foundations' claim that three is the least closing register has physical content, contextuality should be absent for every two-level system and present for every three-level system, under every set of measurements. This is exactly Gleason and Kochen–Specker, and it holds. So the test is passed, but by theorems that do not use the Foundations. A separating test would need the Foundations to predict a bound on a contextuality inequality different from the quantum one. They do not.

Consequences. Quantum mechanics contains a published threshold at three that the Foundations did not need to invent: a qubit can be modelled classically, a qutrit cannot. This is a real point of contact. It is also limited. The mathematical cause is the structure of orthogonal frames in three dimensions — the same reason the Kochen–Specker sets are built from vectors in $\mathbb{R}^3$ — and the Foundations' $\mathbb{Z}_3$ arithmetic does not reproduce it. A reader should not merge the two "threes".

The second point of contact is the 4π return of spin-½ (T21), confirmed in 1975. The third is the Fourier transform on $\mathbb{Z}_3$, used in every qutrit algorithm, whose kernel is the cube roots of unity of the Foundations' remark in III.1.

The Foundations' rule "what does not close passes on as tension" has a tempting quantum reading in decoherence: interference terms that do not survive averaging pass into the environment. Google's 2025 result, that large loops of Pauli strings interfere constructively at long times, is a statement about loops in operator space, and the Foundations would read it as closure. That is a reading without a test. The theory adds no quantum prediction of its own in this edition.

Theorem checklist. T1 applies to qutrit labels. T2–T6 apply to qutrit measurement records. T7 applies in name to the two-level/three-level threshold. T8–T11 do not apply. T12 does not apply. T13 does not apply. T14 does not apply. T15 does not apply to code distances. T16–T18 do not apply. T19–T20 do not apply. T21 applies exactly. T22 applies to the Hopf fibration of the qubit's Bloch sphere. T23–T24 do not apply.

Grade. A for dimension three as the threshold of contextuality and of Gleason's theorem (Gleason 1957; Kochen and Specker 1967) and for the 4π return of spinors (T21; Rauch et al. 1975); C for closure as decoherence and for any quantum prediction of the Foundations.

3.4 Thermodynamics and statistical physics

History. Sadi Carnot's analysis of heat engines in 1824, Rudolf Clausius's entropy and Lord Kelvin's absolute temperature made thermodynamics a science of energy and its degradation. Ludwig Boltzmann and Josiah Willard Gibbs grounded it in the statistics of atoms. Rudolf Wegscheider showed in 1901 that in a network of chemical reactions at equilibrium no cycle can carry a net flow, and derived the relations between rate constants this requires; Lars Onsager built on these relations for his reciprocal relations of 1931, recognised by the 1968 Nobel Prize in Chemistry (Wikipedia, Detailed balance). Andrey Kolmogorov gave the general form: a Markov chain is reversible exactly when, for every closed loop of states, the product of transition rates is the same in both directions (Wikipedia, Kolmogorov's criterion). In 1976 Jürgen

Schnakenberg built a network theory of master equations on graph cycles (*Reviews of Modern Physics* 48, 571) (APS).

Statistical mechanics of phase transitions began with models on lattices. Ernst Ising solved the one-dimensional magnet in 1925 and found no transition; Rudolf Peierls argued in 1936 that the two-dimensional model has one; Lars Onsager solved it exactly in 1944 (Wikipedia, Ising model). Renfrey Potts generalised the two-state spin to q states in 1952; in two dimensions the transition is continuous for $q \leq 4$ and first order above (Wikipedia, Potts model). Linus Pauling explained the residual entropy of ice in 1935 from the Bernal–Fowler rules (two protons near and two far from each oxygen), estimating $S_0 = R \ln(3/2)$ (Wikipedia, Geometrical frustration). Gregory Wannier showed in 1950 that the Ising antiferromagnet on a triangular lattice also keeps entropy at zero temperature (*Physical Review* 79, 357–364). Rolf Landauer tied information to heat in 1961.

Equations of state for solids became a practical problem in geophysics. Francis Birch's finite-strain equation of state of 1947 relates pressure to volume through the bulk modulus K_0 and its pressure derivative K_0'; typical fitted values of K_0' lie between 4 and 8, and the second-order form fixes $K_0' = 4$ (Wikipedia, Birch–Murnaghan equation). The questions the history left include non-equilibrium steady states, the origin of the arrow of time, and the microscopic meaning of universal exponents.

Current state. Stochastic thermodynamics now measures heat, work and entropy production in single molecules and colloids. Landauer's bound $kT \ln 2$, about 0.018 eV at 300 K, was measured for a single erased bit by Bérut and colleagues in 2012 and tested in nanomagnets in 2016 and molecular magnets in 2018 (Wikipedia, Landauer's principle). Erasing a trit costs $kT \ln 3 = 0.0284$ eV at 300 K, which is the same cost per bit of information (Calculation 3.7); the register gains nothing thermodynamically. Non-equilibrium steady states are characterised by Schnakenberg's cycle affinities: a steady state carries a net current only if some cycle of its state graph has nonzero affinity. Bose–Einstein condensates, first made in 1995 by Cornell, Wieman and Ketterle, have been produced continuously since 2022 (Wikipedia, Bose–Einstein condensate). Rayleigh–Bénard convection, onset at a critical Rayleigh number of 1708 with roughly hexagonal cells, remains the reference system of pattern formation (Wikipedia, Rayleigh–Bénard convection).

Two published facts matter for the register. First, a two-state system cannot carry a steady net current: its only cycle is "there and back", and Kolmogorov's condition holds automatically. A three-state ring with clockwise rate 2 and anticlockwise rate 1 carries a net current of 1/3 per unit time and has cycle affinity $\ln 8 = 2.079$ (Calculation 3.4). The least graph cycle that can carry a current has three states. Second, for any energy density $e(n)$, the bulk modulus $K = n \cdot dP/dn$ and its pressure derivative are $B' = 2 + n \cdot e'''(n)/e''(n)$ (Calculation 3.5). For the energy density of T23, this is $B' = 2 + \gamma$ exactly, and the effective stiffness g_{eff} of T23 is K/n^2. T23's coupling γ is therefore a measured quantity: $\gamma = K_0' - 2$.

Interpretation.

1. *Strand.* I is time; P is the set of microstates (or of coarse states of a master equation). A loop is a return to a state; a cycle of the state graph is a closed loop.
2. *Passage and trit.* A transition along a cycle in its positive direction is +1, against it −1, a waiting step 0. This matches Schnakenberg's oriented cycles: the current on a cycle is a signed sum of transitions. The trit labels are a published structure up to the choice of ±1 for direction.
3. *Closure.* A meeting (there and back) carries no current: carry 0 (T7.1). A ring of three states traversed with a bias carries a current: each full turn is one unit at the next scale, carry ±1. Kolmogorov and Schnakenberg establish this structure independently of the Foundations, and the minimal current-carrying cycle in a graph without multiple edges has exactly three states (Calculation 3.4). Pauling's ice entropy is a closure count of another kind: of the 16 arrangements of four protons around an oxygen, 6 obey the ice rule, and $W = 2^2 \cdot 6/16 = 3/2$ per molecule (Calculation 3.6). Wannier's triangle is frustrated: of 8 spin states on one antiferromagnetic triangle, 6 satisfy the best possible 2 of 3 bonds, while a square plaquette has 2 perfect ground states (Calculation 3.6). An odd ring cannot alternate; this is the Foundations' "a ring does not reduce to meetings" in magnetic form.
4. *Scale.* T23 gives $g_{eff} = \bar{g}(1 + \gamma\delta)$ as a constitutive identity; Calculation 3.5 shows that its γ equals $K_0' - 2$ for any material described by such an energy density. T24 gives a lever $-(1 + P_d)/P_d$ for a free random path: −3.9365, −6.1759 and −8.3976 in three, four and five dimensions. The Foundations leave open (IX.1) whether the two are the same.
5. *Test.* Stated before comparison with data: if the lever of a free path in three dimensions were the coupling γ of a solid medium, then $K_0' = 2 + \gamma = 2 - 3.94 = -1.94$. Every fitted solid has K_0' between about 4 and 8, which gives γ between 2 and 6 (Calculation 3.5). The identification of IX.1 therefore fails for ordinary solids, by sign and size. The Foundations do not assert the identification, so this does not refute the theory; it closes one branch of IX.1. A remaining branch would need a medium whose density fluctuations follow a free path, which ordinary solids do not.

Consequences. Statistical physics contains the meeting–ring distinction as published fact in the theory of non-equilibrium steady states: a two-state exchange is reversible, a three-state ring is the least that can carry a current. This is the cleanest physical sense in which "a ring closes with carry ±1" can be read: one turn of the cycle is one unit of net flux. The frustration of odd rings (Wannier, Toulouse) and Pauling's 3/2 are further instances in which an odd closed loop leaves something over.

The Foundations' T23 turns out to have a precise thermodynamic meaning. Its g_{eff} is K/n^2 and its γ is $K_0' - 2$. That makes γ measurable in every solid whose equation of state has been fitted, and it gives the Foundations a real test of IX.1. The test, applied here, rules out the identification of the free-path lever with γ for ordinary solids. This is the kind of result the book should produce: a branch of the theory that could have survived and did not.

The cost of this section's reading is that "tension" in N2 must be read as a current or an affinity, not as stress in a material. The gain is a measured value of γ, for which the Foundations otherwise offer only a state function with no stated value.

Theorem checklist. T1 and T7.1 apply to currents on state cycles (Calculation 3.4). T2 matches the three-state minimum. T3–T5 do not apply. T6 applies as the closure probability of a random window of transitions. T7 applies in full: meetings carry no current, rings do. T8–T10 do not apply. T11 has a loose counterpart in junctions of two driven cycles, without a count. T12–T14 do not apply. T15 does not apply. T16–T18 do not apply. T19–T22 do not apply. T23 applies exactly and is identified with $K_0' - 2$ (Calculation 3.5). T24 applies and is tested against measured K_0' (Calculation 3.5); the identification of IX.1 fails for ordinary solids.

Grade. A for the three-state cycle as the least that carries a steady current (Wegscheider 1901; Kolmogorov; Schnakenberg 1976) and for frustration on the antiferromagnetic triangle (Wannier 1950) and Pauling's residual entropy of ice (1935); B for T23 read as $K_0' = 2 + \gamma$, with the identification $\gamma = -(1 + P_3)/P_3$ of IX.1 failing for ordinary solids.

3.5 Condensed matter (exchange aspects)

History. The exchange interaction — the energy difference between parallel and antiparallel electron spins that comes from the antisymmetry of the wave function — was found by Heisenberg and Dirac in 1926 and explained ferromagnetism and antiferromagnetism. On a lattice where each spin wants to point opposite to its neighbours, a triangle poses a problem: three spins cannot all be antiparallel to each other. Wannier's 1950 solution of the triangular Ising antiferromagnet showed that this frustration leaves a large ground-state degeneracy (Wikipedia, Geometrical frustration). Gérard Toulouse gave the phenomenon its name in 1977. Philip Anderson proposed in 1973 that quantum spins on a triangular lattice could form a "resonating valence bond" liquid — a superposition of pairings that never orders (Wikipedia, Quantum spin liquid).

In 1980 Klaus von Klitzing found that the Hall resistance of a two-dimensional electron gas is quantised in exact fractions of h/e^2, winning the 1985 Nobel Prize; since 2019 the von Klitzing constant $h/e^2 \approx 25{,}812.807\ \Omega$ is exact by definition of the SI (Wikipedia, Quantum Hall effect). In 1982 Daniel Tsui, Horst Störmer and Arthur Gossard found plateaus at fractional filling (*Physical Review Letters* 48, 1559–1562). Robert Laughlin explained them in 1983 with a wave function at filling $1/m$ for odd m and predicted excitations of fractional charge (*Physical Review Letters* 50, 1395–1398) (Wikipedia, Fractional quantum Hall effect). For $m = 3$ the quasiparticles carry one third of an electron's charge. Laughlin, Störmer and Tsui shared the 1998 Nobel Prize.

Andre Geim and Konstantin Novoselov isolated graphene in 2004 and received the 2010 Nobel Prize (Wikipedia, Graphene). Graphene's honeycomb lattice has its electrons' Dirac points at the corners of a hexagonal Brillouin zone, in two inequivalent sets of three, K and K′. The same lattice, rolled into a tube, decides metallic or semiconducting behaviour by the chiral indices mod 3 (Hamada, Sawada and Oshiyama

1992; treated in Chapter 6). Of the 495 tubes with $0 \le m \le n \le 30$, 175 satisfy $n - m \equiv 0 \pmod 3$, about one third, and the condition is the same as the Eisenstein norm $n^2 + nm + m^2$ being divisible by 3 (Calculation 3.8). The question the history left is how strongly interacting electrons organise when no simple order is possible.

Current state. Quantum spin liquids remain elusive in materials. Herbertsmithite, $ZnCu_3(OH)_6Cl_2$, with copper ions on kagome layers, shows no magnetic order down to 50 mK and a spinon gap estimated at 0.07–0.09 meV; signatures matching Kitaev's honeycomb model were reported in 2015; in 2021 toric-code spin liquids were built on a quantum processor and a Rydberg-atom simulator (Wikipedia, Quantum spin liquid). The field still has no single experimental signature that identifies a spin liquid.

Fractional charges and statistics are now measured directly. Fractional charge was detected in a quantum antidot at Stony Brook in 1995 and by shot noise at the Weizmann Institute and near Paris in 1997; in 2020 interferometry experiments in Paris and at Purdue confirmed the braiding statistics of Abelian anyons (Wikipedia, FQHE). Kagome metals AV_3Sb_5 (A = K, Rb, Cs), reported in 2019, combine flat bands, Dirac crossings and superconductivity; CsV_3Sb_5 has been described as a $\mathbb{Z}_2$ topological kagome metal (Wikipedia, Kagome metal). Spin ice, in which each tetrahedron obeys a two-in, two-out rule, is realised in $Ho_2Ti_2O_7$ and $Dy_2Ti_2O_7$ (Wikipedia, Geometrical frustration).

Interpretation.

1. *Strand.* I is the sequence of sites around a plaquette or along an exchange path; P is the set of spin or charge states on a site.
2. *Passage and trit.* For spin-1 systems and three-state clock or Potts models, the site state is a trit. For spin-½ Ising spins it is not; frustration then concerns the product of bond signs around a loop.
3. *Closure.* An antiferromagnetic triangle cannot satisfy all three bonds: the product of three negative bonds is negative (Calculation 3.6). This is the frustration that Toulouse named in 1977, and it is independent of the theory. It matches the Foundations' statement that a ring does not reduce to meetings: three antiparallel pairs cannot close around a triangle. In the fractional quantum Hall effect at $\nu = 1/3$, three quasiparticles of charge e/3 make one electron. In the digit arithmetic of T1, three units of e/3 sum to 3: digit 0, carry 1 (Calculation 3.10). A ring of thirds becomes one unit at the next scale — the same arithmetic as three quarks making one baryon (Chapter 2.1). The physics is entirely different; the bookkeeping coincides.
4. *Scale.* The kagome and triangular lattices are the hexagonal lattice of $\mathbb{Z}[\omega]$ (T14) and its sublattices. Graphene's two triples of Dirac points are the sixth roots of unity split into two cosets of the cube roots. Laughlin fractions 1/m with m odd are not on the ladder beyond 1/3 and 1/9; the hierarchy fractions 2/5, 3/7 are off it. T15 does not organise the fractional quantum Hall plateaus.
5. *Test.* A test that could fail: the Foundations' reading predicts that frustration and fractionalisation are tied to odd closed loops, and especially to loops of three. If a robust spin liquid or fractional state were found on a lattice with no odd loops (a

bipartite lattice) and no frustration from longer-range bonds, the tie would be broken. Kitaev's honeycomb model is a spin liquid on a bipartite lattice, frustrated by bond-dependent exchange rather than by geometry. It weakens the reading: odd geometric rings are sufficient for frustration, not necessary.

Consequences. Condensed matter gives the Foundations two exact published instances. Frustration on triangles is the magnetic form of "a ring does not reduce to meetings". Charge e/3 in the fractional quantum Hall effect is the arithmetic of a ring closing with carry 1 into one electron. Both are facts of the field and both were found without the theory.

What the reading adds is a common pattern: the triangle in magnetism, the third in quantum Hall physics, the three-quark baryon and the triple-alpha nucleus are all "three of a kind make one unit at the next level". The pattern is real in each case, but the mechanisms differ, and the pattern itself predicts no number in any of them. The Kitaev model shows its limit: frustration does not need odd rings.

The register's Z[ω] has a direct physical home here. The triangular, kagome and honeycomb lattices are the Eisenstein lattice and its relatives, and graphene's K and K′ points are its cube-root cosets. The Foundations' T14 — which elements of Z[ω] come to rest and which end in a 60° turn — has no published counterpart in band theory that I can find. It would be worth asking whether the "turn" elements correspond to anything in the electronic structure of hexagonal lattices. That is a question, not a test.

Theorem checklist. T1 and T7.1 apply to charges e/3 (Calculation 3.10). T2–T4 apply to three-state Potts and clock models, whose site variable is a trit. T5–T6 do not add. T7 applies to frustration: an odd antiferromagnetic ring cannot reduce to satisfied pairs. T8–T10 do not apply. T11 does not apply. T12 does not apply. T13 does not apply. T14 applies to the triangular, kagome and honeycomb lattices as Z[ω] and its relatives, and to nanotubes (Calculation 3.8). T15 does not organise quantum Hall fractions. T16–T18 do not apply. T19–T22 apply through hopfions in magnets (2.2). T23 applies as a density-dependent stiffness in any condensed phase (3.4). T24 does not apply.

Grade. A for frustration on the antiferromagnetic triangle (Wannier 1950; Toulouse 1977) and for charge e/3 at $\nu = 1/3$ as three units closing into one (Tsui, Störmer and Gossard 1982; Laughlin 1983); C for any further pattern, with Kitaev's model as a counterexample to "odd rings are needed".

Chapter 4 — The Cosmos

This chapter reads the ring column of level L1: cycles that go round — orbits, resonances, the cycles of the Sun and the history of the universe. The sky is the oldest place where people found the two ladders: Kepler's 2 : 3 between period and distance, the Pythagorean commas in planetary harmonics, the 19-year Metonic cycle of 235 months. The modern record supports a narrower claim than the old harmonies. Mean-motion resonances that hold planets and moons in chains are dominated by small ratios on the ladder 2^a·3^b, and some systems (HD 110067, Kepler-223) lie on it entirely; but the ordinary spacing of the planets is not on it, and resonance gaps at 5 : 2 and 7 : 3 lie off it. Astronomy is grade A for lunisolar convergents (the Metonic 235/19 is a convergent of the year/month ratio) and for ladder resonances as published fact, with C for planetary spacing; astrophysics is C, since stellar and solar cycles are off the ladder; cosmology carries the companion document's prediction $a_0 = cH_0/2\pi$, which restates a coincidence Milgrom noted in 1983 and is graded C as a prediction until it says something the coincidence does not; celestial mechanics carries the solar-edge prediction of about 7,400 AU, which is grade B but is the MOND transition radius of the Sun under another name.

4.1 Astronomy

History. Astronomy began with cycles. Babylonian astronomers of the last centuries BCE recorded that eclipses repeat after 223 synodic months, about 18 years and 11 days; Edmond Halley named the period "saros" in 1686 (Wikipedia, Saros). Meton of Athens introduced the 19-year cycle in 432 BCE: 19 tropical years are almost exactly 235 synodic months (Wikipedia, Metonic cycle). The cycle still fixes the Hebrew calendar and the date of Easter. Lunisolar calendars across the world use the same few ratios because the ratio of year to month, 12.368…, has few good rational approximations.

Johannes Kepler sought the harmonies of the planets in ratios of their speeds, and in 1619 found his third law, $T^2 \propto a^3$, in *Harmonice Mundi* (Wikipedia, Kepler's laws). The exponents 2 and 3 are the first rung of the Foundations' ladder. Johann Titius in 1766 and Johann Bode in 1772 published a doubling rule for planetary distances: (0, 3, 6, 12, 24, …) + 4, divided by 10, in astronomical units. Uranus (1781) and Ceres (1801) seemed to confirm it; Neptune (1846) did not fit, and the rule has no accepted theoretical basis (Wikipedia, Titius–Bode law). Daniel Kirkwood noticed in 1866 that the asteroid belt has gaps where an asteroid's period would be a simple fraction of Jupiter's, and explained them as resonances (Wikipedia, Kirkwood gap). Pierre-Simon Laplace showed that Io, Europa and Ganymede are locked so that their periods stand nearly as 1 : 2 : 4 (Wikipedia, Orbital resonance).

The telescope, photography and spectroscopy turned astronomy into astrophysics in the nineteenth century, and in the late twentieth century planets around other stars

were found. The question the history left was whether the regularities of the solar system are laws or accidents.

Current state. The NASA Exoplanet Archive lists 6,451 confirmed planets as of 15 October 2026 (NASA Exoplanet Archive). Resonant chains are a major result of the exoplanet era. Kepler-223's four planets have periods in the ratio 3 : 4 : 6 : 8. TRAPPIST-1's seven planets have orbit counts near 24 : 15 : 9 : 6 : 4 : 3 : 2 (Wikipedia, Orbital resonance). HD 110067, reported in *Nature* in November 2023 (vol. 623, 932–937), has six transiting sub-Neptunes in a chain of 54 : 36 : 24 : 16 : 12 : 9 (Nature; Wikipedia). Its adjacent period ratios are 3/2, 3/2, 3/2, 4/3 and 4/3 (Calculation 4.10). Not every system is resonant. In 899 Kepler candidates in 365 multi-planet systems, Fabrycky and colleagues found "little statistical preference" for resonance, with an excess of pairs lying just wide of the 3 : 2 and 2 : 1 ratios (arXiv:1202.6328).

The Kirkwood gaps lie at the 4 : 1, 3 : 1, 5 : 2, 7 : 3 and 2 : 1 resonances with Jupiter, among others; the Hilda asteroids sit at 3 : 2 and the Thule group at 4 : 3 (Wikipedia, Kirkwood gap). Saturn's moons Enceladus and Dione are in 1 : 2 resonance, Titan and Hyperion in 3 : 4 (Wikipedia, Orbital resonance). The Vera C. Rubin Observatory's first images in 2025 showed over 2,000 new asteroids; full survey operations were planned to begin in early 2026, and the first alerts were generated in February 2026 (Wikipedia, Rubin Observatory). The open debates are how resonant chains form and break, whether chains are the rule for young systems, and how many of the regularities in our solar system are dynamical and how many are chance.

Interpretation.

1. *Strand.* I is time; P is the set of orbital phases. A loop is a return of a body to the same phase: one orbit. A resonance is a joint return of two or more bodies.
2. *Passage and trit.* For a pair of bodies, each conjunction can be read relative to the first: the resonant angle advancing (+1), retreating (−1) or librating about its centre (0). This is a reading. The published structure is the integer ratio p : q itself.
3. *Closure.* A mean-motion resonance is a closed joint orbit: after q orbits of one body and p of the other, the configuration returns. A Laplace chain is a ring of three bodies whose combined angle φ = λ_Io − 3λ_Europa + 2λ_Ganymede librates about 180°. The coefficients 1, −3, 2 sum to 0: the resonance is a closed word in the integers, not mod 3. The mean periods themselves give n_Io − 3n_Europa + 2n_Ganymede ≈ −0.004 per day (Calculation 4.2), close to zero; the exact closure is in the longitudes.
4. *Scale.* The ladder T15 is the rule to check. Of the resonances listed above, the Galilean 1 : 2 : 4, Pluto–Neptune 3 : 2, Titan–Hyperion 3 : 4, Kepler-223, HD 110067 and Mercury's 3 : 2 spin–orbit lock are all on the ladder; TRAPPIST-1 has one entry off it (15), and the Kirkwood gaps at 5 : 2 and 7 : 3 are off it (Calculation 4.1). Among first-order resonances (k + 1) : k, only 2 : 1, 3 : 2, 4 : 3 and 9 : 8 are on the ladder (Calculation 4.10). The strongest first-order resonances are the first three, which are also on the ladder; the preference is real but is explained by resonance strength, which falls with order and with k. The planets' adjacent period ratios mostly lie off the ladder: Venus/Mercury nearest 13/5, Earth/Venus 8/5, Mars/Earth 11/6; only Neptune/Uranus (≈ 2/1, 1.97 % off) and Pluto/Neptune (3/2, 0.30 % off)

are near ladder fractions (Calculation 4.3). As a baseline, 25 of the 55 reduced fractions with numerator and denominator at most 9 are on the ladder (Calculation 4.3), so “on the ladder” is not rare.

5. *Test.* The lunisolar cycles give a clean test of T17’s logic (best approximations), applied to a different number. The Foundations predict that calendars use convergents of the year/month ratio, not arbitrary fractions. The continued fraction of 365.24219/29.530589 is [12; 2, 1, 2, 1, 1, 17, 3], with convergents 12/1, 25/2, 37/3, 99/8, 136/11, 235/19, 4131/334 (Calculation 4.5). The Metonic 235/19 is a convergent; the 8-year octaeteris (99/8) is one too. The test passed, but it tests Lagrange’s theorem on best approximations, not the Foundations’ ladder: 235 = 5·47 and 19 are off the ladder.

Consequences. Astronomy holds two different structures that the Foundations name. The first is best approximation: lunisolar calendars use convergents of an irrational ratio, which is the logic of T17 applied to year and month instead of 2 and 3. That is published fact and grade A as structure; the numbers themselves (235, 19, the saros 223) are not on the ladder (Calculation 4.8), so the link is to T17’s method, not to its values. The second is resonance: chains of small integer ratios, many of them 2 : 1, 3 : 2 and 4 : 3. These are on the ladder, and HD 110067’s six planets are a striking case. But the reason is that low-order resonances are the strongest, and the lowest-order ratios happen to be 3-smooth. The ladder adds nothing to resonance theory.

What it does add is a sharp question. If T15 had physical content for orbits, resonant chains would avoid ratios like 5 : 4 and 5 : 3 even when they are of low order. TRAPPIST-1 contains 8 : 5 and 5 : 3 between its inner planets (Calculation 4.10), which is a counterexample. The planetary spacing of the solar system is not on the ladder, and the Titius–Bode doubling — the closest historical attempt at a ×2 ladder in the sky — has no accepted basis.

The cost of the reading is the temptation of numerology. The sky offers many ratios, and with a baseline of 25 ladder fractions out of 55 small fractions, a mixed record is what chance gives.

Theorem checklist. T1–T6 do not apply to orbits. T7 applies by name (meeting = 2 : 1, ring = Laplace chain of three). T8–T10 do not apply. T11 does not apply. T12–T14 do not apply. T15 is checked against every published chain and gap listed above, with the off-ladder cases reported (Calculations 4.1, 4.3, 4.10). T16 applies in the sense that no exact ×2/×3 commensurability exists. T17’s method (convergents) applies exactly to lunisolar calendars (Calculation 4.5); T17’s values do not. T18 has no astronomical counterpart that I can verify. T19–T24 do not apply.

Grade. A for lunisolar cycles as convergents (Metonic 235/19; octaeteris 99/8) and for low-order resonances on the ladder as published fact (Laplace; Kepler-223; HD 110067); C for planetary spacing and for any physical preference for the ladder, with TRAPPIST-1’s 8 : 5 and 5 : 3 and the 5 : 2 and 7 : 3 Kirkwood gaps off it.

4.2 Astrophysics

History. Astrophysics began when Joseph von Fraunhofer mapped the dark lines in the solar spectrum (1814) and Gustav Kirchhoff and Robert Bunsen identified them with elements (1859). Samuel Heinrich Schwabe, after 17 years of observing, reported in 1843 that sunspot numbers rise and fall with a period of about ten years (Wikipedia, Solar cycle). George Ellery Hale showed in 1919 that sunspot pairs have opposite magnetic polarity in the two hemispheres and that the polarities swap each cycle, so that the full magnetic cycle lasts about 22 years. The Hertzsprung–Russell diagram (1910s) ordered stars by luminosity and colour.

Nuclear fusion was recognised in the first half of the twentieth century as the energy source of stars. Edwin Salpeter (1952) and Fred Hoyle (1953) explained how helium burns to carbon through three alpha particles (Chapter 2.3), and the theory of the origin of the elements in stars followed within a few years. White dwarfs, neutron stars and black holes became the physics of stellar endpoints. Gravitational waves from merging black holes were detected by LIGO in 2015; Rainer Weiss, Kip Thorne and Barry Barish received the 2017 Nobel Prize (Wikipedia, Gravitational-wave astronomy). The questions the history left include the solar dynamo, the masses of the heaviest black holes and the origin of the heaviest elements.

Current state. Solar Cycle 25 began in December 2019, and a NOAA/NASA panel expected its maximum between 2023 and 2026 with a smoothed sunspot number of 95 to 130; observed cycles have lasted between 9 and 14 years (Wikipedia, Solar cycle). Gravitational-wave astronomy has a catalogue. GWTC-4.0, from the first part of the fourth LIGO–Virgo–KAGRA run (May 2023 to January 2024), adds 128 new candidates to bring the total to 218, including two possible black-hole–neutron-star mergers and several systems above 150 solar masses (LIGO). In June 2023 four pulsar-timing-array collaborations reported evidence for a background of nanohertz gravitational waves, most likely from supermassive black-hole binaries (Wikipedia).

The James Webb Space Telescope, launched on 25 December 2021, has pushed the frontier of early galaxies: JADES-GS-z14-0 at $z \approx 14.18$ (ALMA) was found in 2024, and MoM-z14 at $z = 14.44$ in May 2025 (Wikipedia, JADES-GS-z14-0). Euclid's first data release on 19 March 2025 detected about 26 million galaxies in one week of observation and listed 500 strong-lens candidates (ESA). White dwarfs have become laboratories for constants: Hu and colleagues measured $\Delta\alpha/\alpha$ at G191-B2B at the level of 10^{-5} (Chapter 2.4; arXiv:2007.10905).

Interpretation.

1. *Strand.* I is the history of a star or of the Sun's magnetic field; P is the set of states (polarity, luminosity, composition).
2. *Passage and trit.* For the solar cycle, a polarity reversal is the passage: north-leading (+1), south-leading (−1). There is no published third value. The Hale cycle is a meeting in the Foundations' sense: x then −x, returning after two half-cycles, carry 0.
3. *Closure.* The triple-alpha process inside stars is the published ring of Chapter 2.3: two alphas form a meeting that fails, three a ring that holds. Stellar nucleosynthesis

therefore contains T7 and T7.1 as structure, in carbon. Beyond carbon, the alpha ladder (oxygen-16, neon-20, magnesium-24) adds one alpha at a time; it is not a ladder of rings.

4. *Scale.* The Schwabe period of about 11 years and the Hale period of about 22 years are not on the ladder (Calculation 4.8); they are not integers in any exact sense either, varying from 9 to 14 years. Gravitational-wave catalogues and galaxy surveys have no ladder structure that I can find. The gravitational-wave chirp of a binary is a meeting at every orbit (two bodies), radiating at twice the orbital frequency — a ×2 rung, which is the quadrupole nature of gravitational radiation, published and independent.
5. *Test.* The companion document's white-dwarf prediction ($\Delta\alpha/\alpha \approx 3.08 \times 10^{-8}$ at G191-B2B) is the one astrophysical test; it is discussed in Chapter 2.4. It is about 650 to 900 times below present precision (Calculations 2.9 and 4.9).

Consequences. Astrophysics offers the Foundations the triple-alpha ring (published fact, already credited in Chapter 2), the ×2 of quadrupole radiation, and the Hale meeting of solar polarity. All three are independent of the theory, and none of them gives a number the theory predicts. The solar cycle is off the ladder and irregular, so any reading of it as a closure depth would be forced.

The white-dwarf test is where the Foundations could add a number to astrophysics. Its value depends on an instrument that does not yet exist: an uncertainty of order 10^{-8} on $\Delta\alpha/\alpha$ in a stellar atmosphere. The Hu et al. analysis shows that the limiting error is in laboratory wavelengths, so the path to the test runs through atomic physics in the laboratory, not through the telescope. That is worth stating, because it tells a reader where the theory's first astrophysical test will be decided.

Theorem checklist. T1–T6 do not apply. T7 applies to triple-alpha burning and, in name, to the Hale polarity meeting. T8–T13 do not apply. T14 does not apply. T15 is checked against the solar cycles and fails (Calculation 4.8). T16–T18 do not apply. T19–T22 apply to magnetic flux ropes in the solar corona only as topology; I have not verified a published use of Hopf charge there. T23 is the frame of the white-dwarf prediction. T24 does not apply.

Grade. A for the triple-alpha ring in stellar nucleosynthesis (Salpeter 1952; Hoyle 1953) and for the ×2 of quadrupole radiation as published structure; B for the G191-B2B prediction (untestable at present precision); C for solar and stellar cycles, which are off the ladder.

4.3 Cosmology

History. Albert Einstein applied general relativity to the universe in 1917 and added a cosmological constant Λ to hold it static. Alexander Friedmann (1922) and Georges Lemaître (1927) found expanding solutions; Edwin Hubble's distances to galaxies (1929) showed that recession velocity grows with distance. George Gamow, Ralph Alpher and Robert Herman predicted a relic radiation from a hot early universe in the late 1940s; Arno Penzias and Robert Wilson detected it in 1964–65, and its spectrum is

a near-perfect blackbody at 2.72548 ± 0.00057 K (Fixsen 2009) (Wikipedia, CMB). COBE found its anisotropy in 1992, at about one part in 25,000.

Fritz Zwicky inferred unseen mass in galaxy clusters in 1933, and Vera Rubin and Kent Ford's rotation curves of the 1970s showed that galaxies' outer parts orbit too fast for their visible mass. Two responses followed: dark matter, and modified dynamics. Mordehai Milgrom proposed in 1983 that gravity departs from Newton's law below an acceleration a_0, now fitted at about 1.2×10^{-10} m s^{-2} (Begeman et al. 1991) (Wikipedia, MOND). Milgrom noted early that a_0 is numerically close to $cH_0/2\pi$, and that $2\pi a_0 \approx cH_0 \approx c^2(\Lambda/3)^{\wedge}½$ (Scholarpedia, Milgrom 2014). In 1998 two supernova teams found that the expansion is accelerating, and Λ returned as "dark energy". The Planck satellite's 2018 results gave $H_0 = 67.4 \pm 0.5$ km s^{-1} Mpc^{-1}, with matter at about 31.5 % and dark energy at about 68.5 % of the total (Wikipedia, Planck). The questions the history left are what dark matter and dark energy are, and whether the standard ΛCDM model holds.

Current state. The Hubble tension is the field's sharpest disagreement. The SH0ES distance ladder gave $H_0 = 73.04 \pm 1.04$ (2021); the Chicago–Carnegie programme with JWST gave 70.39 ± 1.94 (2025); a consensus report of the H0DN collaboration on 1 April 2026 gave 73.50 ± 0.81 (Wikipedia, Hubble's law). Against Planck these are 4.9σ, 1.5σ and 6.4σ apart (Calculation 4.6). DESI's second data release (March 2025) combined with supernovae prefers an evolving dark energy over a constant Λ at 2.8 to 4.2σ, depending on the supernova sample (arXiv:2503.14738), and its neutrino-mass bound is in tension with oscillation experiments unless dark energy evolves (DESI).

MOND faces mixed evidence. A 2023 study of 26,615 wide binaries reported MOND-like deviations below 1 nm s^{-2}; Banik and colleagues, analysing Gaia DR3 wide binaries, preferred Newtonian gravity at 19σ and excluded MOND at 16σ in an MCMC fit (*MNRAS*, arXiv:2311.03436). A 2024 weak-lensing study reported flat rotation curves to about 1 Mpc; clusters and the Bullet Cluster remain difficult for MOND (Wikipedia, MOND).

Interpretation.

1. *Strand.* I is cosmic time; P is the set of comoving places. A loop is a return of light or matter to a place — in an expanding universe, rarely literal. The companion document reads the cosmos as the strand at its largest scale.
2. *Passage and trit.* No published trit exists in cosmology. The sign of the deceleration parameter (accelerating, decelerating, coasting) is a three-valued state; reading it as a trit is a reading only.
3. *Closure.* The Foundations' T23 makes the medium's stiffness a state function of density, N4's reading of "context". In cosmology, the analogue is an equation of state w(z) that evolves. DESI's preference for evolving dark energy is compatible with a state-function reading, as with many others; it does not single out the Foundations.
4. *Scale.* The companion document predicts $a_0 = cH_0/2\pi$: 1.0422×10^{-10} m s^{-2} for $H_0 = 67.4$, 1.0824×10^{-10} for 70 and 1.1294×10^{-10} for 73.04 (Calculation 4.4). The measured MOND scale is $1.2 \pm 0.2 \times 10^{-10}$. The prediction lies 6 % to 13 % below

the central value, within its stated uncertainty. But this is Milgrom's coincidence of 1983, stated by him in exactly this form. The Foundations do not yet derive the 2π from the register; the companion document takes it as a winding.

5. *Test.* Stated before data: if $a_0 = cH_0/2\pi$, then a_0 tracks the value of H_0. A determination of a_0 to better than 3 % from rotation curves, together with H_0 to better than 1 %, decides between the low (1.04) and high (1.13) predictions. If a_0 came out at $1.20 \pm 0.02 \times 10^{-10}$, no current H_0 (67 to 74) would fit, since $cH_0/2\pi$ stays below 1.14×10^{-10} for $H_0 \leq 73.5$ (Calculation 4.4), and the prediction would fail. The wide-binary results add a sharper test: if Newtonian dynamics holds for wide binaries at accelerations below a_0, as Banik et al. find, then a_0 is not a universal threshold of local gravity, and the companion document must say why it applies to galaxies but not to binaries.

Consequences. If $a_0 = cH_0/2\pi$ held as a derived law, cosmology would gain a link between local dynamics and the expansion rate that MOND has sought for forty years. That would be a significant result, and Milgrom calls the coincidence "an important clue" (Scholarpedia). The cost is that the Foundations do not yet provide the derivation: the factor 2π is supplied as a closed winding, which is a reading, and the relation is the same one Milgrom wrote down. As things stand the Foundations re-state the coincidence. They have to predict something the coincidence does not — for example the dependence of a_0 on environment through the state function of T23 — before the relation counts as theirs.

The Hubble tension cuts both ways. A prediction $a_0 = cH_0/2\pi$ inherits H_0's uncertainty: the values 1.04 and 1.14×10^{-10} differ by 9 %. If the tension is resolved at the high end, the prediction moves towards the measured a_0; if at the low end, away from it. The wide-binary results are a risk the companion document has not yet addressed.

Theorem checklist. T1–T14 do not apply to cosmological data. T15 does not apply. T16–T18 do not apply. T19–T22 do not apply. T23 applies as a state-function reading of evolving dark energy, without a stated test. T24 does not apply. The companion document's $a_0 = cH_0/2\pi$ is computed for the current range of H_0 (Calculation 4.4).

Grade. C for $a_0 = cH_0/2\pi$ as a prediction of the Foundations (it is Milgrom's 1983 coincidence; a test is stated but it does not separate the theory from the coincidence); C for a state-function reading of evolving dark energy.

4.4 Planetary science and celestial mechanics

History. Celestial mechanics after Newton was the science of perturbations. Lagrange and Laplace showed in the late eighteenth century that planetary mean distances are stable to first order. Urbain Le Verrier and John Couch Adams predicted Neptune from the perturbations of Uranus, and Johann Galle found it in 1846. Mercury's anomalous perihelion advance was explained by general relativity. George William Hill defined the region in which a body's own gravity dominates its primary's — the Hill sphere (Wikipedia, Hill sphere).

Comets led to the outer solar system. Ernst Öpik proposed in 1932 a reservoir of comets at the edge of the solar system, and Jan Oort in 1950 inferred from long-period

comets a spherical cloud with aphelia clustering around 20,000 AU. Jack Hills proposed an inner cloud in 1981 (Wikipedia, Oort cloud). Pluto completes two orbits for three of Neptune's, a 2 : 3 resonance that protects it from close approaches (Wikipedia, Orbital resonance). Mercury's 3 : 2 spin–orbit lock was found by radar in 1965 (Chapter 3.1). Voyager 1 crossed the termination shock at about 94 AU in December 2004 and the heliopause at about 122 AU on 25 August 2012; Voyager 2 crossed the termination shock at 84 AU in August 2007 and the heliopause on 5 November 2018 (NASA). The questions the history left are the edge of the Sun's gravitational domain, the structure of the Oort cloud, and whether an undiscovered planet orbits beyond Neptune.

Current state. The Sun's Hill sphere against the Galaxy's tide has an unstable outer boundary near 230,000 AU (Wikipedia, Hill sphere). The Oort cloud is thought to span about 2,000 to 200,000 AU, with the inner (Hills) cloud from about 2,000 to 20,000 AU and the outer cloud to a tidal truncation radius of 100,000 to 200,000 AU (Wikipedia, Oort cloud). Planet Nine, proposed by Batygin and Brown in 2016 at about 700 AU and 10 Earth masses, has been revised several times; a 2025 estimate by Siraj and colleagues gives 290 ± 30 AU and 4.4 ± 1.1 Earth masses. It has not been detected; the objects 2017 OF201 and 2023 KQ14 have orbits that challenge the hypothesis (Wikipedia, Planet Nine). Precise orbital periods are tabulated by JPL (JPL planets; JPL satellites). The solar gravitational parameter is $GM_\odot = 1.32712440042 \times 10^{20}$ m^3 s^{-2} (Wikipedia), and the astronomical unit has been fixed at 149,597,870,700 m since 2012 (Wikipedia).

Interpretation.

1. *Strand.* I is time; P is phase space of the planets and small bodies. A loop is an orbit.
2. *Passage and trit.* For a resonant pair, libration of the resonant angle about its centre is the neutral trit; circulation in either direction is ±1. This is a reading over a published variable.
3. *Closure.* Pluto–Neptune 2 : 3 is a closed joint orbit on the ladder; Mercury's 3 : 2 spin–orbit lock is another. Measured ratios: Pluto/Neptune 1.50445 (0.30 % from 3/2), Mercury's orbit/spin 1.50002 (Calculations 3.1 and 4.3). The Saturnian pairs Tethys/Mimas 2.00314 and Dione/Enceladus 1.99743 are meetings, ×2 (Calculation 4.2).
4. *Scale.* The companion document predicts a "solar edge" at about 7,400 AU. Its origin is the MOND transition radius $r = \sqrt{(GM_\odot/a_0)}$: with $a_0 = 1.0824 \times 10^{-10}$ m s^{-2} it is 7,402 AU, and with Milgrom's 1.2×10^{-10} it is 7,030 AU (Calculation 4.4). The edge lies inside the inner Oort cloud, about 61 times the heliopause distance and about 31 times smaller than the Hill radius (Calculation 4.7).
5. *Test.* Stated before data: if the solar edge is a physical boundary at about 7,400 AU, the orbits of long-period comets and extreme trans-Neptunian objects should show a change in dynamics there, beyond what Newtonian gravity with the Galactic tide predicts. A sample of comets with well-determined original orbits straddling 7,400 AU could detect or exclude it. The wide-binary results (Banik et al.) already exclude MOND-like effects at comparable accelerations for stellar binaries, which is a strong prior against an edge. If the edge is real, the companion document must explain why binaries do not show it.

Consequences. Planetary science contains ladder resonances as published fact: Pluto–Neptune, Mercury's spin and the Saturnian 1 : 2 pairs. The reason is resonance strength, not the register.

The 7,400 AU solar edge is the companion document's sharpest prediction in this field. It is honest to say what it is: the radius at which the Sun's Newtonian acceleration equals $a_0 = cH_0/2\pi$. MOND makes the same statement with its own a_0: inside this radius the Sun's field is Newtonian, outside it the modified regime would apply. The Foundations add the specific value of a_0 from H_0, which moves the edge by about 5 % from MOND's usual value. A test that could distinguish 7,000 from 7,400 AU would need a dynamical tracer at that distance with an orbit known to a few percent; no such tracer exists today.

The cost is that the prediction inherits all of MOND's difficulties with wide binaries, and adds an apparent conflict with Newtonian fits of long-period comets, which have so far not needed modified dynamics. The gain, if the edge were found, would be large: a boundary of the Sun's gravitational domain set by the cosmic expansion rate.

Theorem checklist. T1–T6 do not apply. T7 applies by name to the resonant pairs. T8–T14 do not apply. T15 is checked against the resonances of the solar system (Calculations 4.1–4.3). T16–T18 do not apply. T19–T22 do not apply. T23 enters through a_0 in the companion document. T24 does not apply. The solar edge is computed in Calculations 4.4 and 4.7.

Grade. A for ladder resonances in the solar system as published fact (Pluto–Neptune 2 : 3; Mercury 3 : 2; Saturnian 1 : 2 pairs); B for the solar edge at about 7,400 AU (testable in principle with comet and trans-Neptunian dynamics, but identical in kind to the MOND transition radius and under pressure from wide-binary results).

Chapter 5 — Engineering of Energy and Signal

This chapter reads the tension column of level L1: the practical sciences that take what does not close in physics and make it close on purpose. It holds the most exact match in this part of the book. Electrical power has run on the cube roots of unity since three-phase systems were built in 1891, and since Fortescue's paper of 1918 engineers have split every three-phase quantity into a part that rotates forward, a part that rotates backward and a part that does not rotate. The sequence of a harmonic is its balanced residue mod 3, and the zero sequence — what fails to close — is exactly what protective relays use to detect a fault (Calculation 5.1). Electrical and power engineering is grade A. Electronics is A for the ring oscillator (odd rings oscillate, even rings latch) and for ternary CMOS, B for ternary energy per bit. Photonics is A for the three-level threshold of lasing and C otherwise. Nuclear engineering is C: the chain reaction is a multiplication, not a closure mod 3. Quantum technology is A for the qutrit register and for error suppression, and C for any claim that the Foundations guide it.

5.1 Electrical and power engineering

History. Alternating current won over direct current at the end of the nineteenth century because transformers could change its voltage. Several engineers found that two or three alternating currents offset in phase produce a rotating magnetic field. Nikola Tesla filed for a polyphase motor patent on 12 October 1887 (US patent 381,968, granted 1888), about two months after Galileo Ferraris published his rotating-field research (Wikipedia, Three-phase electric power). Mikhail Dolivo-Dobrovolsky, working for AEG, built the three-phase generator and squirrel-cage induction motor and studied star and delta connections in 1888–1891. At the International Electrotechnical Exhibition of 1891 a three-phase line carried 221 kW at 15 kV over 176 km from the hydroelectric plant at Lauffen am Neckar with 75 % efficiency (Wikipedia).

Three phases won for three reasons that every textbook still gives. A balanced three-phase load draws constant power, so motors run without pulsation; a three-wire system carries the same power as a single-phase pair with about 0.75 times the conductor; and three phases make a rotating field without a starting circuit (Wikipedia). The constant power is a direct consequence of the cube roots of unity: $\cos^2\theta + \cos^2(\theta - 120°) + \cos^2(\theta - 240°) = 3/2$ for every θ, while a single phase pulses between 0 and 1 (Calculation 5.3).

Analysing unbalanced systems was the problem of the next generation. On 28 June 1918 Charles LeGeyt Fortescue presented "Method of Symmetrical Co-Ordinates Applied to the Solution of Polyphase Networks" at the AIEE convention in Atlantic City, published in *AIEE Transactions* 37, 1027–1140 (Wikipedia, Symmetrical components). Fortescue showed that any set of three phasors is the sum of three balanced sets: a positive sequence rotating like the source, a negative sequence rotating the other way, and a zero sequence of three equal, in-phase phasors. The operator that rotates by

120° is a = e^(2πi/3), a cube root of unity, and the transformation matrix is the discrete Fourier transform on $\mathbb{Z}_3$. In a balanced system $1 + a + a^2 = 0$. The zero sequence does not sum to zero; it sums to three times its phasor.

High-voltage direct current returned for long distances. A 100 kV, 20 MW link connected Gotland to mainland Sweden in 1954, the start of modern HVDC (Wikipedia, HVDC). Converters from AC to DC with six- and twelve-pulse bridges then raised the question of harmonics, which brings the residues mod 3 back into the picture. The questions the history left are the stability of grids with fewer rotating machines, the integration of variable sources, and the protection of networks whose fault currents come from electronics.

Current state. Grids are being rebuilt around renewable generation and converters. On 28 April 2025 at 12:33 CEST the Iberian peninsula lost 31 GW of load — 25.2 GW in Spain and 5.9 GW in Portugal — in under a minute. The final report of ENTSO-E on 20 March 2026 attributed the blackout to “a combination of many interacting factors”: oscillations, gaps in voltage and reactive-power control, and rapid disconnection of generators in Spain. Voltage rose sharply in southern Spain, generation tripped, frequency fell, and the AC and HVDC links to France disconnected. Power was restored to 99 % of demand by 07:00 the next morning (Wikipedia, 2025 Iberian Peninsula blackout). The event became a reference case for grids with low rotating inertia.

Long-distance transmission has reached 1,100 kV DC: a Chinese line completed in 2019 carries 12 GW over about 3,300 km. HVDC lines lose about 3.5 % per 1,000 km against about 6.7 % for AC, and break even with AC at roughly 600–800 km for overhead lines (Wikipedia, HVDC); over 3,300 km this compounds to about 11 % against 20 % (Calculation 5.9). Data centres used about 415 TWh in 2024, 1.5 % of world electricity, and are projected by the IEA to reach about 945 TWh in 2030, growing about 15 % a year (IEA). The open debates are grid-forming inverters, the amount of synchronous inertia a grid needs, and the protection of networks where faults are fed by converters that limit current.

Interpretation.

1. *Strand.* I is time; P is the set of phases of a three-phase system, or of the three conductors. A loop is one electrical cycle, a return of each phase to its starting angle.
2. *Passage and trit.* Fortescue’s three sequences are the trits: positive sequence +1, negative sequence −1, zero sequence 0. For a balanced set of harmonic order h, the sequence is fixed by h mod 3: harmonics 1, 4, 7, 10, 13 are positive, 2, 5, 8, 11 negative, and 3, 6, 9, 12 (the “triplen” harmonics) zero sequence (Calculation 5.1). In the language of T1, the sequence of harmonic h is the balanced digit of h. This is a published structure of the field — the standard harmonic-sequence table — and the identification is exact.
3. *Closure.* Positive and negative sequences sum to zero over the three phases: they close. The zero sequence sums to three times its phasor: it does not close (Calculation 5.1). In the Foundations’ terms, a closed window is a balanced set ($1 + a + a^2 = 0$), and the zero sequence is tension that passes on (the reading N2). In

practice it passes on literally: zero-sequence current flows in the neutral or to earth, and a delta winding traps it in a circulating loop. When phase b is shorted to earth, a balanced set of unit phasors becomes positive sequence 2/3, negative 1/3 and zero 1/3 (Calculation 5.2). Protective relays use exactly this: zero-sequence current is near zero in normal operation and large in an earth fault (Wikipedia, Symmetrical components).

4. *Scale.* A six-pulse converter draws harmonics only of order 6k ± 1: 5, 7, 11, 13, 17, 19, 23, 25, with amplitudes 1/h (Calculation 5.4). The triplen harmonics vanish, and so do the even ones. Twelve-pulse converters, made by two six-pulse bridges shifted by 30°, remove 5 and 7 as well and leave 12k ± 1. The pulse numbers 6, 12, 24 are on the ladder 2^a·3^b; each doubling of the pulse number removes the next pair of harmonics. This is a published design rule: converters climb the ×2 ladder on a ×3 base.
5. *Test.* The interpretation is an identity, so it cannot fail as such. A test with content: the Foundations' T11 says that joining two different closed windows always produces tension. In a power system, joining two balanced three-phase sources with different phase sequences (one positive, one negative) must produce a non-closing current. Every engineer knows this: connecting generators with the wrong phase rotation produces large unbalanced currents. The test is passed, by practice older than the theory.

Consequences. Electrical engineering has used the Foundations' algebra for a century. The direction of rotation is the trit; balance is closure; the zero sequence is tension, and engineers treat it exactly as the Foundations' reading N2 treats what does not close: as the signal that something is wrong and the quantity that must be diverted. The harmonic sequence rule — h mod 3 decides the rotation of a harmonic — is the balanced digit of T1 in a working table.

What the reading adds is small and specific. It explains in one sentence why triplen harmonics are dangerous in four-wire systems: they do not close across the phases, so they add in the neutral. Engineers already know this. It also frames the pulse number of converters as a ladder, which is accurate but adds no design rule.

The Iberian blackout is a reminder of where the analogy stops. Its causes were voltage control and oscillations in a system with little rotating inertia — dynamics that symmetrical components do not describe. The Foundations' T23, a stiffness that depends on the state of the medium, has a loose analogue in grid inertia that depends on the generation mix, but no test is stated, and I give it C. The Foundations' register is in the grid's phasors, not in its stability.

Theorem checklist. T1 applies exactly: the sequence of harmonic h is the balanced digit of h (Calculation 5.1). T2 applies: three phases are the least closing set of equal rotations. T3–T4: of the 27 ordered triples of sequence labels, 9 sum to zero; engineers do not use this count. T5 does not apply. T6 does not apply to deterministic phasors. T7 applies: positive and negative sequences close, the zero sequence does not. T8–T10 do not apply. T11 applies to paralleling sources of different phase sequence. T12 does not apply. T13 does not apply. T14 applies: the phasor operators 1, a, a^2 are units of Z[ω]. T15 applies to converter pulse numbers 6, 12, 24 (Calculation 5.4). T16–T18 do not

apply. T19–T22 do not apply. T23 has a loose analogue in grid inertia (C). T24 does not apply.

Grade. A for Fortescue's symmetrical components as the trit (positive, negative, zero sequence), for the harmonic sequence rule h mod 3, for the zero sequence as non-closure detected in earth faults, and for constant three-phase power from $1 + a + a^2 = 0$; C for any reading of grid stability through T23.

5.2 Electronics

History. The vacuum tube, the transistor (1947) and the integrated circuit (1958–59) built electronics. Gordon Moore's observation of 1965, that the number of components on a chip doubles at a regular interval, set a ×2 ladder for half a century. Complementary MOS logic (CMOS) became dominant because it draws almost no current when idle.

The ring oscillator is the simplest circuit that keeps time. An odd number of inverters connected in a loop has no stable state: a signal is inverted once per trip around the ring and keeps flipping. An even number returns the signal unchanged and settles into one of two stable states, which is the basic memory element of static RAM. An unbuffered ring needs at least three inverters to oscillate; the period is twice the sum of the stage delays, $f = 1/(2n \cdot t_d)$ (Wikipedia, Ring oscillator). Biologists later built the same distinction from genes: Elowitz and Leibler's "repressilator" (2000) is a ring of three repressors that oscillates; Chapter 10 returns to it.

Ternary electronics had its own history. Setun, the balanced-ternary computer at Moscow State University, was built from ferrite and diode elements from 1958 (Chapter 1.6). CMOS and totem-pole TTL outputs can be high, low or high-impedance, a three-state output used on every shared bus (Wikipedia, Ternary numeral system). True ternary logic in silicon needed a reliable middle state, which multi-threshold designs struggled to provide. The question the history left was whether a third logic level could be made cheap enough to pay.

Current state. Transistor counts are in the hundreds of billions. Nvidia's GB200 Grace Blackwell (2024) has 208 billion transistors, its Rubin accelerator (2026, TSMC 3 nm) 336 billion; Apple's M3 Ultra (March 2025) has 184 billion; Cerebras's wafer-scale engine of 2020 has 2.6 trillion (Wikipedia, Transistor count). Jeong and colleagues showed in 2019 a wafer-scale ternary CMOS with a single threshold voltage, using band-to-band tunnelling to give a third state with sub-picoampere current at 0.5 V, and built ternary logic and latch circuits with it (*Nature Electronics* 2, 307–312) (KNU). Computing-in-memory chips run ternary-weighted neural networks at 16.95 TOPS/W (Nature Electronics). Batteries are changing alongside: CATL's Naxtra sodium-ion cell, launched in April 2025, reaches about 175 Wh/kg, close to lithium iron phosphate at 185 Wh/kg (Wikipedia, Sodium-ion battery).

The open debates are the end of classical scaling, the energy cost of AI hardware (Chapter 5.1), and whether non-binary logic, analogue in-memory computing or photonic computing will take part of the load. Landauer's bound — kT ln 2 per erased

bit — remains about a billion times below the energy per operation of present computers (Chapter 3.4).

Interpretation.

1. *Strand.* I is the sequence of gate stages along a signal path; P is the set of nodes. A loop is a feedback path returning to a node.
2. *Passage and trit.* An inverter maps a binary level to its opposite: a passage that reverses (−1). A buffer passes it unchanged (+1). A high-impedance output is the neutral 0. Tri-state logic is the trit of T1 in hardware, and ternary CMOS makes the middle level a logic value.
3. *Closure.* A ring of n inverters has stable states only when n is even: 2 stable states for every even n, none for odd n (Calculation 5.5). The even ring is a chain of meetings (each pair of inverters cancels) and latches; the odd ring cannot pair off and oscillates. This is the published rule of ring oscillators and of SRAM, independent of the theory, and it matches T7 exactly in structure: a word of reversals reduces to meetings only if its length is even; an odd remainder must go round. The minimal unbuffered oscillator has three stages: the ring.
4. *Scale.* Moore's law is a ×2 ladder in time; it is a published empirical regularity, now slowing. Counting digits: 10^6 states need 20 bits or 13 trits, with radix economy 40 against 39 (Calculation 5.8). A 13-trit balanced word covers ±797,161 (T12.3). T13's necklaces count the distinct states of a ring of n cells up to rotation, which is the number of distinct circular shift-register contents.
5. *Test.* Stated before data: if ternary CMOS at the same technology node delivers less energy per bit of information than binary CMOS on the same task, III.1's radix-economy remark has physical weight. If it delivers more, the register's case in electronics rests on facts (a)–(d) of III.2 alone. No same-node comparison per bit has been published to my knowledge.

Consequences. Electronics contains the meeting–ring distinction as a published design rule: even rings store, odd rings oscillate, and three is the least ring that oscillates without buffering. This is the same rule biology rediscovered in the repressilator, and it is graded A for both. The Foundations add a vocabulary that unifies these cases, and one practical observation: in a ring of inverters the period is set by the number of stages, so a ring of 3k stages is the natural clock of a ternary counter. That is a design remark, not a result.

Carbon-nanotube electronics carries the closure rule in its materials: a tube is metallic or quasi-metallic exactly when its chiral indices satisfy $n - m \equiv 0 \pmod 3$ (Hamada, Sawada and Oshiyama 1992), and of the 495 tubes with $0 \le m \le n \le 30$, 175 qualify, 30 of them armchair (Calculation 5.10). Sorting nanotubes for transistors is sorting by a trit; Chapter 6 treats the physics.

The tri-state bus and ternary CMOS show that the register is buildable. Whether it is worth building is a question of device physics and economics. The tunnelling device of 2019 makes the middle state cheap at rest; the costs at switching speed are not in the paper's abstract. Until a same-node comparison exists, the Foundations' preference for three in hardware is an engineering hypothesis.

Moore’s ×2 ladder is a social and economic regularity, not a closure, and the Foundations should not claim it.

Theorem checklist. T1 and T7.1 apply to a ternary adder. T2–T6 apply to ternary signals. T7 applies exactly to rings of inverters (Calculation 5.5). T8–T10 do not apply. T11 does not apply. T12 and T12.3 apply to ternary words (Calculation 5.8). T13 applies to circular shift registers. T14 applies to nanotube devices through the Eisenstein lattice (Calculation 5.10). T15 matches Moore’s ×2 only as a regularity, not a closure. T16–T24 do not apply.

Grade. A for the ring oscillator rule (odd rings oscillate, even rings latch; minimum three stages) and for tri-state and ternary CMOS as the register in hardware (Jeong et al. 2019); B for ternary energy per bit at equal node.

5.3 Photonics

History. Photonics grew from the laser. Theodore Maiman’s ruby laser of 1960 used three energy levels of chromium ions: pumped from the ground state to a broad upper band, the ions decay quickly to a metastable level and lase back to the ground state (*Nature* 187, 493–494). The theory of population inversion explains why two levels are not enough: optical pumping between two levels can at best equalise the populations, where absorption and stimulated emission balance and there is no gain; three levels allow inversion with strong pumping; four levels allow it easily, which is why most practical lasers use four (Wikipedia, Population inversion).

Optical fibre and optical amplifiers carried the internet. Nonlinear optics added frequency conversion: second-harmonic generation (Franken, 1961) and the optical parametric oscillator (Giordmaine and Miller, 1965) (Wikipedia, OPO). The optical frequency comb, proposed in 2000 and honoured in 2005, linked optical and radio frequencies; it is self-referenced by the f − 2f method, which needs an octave of spectrum, or by 2f − 3f, which needs two-thirds of an octave (Wikipedia, Optical frequency comb; NTT). Lasers powerful enough to compress matter were built for fusion research. The question the history left is how far light can replace electrons in computing and communication.

Current state. Microresonator combs are on chips: a single laser coupled into a microring generates a comb by four-wave mixing, and soliton states give stable pulse trains (Wikipedia, Optical frequency comb). Attosecond pulses, honoured by the 2023 Nobel Prize, resolve electron motion. Optical clocks with comb readout reach the 19th decimal place (Chapter 2.4). The largest laser system, the National Ignition Facility, delivered 2.05 MJ of laser energy on 5 December 2022 and got 3.15 MJ of fusion energy back, the first ignition; on 7 April 2025 it reached 8.6 MJ from 2.08 MJ, a target gain of 4.13 (LLNL). Structured light carries topological textures such as optical hopfions (Chapter 3.2). The open debates are integrated photonics for AI hardware, the energy efficiency of optical interconnects, and quantum photonic computing.

Interpretation.

1. *Strand.* I is the sequence of transitions of an emitter; P is its energy levels. A loop is a cycle of pump, decay and emission returning to the ground state.

2. *Passage and trit.* A transition up (pump) is +1, down (emission) −1, a non-radiative relaxation 0. This is a reading over a published scheme.
3. *Closure.* A two-level cycle is a meeting: up and down, with no net gain possible at equal populations. A three-level cycle — pump up, relax, emit down — is the least closed cycle that can produce gain (Maiman 1960). This matches T7's meeting and ring in structure and is a published fact independent of the theory. It also matches Chapter 3.4: a three-state ring is the least that carries a steady current; a laser is that current, carried as light.
4. *Scale.* Harmonic generation gives the ×2 and ×3 rungs (second and third harmonic). The 2f − 3f comb method uses the first near-meeting of the two ladders, 3/2 (T17, closure depth 1), because it lacks bandwidth for an octave. NIF's gain record, 4.13, and its earlier 1.54 and 2.44 are not ladder numbers (Calculation 5.7).
5. *Test.* A test that could fail: the reading predicts that no laser operates on a closed two-level cycle with continuous optical pumping alone. That is established physics, so the test is passed by others. A separating test would need the Foundations to predict which three-level or four-level schemes work better; they do not.

Consequences. Photonics contains one clean published instance of the meeting–ring distinction: the threshold of lasing at three levels. The reason is rate equations and detailed balance, the same reason a three-state cycle is needed for a steady current in Chapter 3.4. The Foundations unify the two cases under T7.1 — a meeting closes with carry 0, a ring with carry ±1, and the carry here is a photon emitted per cycle. That is a fair reading and an instructive one, but it adds no number to laser physics.

The 2f − 3f comb is a small example of engineers using the ladder's first rung because it is the lowest-order option. It does not support the Foundations beyond showing that 3/2 is useful, which musicians knew before physicists.

Fusion lasers are covered under nuclear engineering. Their gain is a ratio of energies, not a closure.

Theorem checklist. T1–T6 do not apply to photon counts, which are conserved in the integers. T7 applies by name to the two-level/three-level threshold. T8–T14 do not apply. T15 applies to harmonic orders. T16–T18 apply to the 3/2 ratio of 2f − 3f self-referencing. T19–T22 apply to optical hopfions (Chapter 3.2). T23 is optics' own intensity-dependent response, without a test. T24 does not apply.

Grade. A for the three-level threshold of population inversion (Maiman 1960) as a published meeting/ring distinction; C for the ladder in harmonic generation and comb self-referencing, and for the rest of photonics.

5.4 Nuclear engineering

History. Fission was discovered in 1938. Leó Szilárd had conceived a neutron chain reaction on 12 September 1933. Enrico Fermi's team achieved the first self-sustaining chain reaction in Chicago Pile-1 on 2 December 1942 (Wikipedia, Nuclear chain reaction). The chain reaction is controlled through the multiplication factor k_eff, the ratio of neutrons produced to neutrons lost: below 1 subcritical, at 1 critical, above 1 supercritical. Control is possible because a small fraction of neutrons — less than one

percent — are delayed; a reactor must stay below prompt criticality, where prompt neutrons alone would sustain the chain (Wikipedia). Each fission releases on average between about 2.5 and 3 neutrons, depending on conditions.

Paul Kuroda predicted in 1956 that natural reactors could have existed; evidence of one was found at Oklo, Gabon, in September 1972, active about two billion years ago. Ternary fission, in which a light third particle is emitted, was detected through its helium-4 and tritium in 1957; it occurs in 0.2–0.4 % of fissions, with an alpha particle as the third fragment about 90 % of the time (Wikipedia, Ternary fission). Fusion research developed two routes: magnetic confinement (tokamaks, stellarators) and inertial confinement by lasers. The questions the history left are cost, waste, safety and whether fusion can deliver net power.

Current state. The world has about 440 operable power reactors with about 400 GWe of net capacity; nuclear power generated 2,702 TWh in 2025, about 9 % of world electricity, and 79 reactors were under construction according to the World Nuclear Association's table updated on 23 October 2026 (World Nuclear Association). Small modular reactors have been built only in Russia (the floating Akademik Lomonosov, two 30 MWe reactors, commercial since May 2020) and China (the HTR-PM pebble-bed reactor, grid-connected in December 2021; the 125 MWe Linglong One expected by the end of 2026). Ontario took a final investment decision in 2025 to build a BWRX-300 at Darlington, forecast at CA$7.7 billion (Wikipedia, Small modular reactor).

In fusion, the National Ignition Facility has repeated ignition many times since December 2022, with a record yield of 8.6 MJ and a target gain of 4.13 on 7 April 2025 and 7.9 MJ on 20 June 2026 (LLNL). ITER's 2024 rebaseline puts the start of research operation in 2034, deuterium–deuterium plasmas in 2035 and deuterium–tritium operation in 2039; its goal is Q = 10, 500 MW of fusion heat from 50 MW of heating for 400–600 seconds (Wikipedia, ITER). Wall-plug efficiency, not target gain, is the measure that matters for power plants, and NIF's lasers draw far more energy from the grid than the 2 MJ they deliver.

Interpretation.

1. *Strand.* I is the sequence of neutron generations; P is the set of fissile nuclei and the reactor's regions. A loop is a neutron returning to cause fission in the fuel.
2. *Passage and trit.* There is no published trit. Each generation multiplies the neutron population by k_eff; the three regimes ($k < 1$, $k = 1$, $k > 1$) are a three-valued state, which could be read as −1, 0, +1. That reading is a sign function, not a residue.
3. *Closure.* Criticality, $k = 1$, is the closed state: each generation reproduces itself. The Foundations' closure is a sum mod 3; criticality is a product equal to 1. They are different operations, and only a loose analogy links them. Ternary fission is a literal split into three, but its rate (2–4 per thousand, mostly with an alpha) has no ladder or closure meaning (Calculation 5.11).
4. *Scale.* The chain reaction multiplies by about 2.5 per generation, not by 2 or 3; it is not on the ladder. The delayed fraction, below 1 %, is what makes the chain controllable; it plays the role the Foundations give to "what does not close at one scale is the material of the next" only in a metaphorical sense. In fusion, the triple-

alpha process of stars (Chapter 2.3) is not used in reactors, which burn deuterium and tritium.

5. *Test.* I find no test the Foundations could state for nuclear engineering that would separate them from reactor physics. The companion document makes no prediction for fission or fusion yields.

Consequences. Nuclear engineering does not contain the Foundations' structure. Its central quantity is a multiplication factor, its control rests on delayed neutrons, and its numbers are set by cross-sections. The tempting readings — criticality as closure, ternary fission as a ring, three regimes as a trit — are re-description, and this book does not count them.

What follows for the Foundations is a limit worth recording. The register applies where quantities add mod 3 or rotate by thirds. A chain reaction multiplies, and multiplication by a non-integer factor near 2.5 has no residue structure. A theory that claimed to read all of physics would have to say something here; the Foundations, honestly read, say nothing.

Theorem checklist. T1–T14 do not apply: the chain reaction multiplies. T15 does not apply: neutrons per fission are about 2.5. T16–T22 do not apply. T23 has a counterpart in density-dependent reactivity coefficients, which reactor physics already computes. T24 does not apply to neutron transport in any tested way, though neutron diffusion is a random walk; the lever of a free path has no published use there.

Grade. C for nuclear engineering as a whole.

5.5 Quantum technology

History. Quantum technology as a field begins with the idea, from the early 1980s, that quantum systems are best simulated by quantum computers, and with Peter Shor's factoring algorithm of 1994, which made the idea a threat to cryptography; Shor received the 2025 Shannon Award (Wikipedia, Shannon Award). Quantum error correction followed, and the surface code made it local: every check involves only neighbouring qubits on a grid. The first working qubits came from trapped ions, nuclear spins and superconducting circuits. The physics behind the last was recognised in 2025, when John Clarke, Michel Devoret and John Martinis received the Nobel Prize for the discovery of macroscopic quantum tunnelling and energy quantisation in an electric circuit (Wikipedia, Nobel laureates).

Quantum sensing grew in parallel: atomic clocks, magnetometers, gravimeters and interferometers. Quantum communication uses entanglement and the no-cloning of quantum states. The 2022 Nobel Prize to Aspect, Clauser and Zeilinger recognised the experiments on entanglement that underlie it (Wikipedia, Bell test). In 2024 the United Nations declared 2025 the International Year of Quantum Science and Technology (Wikipedia, IYQ). The question the history left is when error-corrected quantum computers will solve useful problems that classical computers cannot.

Current state. Error correction has crossed its threshold. Google's Willow chip (105 qubits, announced 9 December 2024) ran surface codes of distance 3, 5 and 7; each step of two in distance cut the logical error by a factor of 2.14 ± 0.02, to 0.143 % per

cycle at distance 7 on 101 qubits (arXiv:2408.13687; Google). Extrapolated backwards, that suppression implies about 0.31 % at distance 5 and 0.66 % at distance 3 (Calculation 5.6). IBM announced on 10 June 2025 a plan for “Starling”, a fault-tolerant machine with 200 logical qubits running 100 million gates by 2029, preceded by the 120-qubit Nighthawk and the Loon test chip for qLDPC codes (IBM). In October 2025 Google reported out-of-time-order correlator measurements it describes as a promising route to practical quantum advantage (Nature).

Qutrits and qudits are maturing. Superconducting qutrits have been benchmarked (2021) and given high-fidelity entangling gates (2022); qudit error correction beyond break-even was reported in *Nature* in 2025 (Wikipedia, Qutrit). Quantum sensing has a new instrument: the thorium-229 nuclear isomer at 8.355733554021(8) eV, excited by laser in 2024 and measured to parts per trillion; the first optical nuclear clocks were announced in June 2026 (Wikipedia, Thorium-229). Optical atomic clocks lead the plan to redefine the second around 2030–2034 (Wikipedia, Atomic clock). The open debates are which qubit platform scales, how much overhead error correction needs, and which applications justify the cost.

Interpretation.

1. *Strand.* I is the sequence of gate operations and measurements; P is the set of states of the register. A loop is a return of the register to a state, as in a stabiliser measurement that confirms the code space.
2. *Passage and trit.* For qutrits, each basis state is a trit, and the qutrit Fourier transform uses the cube roots of unity (Calculation 3.12). For qubits, the trit appears only in the syndrome of some codes; the surface code’s stabilisers are parity checks, mod 2.
3. *Closure.* A stabiliser code accepts a state when all checks return +1: a closure condition. For qubit codes the checks are mod 2; for qutrit stabiliser codes they are mod 3, and the [3, 2] zero-sum code of Chapter 1.5 is the smallest classical example of the latter. These are published structures.
4. *Scale.* Error suppression by a constant factor per distance step is a ladder in the code distance: $\Lambda \approx 2.14$ per step of two. It is not 2^a·3^b; it is an empirical factor of the hardware. Thorium-229’s isomer energy has no ladder meaning. The 4π return of spinors (T21) underlies every spin qubit.
5. *Test.* A test that could fail: if the Foundations’ register had physical advantage, qutrit codes at equal physical error rates should reach the same logical error with fewer physical particles than qubit codes. Theory predicts some qudit advantages already; the 2025 break-even result is a first data point. The Foundations add no number that would distinguish their expectation from standard coding theory, so the test is not theirs.

Consequences. Quantum technology contains the register (qutrits, $\mathbb{Z}_3$ Fourier transforms, mod-3 stabiliser codes) as published fact, and contextuality appears first at dimension three (Chapter 3.3), which is one reason qutrits are interesting for quantum computation. None of this comes from the Foundations.

What the reading adds is a pointer, not a result. The Foundations' T11 counts seam failures when two closed windows are joined: 0 open windows in 9 of 81 ordered pairs, 1 in 36, 2 in 36. In a qutrit code built from zero-sum blocks, this is the count of syndrome failures at block boundaries. A coding theorist would derive it in a line; it may help in designing sliding-window decoders for ternary codes, but I know of no use.

The larger point is cautionary. Quantum technology is a field where claims of advantage are tested by benchmarks and error budgets. The Foundations' claims about three should be held to that standard. Today they are matched by the qutrit literature, not led by it.

Theorem checklist. T1–T6 apply to qutrit registers. T7 does not apply to stabilisers. T8–T10 do not apply. T11 applies as a count of syndrome failures at block joins of zero-sum codes. T12–T13 do not apply. T14 does not apply. T15 does not apply to code distances. T16–T18 do not apply. T19–T20 apply to topological quantum computing with anyons, which is beyond this section. T21 applies to spin qubits. T22 does not apply. T23–T24 do not apply.

Grade. A for the qutrit register, $\mathbb{Z}_3$ Fourier transforms and mod-3 stabiliser codes as published structure, and for error suppression below threshold as a measured fact (Google Quantum AI 2024); C for any claim that the Foundations guide quantum technology.

Part III — Matter

Chapter 6 — The Elements, Molecules and Materials

This chapter reads the closed units of matter: elements, molecules, crystals, materials and nanostructures. Three links carry real weight because the fields published them long before this theory existed. A carbon nanotube conducts when $n - m \equiv 0 \pmod 3$, which is the residue of its chiral vector at the prime above 3 in $\mathbb{Z}[\omega]$ (Foundations T14); a close-packed polytype repeats after N layers exactly when its Hägg signs sum to 0 mod 3, and needs three times as many layers otherwise; and the rotations a periodic crystal allows, 1, 2, 3, 4 and 6, all lie on the ladder $2^a \cdot 3^b$ (T15). The seven observed period lengths of the periodic table also lie on that ladder, but so would many even numbers by chance, and the blocks of 10 and 14 do not; the period table earns a B, through a stated test on period 8, not an A. Grades: the periodic system B for period 8, C for the seven observed lengths, A for Döbereiner's triads as spans of two equal periods; structural and inorganic chemistry A for the ice rules as meetings, C otherwise; organic chemistry A for knots and enumeration, C for aromaticity; crystallography A; materials science A for the density-dependent stiffness of T23 and the Shockley-partial ring, B (refuted for ordinary solids) for the sign of IX.1; nanoscience A, with a partly refuted B for ribbons.

6.1 The Elements and the Periodic System

History. The periodic system began with a triple. In 1817 a letter by Wurzer reported that Johann Wolfgang Döbereiner had found strontium oxide to weigh almost exactly the mean of calcium oxide and barium oxide: 50 against 27.55 and 72.5 on a hydrogen scale (Döbereiner's triads). In 1829 Döbereiner published "Versuch zu einer Gruppirung der elementaren Stoffe nach ihrer Analogie" in the *Annalen der Physik und Chemie*. There he grouped elements in threes: lithium, sodium, potassium; calcium, strontium, barium; chlorine, bromine, iodine; sulphur, selenium, tellurium. In each triad the middle element had roughly the mean weight of the outer two. The rule failed for other threes, such as fluorine, chlorine and bromine.

Others looked for longer patterns. Alexandre-Émile Béguyer de Chancourtois wound the elements round a cylinder in 1862, his "telluric helix". John Newlands presented a "law of octaves" in 1864–65: every eighth element repeated the properties of the first. The Chemical Society refused to publish it (History of the periodic table). Lothar Meyer published a table of 28 elements in 1864 and of 55 in 1870.

Dmitri Mendeleev arranged 63 elements by atomic weight in 1869. He left gaps and predicted their occupants. Gallium (1875), scandium (1879) and germanium (1886) filled three of them. Ramsay and colleagues isolated argon, krypton, neon and xenon in 1894–98, and Mendeleev added them as group 0 in 1902. In 1913 Henry Moseley's X-ray spectra showed that atomic number, not weight, fixes an element's place. Van den Broek had proposed this earlier the same year.

The reason for the periods came from quantum theory. Niels Bohr's atom of 1913 and his building-up principle of the early 1920s placed electrons in shells. Wolfgang Pauli's

exclusion principle (1924–25) allowed two electrons per orbital. A shell with principal number n holds n^2 orbitals and $2n^2$ electrons. The order in which subshells fill is the n + ℓ rule. Charles Janet drew a table with one row per n + ℓ in 1928; Karapetoff published the rule in 1930; Erwin Madelung stated it in 1936, and Klechkovsky gave a Thomas–Fermi explanation in 1961 (Aufbau principle). Glenn Seaborg proposed the actinide series in 1945, from his work on americium and curium. That moved thorium to uranium out of the transition metals and gave the table its modern shape.

Superheavy synthesis followed. Elements 113, 115, 117 and 118 were named nihonium, moscovium, tennessine and oganesson on 28 November 2016. RIKEN found 113; Dubna, Oak Ridge, Vanderbilt and Livermore found 115 and 117; Dubna and Livermore found 118 (IUPAC). “For now, we can all cherish our periodic table completed down to the seventh row,” said IUPAC’s Jan Reedijk. The open questions were where period 8 begins, how long it is, and whether the table keeps its shape at all.

Current state. No element beyond 118 has been made. The route used for 113–118, a calcium-48 beam on an actinide target, ends at oganesson: going further would need einsteinium or fermium targets, which exist only in tiny amounts. Lawrence Berkeley National Laboratory therefore tested a titanium-50 beam. After 22 days on a plutonium target it made two atoms of livermorium, element 116, presented at Nuclear Structure 2024. Lead author Jacklyn Gates estimated element 120 to be 10 to 20 times harder. The experiment needs about 45 mg of californium-249, and RIKEN planned to try element 119 with vanadium ions (Chemistry World). Berkeley’s planning note of 2023 said a run for 120 “would need to run for several years” to see one atom or more (Berkeley Lab Nuclear Science). On 2 September 2026, at the INTDS2026 conference at RIKEN, Kristian Myhre of Oak Ridge described the fabrication of the californium-249 target for the Berkeley attempt (INTDS2026 programme). No discovery has been announced.

Theory is ahead of experiment and divided. Relativistic effects grow with nuclear charge. The Madelung rule already fails for twenty ground states among the known elements: Cr, Cu, Nb, Mo, Ru, Rh, Pd, Ag, Pt, Au and Lr in the d-block, and La, Ce, Gd, Ac, Th, Pa, U, Np and Cm in the f-block (Aufbau principle). Element 121 is expected to break it again. Pekka Pyykkö’s table of 2011, built on Dirac–Fock calculations and extending the work of Fricke, Greiner and Waber (1971), gives a rough filling order 8s < 5g ≤ $8p_{1/2}$ < 6f < 7d < 9s < $9p_{1/2}$ < $8p_{3/2}$. He places some elements out of numerical order. He also writes: “To obtain the next, 50-electron magic number, the first chance would be the 6g shell which, however, remains outside the elemental range” (Pyykkö 2011). Fricke and colleagues end period 8 at the noble metal element 164. Kulsha extends Nefedov’s proposal to a period 8 of 54 elements, with 172 as the next noble element (Extended periodic table). The island of stability is hypothesised near element 126 and a second one near 164. The nuclear magic numbers 2, 8, 20, 28, 50, 82 and 126 are a separate sequence; predicted next ones include 114 protons and 184 neutrons (Magic number).

Interpretation.

1. *Strand.* I is the order of the elements by atomic number Z. P is the set of chemical types, read as columns of the table: the valence configuration an element repeats. σ sends Z to its column. The strand returns to a column each time the table wraps to a new row.
2. *Passage and trit.* A return is a new period reaching the column it left. The first passage through a column (hydrogen or helium, lithium, beryllium and so on) fixes its direction (N3 as a definition). The trit of a later passage is not well defined by chemistry alone: an element can resemble its column (1), resemble the column it would mirror under a filled-shell rule (T), or be anomalous (0). This is a reading; the theorems below do not depend on it.
3. *Closure.* A period closes at a noble gas, a filled outer shell. The closed unit is the period, and its length is the count to read. The seven observed lengths are 2, 8, 8, 18, 18, 32, 32. All seven lie on the ladder $2^a \cdot 3^b$: $2 = 2$, $8 = 2^3$, $18 = 2 \cdot 3^2$, $32 = 2^5$ (Calculation 6.1). Chemistry explains them as $2n^2$ for n = 1 to 4: the factor 2 is spin, n^2 is the count of orbitals in a shell. $2n^2$ lies on the ladder exactly when n does, and 1, 2, 3 and 4 are all 3-smooth. The first n that is not is 5. The block sizes are $2(2\ell + 1) = 2, 6, 10, 14, 18$. Here 10 and 14 are not on the ladder. The d- and f-blocks are therefore not ladder numbers; only their sums with s and p are.
4. *Scale.* Every length after the first occurs twice: 8, 8; 18, 18; 32, 32. That is a meeting at the scale of periods: two equal blocks side by side. Döbereiner's triads sit on this doubling. In atomic number his four triads are 3, 11, 19; 20, 38, 56; 17, 35, 53; 16, 34, 52. Each spans two equal gaps (8 and 8, or 18 and 18), so the middle Z is exactly the mean. The failed triad F, Cl, Br has gaps 8 and 18 (Calculation 6.2). So the triad, a ring of three elements, holds only when it lies across a meeting of two equal periods. In weight the triads hold to within 0.13 % for Li–Na–K and within 1.1–1.6 % for the others. Nuclear magic numbers are a check off the ladder: 2 and 8 are on it, but 20, 28, 50, 82 and 126 are not (Calculation 6.1).
5. *Test.* By the Madelung rule period 8 would be 8s (2) + 5g (18) + 6f (14) + 7d (10) + 8p (6) = 50 elements. $50 = 2 \cdot 5^2$ is not on the ladder. The Foundations, read with T15, predict that no regular closed period of 50 forms. The relativistic tables agree that the Madelung period of 50 breaks; Pyykkö says a 50-electron shell would need 6g, outside the range. They disagree on what replaces it: Fricke's period 8 of 46 elements (119–164) is off the ladder; Kulsha's of $54 = 2 \cdot 3^3$ is on it. The test, stated before any atom of 119 exists: if the eighth period is closed by a noble-type shell, its length lies on the ladder (for example 54, 48 or 36), not 46 or 50. A computed or observed closure at 46 or 50 refutes the reading.

Consequences. If the reading holds, the ladder adds one constraint chemistry did not state: the length of a closed period is a product of meetings and rings. That is a statement about closures, not about orbitals, and it says nothing about the d- and f-blocks taken alone. It costs little and claims little. The seven observed lengths are not strong evidence. Among even numbers from 4 to 32, 8 of 15 lie on the ladder; three distinct lengths would all land there by chance with probability about 0.15 (Calculation 6.1). The fit follows from the arithmetic fact that 1, 2, 3 and 4 are 3-smooth. The

Döbereiner observation is sharper and costs nothing: a vertical triad is exact in Z exactly when it spans two periods of the same length, and periods come in pairs. That is chemistry's own pairing of periods, read as meetings; it was known in the form "the period lengths repeat". The period-8 test is the only part that can fail, and it cannot be decided by experiment soon: chemistry beyond element 120 is out of reach. It can be decided earlier by computation, if relativistic methods converge on where the next noble-type closure lies. The comma (T18) and the closure depths (T17) have no role here; the Z[ω] split (T14) has none either.

Grade. B for the period-8 test (closed period length on the ladder, 46 and 50 refuting); C for the seven observed lengths, which lie on the ladder but would do so with probability about 0.15 by chance; A only for the fact, published in chemistry, that triads are exact when they span two equal periods.

6.2 Structural and Inorganic Chemistry

History. Structural chemistry asks how atoms are arranged in space and why. Alfred Werner's coordination theory of 1893 placed six ligands round a metal at the corners of an octahedron. It explained the isomer counts of cobalt ammines and won the 1913 Nobel Prize. Gilbert Lewis's electron-pair bond of 1916 gave the bond a count. Linus Pauling's *The Nature of the Chemical Bond* (1939) joined Lewis's pairs to quantum mechanics: hybrid orbitals, resonance and electronegativity. Pauling's rules for ionic crystals (1929) linked coordination to radius ratio. Werner's own test was a count. An octahedral complex MA_4B_2 has two isomers, cis and trans; a hexagonal-planar one would have three, and a trigonal-prismatic one three up to reflection. Werner found two, and later resolved optically active cobalt complexes with no carbon at all, which ended the idea that chirality belonged to organic chemistry. Counting isomers against geometry has stayed a tool of structural chemistry since.

Water ice gave structural chemistry one of its first counting arguments. In 1933 John Desmond Bernal and Ralph Fowler stated the ice rules: each oxygen is bonded to two hydrogens, and there is exactly one hydrogen between each pair of oxygens (Ice rules). In 1935 Pauling used them to count the configurations of ice and so explain why ice keeps entropy at absolute zero. His estimate, 3.37 J mol^{-1} K^{-1}, agrees with the measured 3.4 ± 0.1 J mol^{-1} K^{-1} = R ln(1.50 ± 0.02) (Ice Ih).

Twentieth-century inorganic chemistry built structures beyond the molecule. Cluster chemistry, Wade's electron-counting rules and organometallic sandwich compounds (ferrocene, 1951) followed. Supramolecular chemistry, honoured by the Nobel Prize of 1987, built structures held by weak bonds. Mechanically interlocked molecules followed: catenanes, rotaxanes and knots. In 2004 Fraser Stoddart's group at UCLA made molecular Borromean rings in one step from 18 components, by coordination chemistry (Borromean rings). Seven years earlier, Mao, Sun and Seeman had made Borromean rings from DNA. In 1989 Richard Robson combined copper ions with four-armed organic linkers into an open crystal. It collapsed easily, but it began metal–organic frameworks. The questions left open were how far the design of frameworks could go, and how to predict the structure of a solid from its composition.

Current state. The 2025 Nobel Prize in Chemistry went to Susumu Kitagawa, Richard Robson and Omar Yaghi “for the development of metal–organic frameworks” (Nobel Prize press release). Between 1992 and 2003 Kitagawa showed that gases flow in and out of such frameworks and that they can be flexible. Yaghi made very stable frameworks and modified them by design. Chemists have since built tens of thousands of different MOFs. Uses listed by the committee include carbon dioxide capture, water harvesting from desert air, storage of toxic gases, catalysis, PFAS separation and the breakdown of pharmaceutical traces.

The record of structures keeps growing. On 1 January 2025 the Cambridge Structural Database held 1,362,809 structures of 1,217,887 different compounds from 2,557 sources; 45.4 % were organic and 46.1 % contained a transition metal (CSD statistics 2025). The database has added over 50,000 structures a year since 2014 (CCDC).

Ice remains a frontier of structural chemistry. Twenty-two crystalline phases have been observed, and the table now lists ice XXII (2026), formed from ice X at 298 K and 310 GPa, and an unnamed hexagonal close-packed phase (2026). Ice XXI was reported in 2025 (Phases of ice). Seven phases come in clear pairs of a proton-disordered form and its ordered counterpart: Ih/XI, Ic/XIc, III/IX, V/XIII, VI/XV, VII/VIII and XII/XIV. Ice II is ordered with no disordered partner; the ordered partner of ice IV has never been reported. The debates are about which phases are truly distinct, and how proton order sets in. Ice X was reported in 2022 but remains contested, and the hydrogen order of ice XIX was still debated in 2022. In frameworks, the frontier is control of flexibility and of defects: the same press release notes that Kitagawa predicted frameworks could breathe, and that the field now designs frameworks for uses as different as water harvesting and PFAS separation. For the structure database, the trend is stable growth. The CSD needed over fifty years to reach one million structures and only twenty months more to reach 1.1 million, in January 2021 (CCDC).

Interpretation.

1. *Strand.* I is the sequence of bonds traced through a network: hydrogen bonds in ice, metal–linker bonds in a framework, bonds round a ring of atoms. P is the set of atoms or nodes. A loop is a ring of bonds that returns to an atom.
2. *Passage and trit.* In ice, a hydrogen bond has a direction: its proton sits near one oxygen or the other. Read from an oxygen, a bond is +1 if the proton points in and −1 if it points out. No bond is 0 in a perfect ice. This is a two-valued reading; the 0 enters only at defects. For coordination complexes the strand runs over the ligand sites, and a site carries one of a few ligand kinds. Werner’s isomer count is then a count of configurations up to the rotations of the skeleton, the same Burnside count that gives g(n) in T13. Computed with the rotation groups: the octahedron (24 rotations) gives 2 isomers of MA_4B_2; the planar hexagon (12 symmetries) gives 3; the trigonal prism (6 rotations) gives 4, which is 3 when mirror images are identified (Calculation 6.17). Werner’s two isomers select the octahedron.
3. *Closure.* The Bernal–Fowler rule demands two in and two out at every oxygen. Among the $2^4 = 16$ arrangements of four arrows at an oxygen, exactly 6 have arrow sum zero (Calculation 6.3). By T7, a word of ±1 with sum zero and nonzero steps decomposes into meetings (x, −x). So every allowed oxygen is two meetings.

Pauling's count is then $4 \times 6/16 = 3/2$ configurations per molecule, and $R \ln(3/2) = 3.371$ J mol^{-1} K^{-1} (Calculation 6.3). The ice rule is a closure with winding number 0: a meeting rule, not a ring rule. It is closure over the integers, not only mod 3. A defect (three in, one out) is an open vertex: tension that moves through the lattice.

4. *Scale.* The pairs of disordered and ordered ice phases are seven meetings of a kind: one structure, two states of proton order. The count of phases, 22, is off the ladder. The Borromean rings are three rings linked as a whole while no two are linked. They are a closure of three that does not reduce to meetings. The octahedron of Werner has six ligands, $2 \cdot 3$ on the ladder; the tetrahedron has four, 2^2. Of the common coordination numbers 2, 3, 4, 5, 6, 7, 8, 9 and 12, seven lie on the ladder and 5 and 7 do not (Calculation 6.13). Small numbers are mostly 3-smooth, so this is a reading. The theorem check for this discipline: T7 and T7.1 apply to the ice rule; T4's window rule and T6's baseline do not, because the arrows are two-valued; T13 does not count ice states, which are not rotation classes; T11, T12, T14, T16–T18 have no object here; T19 covers knotted molecules but the Borromean rings are a link, not a knot, and the Foundations prove nothing about links.
5. *Test.* T7 makes one claim here that can fail: a vertex rule with nonzero arrows can be satisfied only by meetings (sum 0 over the integers) or by rings (sum ± 3). In a lattice of coordination 3 with ± 1 arrows, the only closed vertices would be all-in or all-out, a ring. Three-coordinated ice-rule systems cannot satisfy a zero-sum rule at every vertex, and must carry a defect at each. If a three-coordinated network were found with a zero-sum ± 1 rule satisfied at every vertex, T7 would be refuted. It cannot be: an odd number of ± 1 terms never sums to 0. So this test fixes the reading rather than testing it.

Consequences. The ice rules give the Foundations a published structure: a local closure rule whose count, 3/2 per molecule, matches calorimetry. That match was Pauling's, not this theory's. What the theory adds is the classification: the allowed vertices are pure meetings, and defects are open windows. This says the defects carry the tension and the ordered phases are the meetings fixed. It adds no number. For MOFs and other frameworks the theory adds nothing beyond geometry: node valences of 3, 4 and 6 are lattice facts, not consequences of closure. Seam rules (T11) and the $\mathbb{Z}[\omega]$ split (T14) do not apply. Balanced ternary addresses (T12) and the closure depths (T17) have no use here. The honest result is one structure already in the field, and readings elsewhere. One small gain is a vocabulary for defects. In the ice rule, a Bjerrum defect (a bond with two protons or none) and an ionic defect (three in, one out) are both open vertices. The Foundations say that what does not close passes on as tension; in ice, defects do pass on, hopping through the lattice and carrying charge. That is the known physics of proton conduction in ice. The theory renames it; it does not predict its rate.

Grade. A for the ice rules as meeting closure (Bernal and Fowler 1933, Pauling 1935, matched by measurement); C for coordination geometry and frameworks.

6.3 Organic Chemistry

History. Organic chemistry became structural in 1858, when August Kekulé and Archibald Couper made carbon tetravalent and able to bond to itself. In 1865 Kekulé proposed the ring structure of benzene. In 1874 van ’t Hoff and Le Bel placed carbon’s four bonds at the corners of a tetrahedron. That explained optical activity: a carbon with four different substituents has two mirror forms. Pasteur had separated such mirror crystals of tartaric acid in 1848. Emil Fischer used the tetrahedral carbon to solve the sugars. An aldohexose has four stereocentres and so $2^4 = 16$ stereoisomers, eight mirror pairs. In 1891 Fischer fixed which of them is glucose. Each stereocentre is a two-valued choice, and the count of a chain of independent choices climbs the ×2 ladder of T15, the meeting ladder.

Counting isomers turned organic chemistry into combinatorics. Arthur Cayley counted alkane isomers as trees in 1875, in the *Chemische Berichte* (OEIS A000602). The counts for one to fourteen carbons are 1, 1, 1, 2, 3, 5, 9, 18, 35, 75, 159, 355, 802, 1858. George Pólya’s enumeration theorem of 1937 counted substitution patterns up to the symmetry of the skeleton. It is the same Burnside counting that gives the necklace count g(n) of Foundations T13.

Aromaticity got its rule from quantum theory. Erich Hückel worked out the basis in 1931: a planar cyclic conjugated ring with 4n + 2 π-electrons is aromatic. The short form “4n + 2” is attributed to Doering in 1951 (Hückel’s rule). Benzene has 6, the cyclopropenyl cation 2, the cyclopentadienyl anion and the tropylium cation 6, and [18]annulene 18.

The second half of the century made molecules into machines and knots. Fullerenes were found in 1985 by Kroto, Heath, O’Brien, Curl and Smalley; Kroto, Curl and Smalley received the Nobel Prize in 1996 (Fullerene). In 1989 Jean-Pierre Sauvage and colleagues made the first synthetic molecular knot, a trefoil, by winding two strands round copper(I) ions (Molecular knot). Sauvage shared the 2016 Nobel Prize for molecular machines. Asymmetric catalysis, which makes one mirror form in excess, was honoured in 2001. Woodward and Hoffmann’s orbital-symmetry rules (1965) made pericyclic reactions predictable from the parity of electron counts, a two-valued rule. Corey’s retrosynthesis turned synthesis into planning on a tree. The open questions were how to control chirality, how to tie arbitrary knots, and why life uses only one hand.

Current state. The 2026 Nobel Prize in Chemistry, announced on 7 October 2026, went to Henri Kagan and Kenso Soai “for the discovery of non-linear effects and autocatalysis in asymmetric organic synthesis” (Khaleej Times). Kagan and colleagues described non-linear effects in asymmetric catalysis from 1981, with a 1986 paper on asymmetric oxidations and aldolisations (Non-linear effects). Soai reported asymmetric autocatalysis with amplification of enantiomeric excess in *Nature* 378, 767–768 (1995) (Soai reaction). Chapter 7 treats the kinetics.

Molecular knots have grown in complexity. Synthesised knot types now include 3_1 (1989), 4_1 (2014), 5_1 (2012), 5_2 (2020), 7_1 (2020), 7_4 (2021), 8_{18} (2018) and 8_{19} (2017) (Molecular knot). David Leigh’s group made the 8_{19} knot in January 2017. The Knot

Atlas identifies 8_{19} as the torus knot T(3, 4), with determinant 3 and Jones polynomial $-q^8 + q^5 + q^3$ (Knot Atlas 8_{19}).

Chirality has become a policy question. In December 2024, 38 scientists warned in *Science* against creating mirror-image bacteria. Later, 96 biotechnology experts endorsed the warning, UNESCO's bioethics committee recommended a precautionary moratorium, and in 2026 the UN Scientific Advisory Board called for coordinated multilateral action (Mirror life). There are 1,812 non-isomorphic C_{60} fullerenes; C_{60} is the smallest with no adjacent pentagons (Fullerene). Spherical aromaticity has its own count: Hirsch's rule (2000) makes closed-shell spherical compounds aromatic at $2(n + 1)^2$ π-electrons; Poater and Solà extended it in 2011 to open-shell spheres with $2n^2 + 2n + 1$ (Hückel's rule). The knot record keeps rising: the same knot table cites a 2018 paper on a molecular knot of sixteen crossings. Kagan's model of non-linear effects, central to the 2026 prize, works with pairs of chiral ligands on a metal: homochiral pairs (R, R or S, S) and heterochiral pairs (R, S). The heterochiral complex is the key: if it is less reactive and is held in a reservoir, the product's enantiomeric excess exceeds the ligand's (Non-linear effects).

Interpretation.

1. *Strand.* I is the backbone of a molecule traced atom by atom; P is the set of atoms. In a ring molecule the strand returns to its start. In a knotted molecule the strand passes over and under itself in space, the embedding hypothesis of Foundations Part VII.
2. *Passage and trit.* At a crossing of a knotted molecule, over is +1, under is −1, no crossing is 0. In substitution patterns, a ring position carries one of a few substituents; with three kinds, each position is a trit.
3. *Closure.* T19 proves that three crossings are the least that cannot be undone, and that the first knot is the trefoil. Sauvage's trefoil of 1989 is that knot, made in a flask. T20 proves it is chiral. Fox three-colouring, the window rule of T4, separates knots: a knot is three-colourable when its determinant is divisible by 3. Among the synthesised knots, 3_1, 7_4, 8_{18} and 8_{19} are three-colourable; 4_1, 5_1, 5_2 and 7_1 are not (Calculation 6.4). The torus knots made so far that are three-colourable, T(2, 3) and T(3, 4), have both indices on the ladder; T(2, 5) and T(2, 7) do not and are not. For 8_{19}, $3|V(e^{i\pi/3})|^2 = 9$, as the Foundations' bracket relation requires (Calculation 6.4).
4. *Scale.* Isomer counting is the Burnside count of T13. For benzene with k substituents of one kind, the classes under the ring's symmetry are 1, 1, 3, 3, 3, 1, 1: 13 in all. Under rotation only, as in T13, there are 14 (Calculation 6.5). With three kinds per position, the rotation classes of six are $g(6) = 130$, the number in T13. Hückel's numbers $4n + 2$ are 2, 6, 10, 14, 18, …; only 2, 6 and 18 lie on the ladder. Hirsch's $2(n + 1)^2$ gives 2, 8, 18, 32, 50, the same sequence as the period lengths, with the same break at 50 (Calculation 6.5). Of Cayley's alkane counts, 1, 2, 3, 9 and 18 lie on the ladder; that is what small numbers do.
5. *Test.* The knot statements are theorems. A test that could fail: if knotted molecules are made by templating strands round metal ions, the T19–T20 reading says the yield of a given knot should track its minimal crossing number, with three-

colourable knots needing no extra template symmetry. No such claim is stated precisely enough to test here, so the test is left open. For aromaticity, no test is stated. The theorem check: T19 and T20 apply to knots, T13 to enumeration, T4 through Fox colouring. T2–T3 and T6 have no natural window in a molecule. T15 fails as a rule for aromatic counts. T12, T14, T16–T18, T21–T24 have no object in organic structure. The Hirsch sequence repeats the period lengths because both are $2n^2$; it brings no new evidence.

Consequences. Organic chemistry contains two structures of the Foundations as published fact. Knots in molecules obey T19 and T20 because those are theorems of knot theory, and chemists made the first knot in 1989. Counting molecules up to symmetry is Pólya's theorem, of which T13's necklace count is the cyclic case. Neither is new to chemistry. The theory adds the claim that the trefoil and the necklace count are instances of the same register rule as the window of T4. That is a unification of mathematics, not a chemical prediction. For Hückel's rule the theory adds nothing: 4n + 2 is a rule of ring orbitals, and its members fall on and off the ladder without pattern. For chirality the theory has one structural link, the chirality of the first knot (T20), and no account of why life chose one hand. The Kagan and Soai prize concerns the kinetics of amplification, read in Chapter 7. Seams (T11), the comma (T18) and the lever (T24) do not apply. The mirror-life debate shows where the theory might say something and does not. T20 proves that the first knot has two hands that cannot be deformed into each other. It does not say which hand a world should use, nor that a mirror world would be dangerous. The policy question rests on immunology and ecology, which are Chapter 11's subjects. One link to the next chapter is structural. Kagan's heterochiral complex pairs a ligand with its mirror image, R with S. In trits that is a meeting $(x, -x)$, closed with carry 0 (T7.1). The homochiral complexes, R with R, are open pairs. Chapter 7 shows that the non-linear effect arises when the closed meetings are taken out of the catalytic cycle.

Grade. A for molecular knots (T19–T20; Dietrich-Buchecker and Sauvage 1989) and for isomer enumeration (Cayley 1875; Pólya 1937; T13); C for aromaticity and the hand of life.

6.4 Crystallography

History. René-Just Haüy proposed in the 1780s that crystals are built from small identical units. Auguste Bravais counted the distinct lattices in three dimensions in 1848: fourteen. In 1891 Evgraf Fedorov and Arthur Schoenflies independently counted the space groups: 230. The 32 crystallographic point groups had been found by Hessel in 1830. All of this was mathematics before anyone could see an atom.

Max von Laue showed in 1912 that crystals diffract X-rays. William Henry and William Lawrence Bragg turned diffraction into structure determination in 1913. Crystallography then became the main way to know where atoms sit. Ramsdell in 1947 introduced the notation used for the stacking variants of silicon carbide: a number for the layers in the repeat and a letter for the lattice, H, C or R (Polymorphs of SiC). Hägg (1943) wrote a stacking as a sequence of + and − signs, and Zhdanov (1945) shortened

it to the lengths of runs of equal sign. The IUCr nomenclature report of 1984 records both (Guinier et al. 1984).

One theorem governed all of this. A periodic lattice in two or three dimensions can only have rotation axes of order 1, 2, 3, 4 and 6. The trace of a rotation in a lattice basis must be an integer, and 2 cos(2π/n) is an integer only for these n (Crystallographic restriction). On 8 April 1982 Dan Shechtman saw a diffraction pattern with a forbidden symmetry in an aluminium–manganese alloy. He was removed from his research group for defending it. He received the 2011 Nobel Prize in Chemistry “for the discovery of quasicrystals” (Nobel Prize 2011). The International Union of Crystallography changed its definition of a crystal in 1992 to admit them. The questions left were how to classify aperiodic order and how to solve structures that resist crystallisation.

Current state. Crystallography is now a data science. The Cambridge Structural Database held 1,362,809 small-molecule structures on 1 January 2025 (CSD statistics 2025). In November 2023 Google DeepMind’s GNoME project predicted 2.2 million new crystals, of which 380,000 were reported as the most stable candidates; the A-Lab at Berkeley synthesised “more than 41 new materials” in a companion *Nature* paper (Google DeepMind). How many of those are new and correctly characterised has been disputed.

Polytypism is an active field. More than 250 polymorphs of silicon carbide had been identified by 2006 (Polytypism). The common ones are 3C (ABC), 2H (AB), 4H (ABCB), 6H (ABCACB) and 15R. Their band gaps differ: experimental values are 2.390 eV for 3C, 3.330 eV for 2H, 3.263 eV for 4H and 3.023 eV for 6H (arXiv 2306.17756). That paper treats each layer as a pseudo-spin: +1 when the next layer goes A → B, B → C or C → A, and −1 for the reverse, and fits gaps of long-period polytypes with an Ising-type model to a mean error of about 0.04 eV. In the International Tables, Pandey and Krishna record that “the rhombohedral stackings have three identical sets of Zhdanov symbols in an identity period” (International Tables Vol. C, §9.2). Quasicrystal research continues on stability, on higher-dimensional descriptions, and on the 4D and 5D restriction, which admits orders 5, 8, 10 and 12 as well (Crystallographic restriction). Hexagonality, the share of layers whose two neighbours sit in the same position, is the standard single number for a SiC polytype: 100 % for 2H, 66.6 % for 9R and 40 % for 15R in the tabulation of the SiC page cited above. It orders the measured gaps: 3C (0 %) 2.390 eV, 6H (33 %) 3.023 eV, 4H (50 %) 3.263 eV and 2H (100 %) 3.330 eV, the experimental values quoted in the arXiv paper above.

Interpretation.

1. *Strand.* I is the sequence of close-packed layers along the stacking axis. P is the set of three lateral positions a layer can take, A, B and C. These form $\mathbb{Z}_3$: B is A shifted by one third of a lattice vector, C by two thirds.
2. *Passage and trit.* Each step from one layer to the next is a trit. A → B, B → C, C → A is +1; the reverse is −1 (T). Hägg’s + and − signs are exactly these trits; the pseudo-spin of the SiC literature is the same assignment. A step of 0 (A on A) does not occur in close packing, so the stacking word has only nonzero trits.

3. *Closure.* After N layers the position is the sum of the N trits mod 3 (T1). The stacking returns to its starting position, and so repeats with N layers, exactly when the sum is 0 mod 3. Otherwise the pattern drifts by one position each repeat and closes only after 3N layers. That is the hexagonal–rhombohedral split. 2H (+−), 4H (++−−) and 6H (+++−−−) have sum 0 and repeat in 2, 4 and 6 layers. 9R (Zhdanov 21, ++−), 15R (Zhdanov 32, +++−−) and 21R (Zhdanov 34) have sums ±1 and repeat in 9, 15 and 21 layers (Calculation 6.6). This is T1 and T7.1 in a crystal: an open sum keeps a nonzero residue and carries the stacking to a period three times longer. It matches Pandey and Krishna's "three identical sets of Zhdanov symbols". 3C, with Hägg word +, is the degenerate case: three layers, cubic lattice.
4. *Scale.* The allowed rotation orders 1, 2, 3, 4, 6 all lie on the ladder $2^a \cdot 3^b$ (Calculation 6.7). In four and five dimensions the allowed orders include 5 and 10, which are off the ladder. So the ladder statement holds in three dimensions and fails above: the restriction is geometric, not a consequence of T15. The counts of crystallography are mostly off the ladder: 7 systems, 14 Bravais lattices, 230 space groups, 17 wallpaper groups. Only the 32 point groups land on it (Calculation 6.7).
5. *Test.* For random stacking words the share that close is near 1/3, the T6 baseline: 22 of 64 words of length 6, 1,366 of 4,096 of length 12 (Calculation 6.6). If observed polytypes followed chance, about one third would be H and two thirds R. A test stated before data: count H and R polytypes of SiC among those with a given layer number N not divisible by 3. Any deviation from 1/3 is a fact about growth, not about the register. The register makes one prediction that could fail: every polytype with Zhdanov sum $\not\equiv 0 \pmod 3$ must have an identity period of 3N layers. A polytype with nonzero sum and period N would refute it. None can exist, because the arithmetic forces it; so this fixes the reading. The useful test separates residue from hexagonality. Computed from the Hägg words, 6H has hexagonality 1/3 and residue 0; 21R has 2/7 and residue 2; 15R has 2/5 and residue 1; 4H has 1/2 and residue 0 (Calculation 6.14). The computed values for 9R and 15R agree with the tabulated 66.6 % and 40 %. Test, stated before looking at gap data: plot measured gaps of SiC polytypes against hexagonality. If the R-type (residue $\neq 0$) polytypes lie systematically on one side of the curve through the H-type ones, the residue is a physical variable; if they scatter on both sides, it is not.

Consequences. Two structures are in crystallography as published fact. The restriction theorem gives rotations of orders 1, 2, 3, 4, 6, all on the ladder, though only because of the geometry of three dimensions. The stacking rule gives a residue mod 3 that decides whether a polytype repeats after N layers or 3N. The literature states the second as "three identical sets of Zhdanov symbols" and as the pseudo-spin convention; the Foundations name it as T1's residue and T7.1's carry. What the theory adds is a reason to look at the residue of a stacking word as a physical variable. A test that adds something: in SiC, band gap correlates with hexagonality (the share of h-type layers). The theory suggests that R-type polytypes, whose stacking word carries a residue, might differ from H-type polytypes of the same hexagonality. That is a B-level question, and the Ising-type models already fit gaps to 0.04 eV without such a term. If a residue term improves the fit, the reading gains; if not, it adds nothing. Quasicrystals mark the

limit: five-fold order is admitted only by giving up periodicity. The theorem check for crystallography: T1, T7 and T7.1 apply to stacking; T6 gives the baseline share; T15 applies to rotation orders and fails for the counts of groups; T14 applies to the hexagonal plane lattice, which is Z[ω], so every hexagonal crystal is a sublattice question in the Eisenstein integers; T11 has a natural object in a stacking fault, where two closed stackings meet. A fault joining 2H and 3C stacking is a seam, and the Foundations say it must carry tension. Stacking faults do carry energy; that is known, and the seam rule adds the count of open windows at the joint, not the energy.

Grade. A for the restriction (orders 1, 2, 3, 4, 6 on the ladder) and for the H/R stacking rule as residue mod 3; B for a residue term in polytype band gaps; C for the counts of systems and groups, which are off the ladder.

6.5 Materials Science

History. Materials science grew from metallurgy. Henry Bessemer's converter of 1856 made cheap steel. Sorby's microscopy of etched steel in the 1860s showed that properties follow microstructure. In 1934 Taylor, Orowan and Polanyi independently explained plasticity by dislocations, line defects that let crystals slip at a fraction of their ideal strength. J.M. Burgers defined the vector that bears his name in 1939: draw a circuit of lattice steps round a dislocation, and the step needed to close it is the Burgers vector. Twin boundaries, where one crystal is the mirror of its neighbour, were known as annealing twins in copper and brass. The coincidence site lattice, with its index Σ, became the standard way to label grain boundaries: Σ is the reciprocal of the share of lattice sites the two grains have in common (Grain boundary).

Equations of state described how materials stiffen under pressure. Francis Murnaghan (1944) and Francis Birch (1947) wrote the bulk modulus as a function of pressure, with K_0 the modulus at zero pressure and K_0' its pressure derivative (Birch–Murnaghan). Semiconductor physics made silicon the material of the century. The lithium-ion battery followed: Whittingham's titanium disulphide cathode in the 1970s, Goodenough's 4 V cobalt oxide in 1980, and Yoshino's commercially viable cell in 1985, honoured by the Nobel Prize in 2019 (Nobel Prize 2019). The Materials Genome Initiative of 2011 made computation part of design. Polymers, ceramics and composites joined metals as engineered materials after 1945. The questions were how to predict a material before making it, and how to engineer its defects.

Current state. Machine learning has entered materials discovery. GNoME's 2.2 million predicted crystals and 380,000 stable candidates were announced on 29 November 2023 (Google DeepMind). The 2025 Nobel Prize for metal–organic frameworks rewarded porous materials designed from molecular parts (Nobel Prize 2025).

Equation-of-state data are mature. Tabulated values of K_0' illustrate the range: NaF 5.28, NaCl 5.39, NaBr 5.46, NaI 5.59, MgO 4.7, calcite 4.63, magnesite 3.08 and 3C-SiC 4.0. Many substances have K' of about 3.5, and theory predicts $K' \to 5/3$ at extreme compression, the Thomas–Fermi limit (Murnaghan equation of state).

Grain-boundary science studies “special” boundaries whose energies are markedly lower than general ones. Coherent twins (Σ3) are the main example; in a Σ3 boundary one atom in three is shared between the two lattices. The same source warns that “no general and useful criterion for low energy can be enshrined in a simple geometric framework” (Grain boundary). Battery materials are the largest applied frontier: Chapter 9 gives the prices and the sodium-ion turn. The open debates are whether AI-predicted materials are real and new, how to make defects useful rather than harmful, and how far equations of state can be trusted outside measured ranges. The range of K_0' matters in practice. Geophysics uses it to extrapolate mantle minerals to pressures that cannot be held in the laboratory for long, and the choice between second-order Birch–Murnaghan (which fixes K_0') and third-order fits changes the inferred density at depth. The source lists its values as illustrations only, to be treated with caution; the values differ between studies by a few tenths.

Interpretation.

1. *Strand.* I is the history of a material under load or growth: successive states of density as pressure rises, or successive grain boundaries crossed by a path through a polycrystal. P is the set of states (density) or of grain orientations. For dislocations, I is a Burgers circuit, a closed walk of lattice steps, and P is the set of lattice sites it visits.
2. *Passage and trit.* For compression, a passage through a density compares the stiffness there with the stiffness at the reference density: stiffer (+1), softer (−1), the same (0). For grain boundaries, a twin operation (60° about ⟨111⟩) is a step; doing it about the same axis twice returns the grain, a meeting.
3. *Closure.* Foundations T23 takes the energy density $e(n) = (g_2/2)n^2 + (g_3/6)n^3$ and shows that the stiffness $g_eff = d^2e/dn^2$ is a state function: $g_eff = \bar{g}(1 + \gamma\delta)$, with $\gamma = g_3\bar{n}/\bar{g}$. For a solid, pressure is $P = n\,e' - e$ and the bulk modulus $K = n\,dP/dn$. With T23’s energy density, $K = n^2\,g_eff$ exactly, and $K' = dK/dP = 2 + \gamma$ exactly (Calculation 6.8). So the pressure derivative of the bulk modulus, measured for decades, is γ shifted by 2. The tabulated values give γ = 1.08 (magnesite) to 3.59 (NaI); 3C-SiC gives γ = 2.00. This is N4, the context reading, as published data: stiffness depends on density, and the field measures how. For grain boundaries, a twin boundary is Σ3. Two twins about different ⟨111⟩ axes meet in a Σ9 boundary; three give Σ27. Two twins about the same axis give Σ1, no boundary (Calculation 6.9). The twin is a meeting under repetition about one axis, and a ladder 3^n across different axes.
4. *Scale.* The $\Sigma 3^n$ family is the ring ladder (T15 with a = 0): each new twin variant multiplies Σ by 3. A boundary network whose triple junctions combine Σ3, Σ3 and Σ9 closes in the sense of the CSL product rule. Dislocations give a sharper case. A Burgers circuit is a loop round a dislocation that fails to close; the closure failure is the Burgers vector, which is the field’s own definition of tension in N2’s sense. At a node, Burgers vectors sum to zero. In an fcc crystal a perfect dislocation a/2⟨110⟩ splits into two Shockley partials a/6⟨112⟩, each of which shifts the stacking by one position, A → B: a trit step. The three Shockley partials of one {111} plane sum to zero, at 120° to one another. Three partials of the same kind add to a full lattice

vector a/2⟨121⟩ (Calculation 6.15). That is T7.1 in a crystal: 1 + 1 + 1 = 10, a ring that closes at its own scale and carries one lattice translation to the next. T23's γ is a state function (N4) and changes as the material compresses: the Thomas–Fermi limit K′ = 5/3 gives γ = −1/3, so γ changes sign between ordinary and extreme compression.

5. *Test.* Open question IX.1 of the Foundations asks whether the lever of a free path, −(1 + P)/P = −3.9365 in three dimensions (T24), is the γ of the medium. For ordinary solids at ambient pressure, γ = K_0' − 2 is positive, between about 1 and 3.6 in the table above. If the medium is an ordinary solid and the identification is meant at its reference state, the identification fails in sign. The test was stated before looking only in the sense that IX.1 is in the Foundations; the data are old. It now stands as refuted for these solids, unless a medium is found whose K_0' − 2 is near −3.94, which would need $K_0' \approx -1.94$. A negative K_0' is rare and occurs only near instabilities. Any such material is a place to look.

Consequences. T23 is not a new law for materials science. The field already writes stiffness as a function of density, and K_0' is its measured lever. The theory adds the exact translation γ = K′ − 2 and therefore a sharp sign check on its own open question IX.1. The check comes out against the simplest identification: ordinary solids have γ > 0, the free path has γ < 0. That is a real result: a part of the theory meets data and loses. It costs nothing in materials science and tells the theory where not to look. For grain boundaries, the $\Sigma 3^n$ ladder is published crystallography (CSL theory), and the Foundations name it as the ring ladder; that adds a name, not a prediction. A test that could add something: grain-boundary networks engineered to have many Σ3 boundaries should show triple junctions of type (Σ3, Σ3, Σ9) far more often than (Σ3, Σ9, Σ27). That is already a geometric consequence of CSL algebra. The theory adds nothing beyond it. The Shockley partials give the one ring in this section that is the field's own result: three partial shifts of the same kind make one full lattice translation, and the three variants of a partial sum to zero. Metallurgists use this to analyse dislocation reactions and stacking faults; the Foundations read it as T7.1, a ring carrying one unit to the next scale. That is a renaming, not a prediction, but it is exact. The theorem check for materials science: T23 applies exactly; T24 and IX.1 meet data and fail in sign for solids; T15 applies to the $\Sigma 3^n$ series; T7 applies to twinning as meetings. T2–T6, T11–T14 and T16–T22 have no clear object in bulk materials beyond what Sections 6.4 and 6.6 cover.

Grade. A for density-dependent stiffness as published structure (K_0' measured; γ = K′ − 2 computed exactly), for the $\Sigma 3^n$ series and for the Shockley-partial ring (three partials make one lattice vector); B, and refuted for ordinary solids, for the identification of IX.1 at the reference state.

6.6 Nanoscience

History. Richard Feynman's lecture "There's Plenty of Room at the Bottom" (1959) imagined writing and building at the scale of atoms. The scanning tunnelling microscope of Binnig and Rohrer (1981) made atoms visible and movable. In 1985 C_{60}

was found in carbon soot (Fullerene). Euler's formula requires every closed cage of pentagons and hexagons to have exactly 12 pentagons.

Sumio Iijima's report of multi-walled carbon nanotubes in 1991 started a field (Carbon nanotube). A single-walled tube is a graphene sheet rolled along a chiral vector $C = n\,a_1 + m\,a_2$. In 1992 two groups predicted from band theory which tubes conduct: Hamada, Sawada and Oshiyama in *Physical Review Letters* 68, 1579, and Saito, Fujita, Dresselhaus and Dresselhaus in *Physical Review B* 46, 1804. The rule, as later quoted: a tube is metallic "where $m - n$ is a multiple of three" (arXiv cond-mat/9212036). Curvature later refined it: tubes with $n - m$ a multiple of 3 and $n \neq m$ have a very small gap, while armchair tubes ($n = m$) are truly metallic.

Graphene was isolated in 2004. In 2006 Son, Cohen and Louie computed the gaps of graphene ribbons with armchair edges and found "three distinct family behaviors" by width $N = 3p$, $3p + 1$ and $3p + 2$ (arXiv cond-mat/0611602). Ned Seeman's structural DNA nanotechnology made DNA Borromean rings in 1997, and DNA origami (2006) made nanoscale shapes to order. Colloidal quantum dots, whose colour depends on size, were developed in the 1980s and made nearly perfect by Bawendi in 1993. The US National Nanotechnology Initiative of 2000 made the field a funding category. The open questions were how to make tubes of one chirality, and how to place atoms with the precision Feynman imagined.

Current state. The 2023 Nobel Prize in Chemistry went to Moungi Bawendi, Louis Brus and Aleksey Yekimov "for the discovery and synthesis of quantum dots" (Nobel Prize 2023). Quantum dots are in QLED screens and some LED lamps and guide surgeons in removing tumours.

Nanotube sorting by electronic type is routine but not solved. Methods include polymer-assisted sorting, density-gradient ultracentrifugation, size-exclusion, ion-exchange and gel chromatography, and aqueous two-phase extraction (Carbon nanotube). Small-diameter zigzag and chiral tubes that should be metallic have a finite gap; armchair tubes stay metallic.

For graphene ribbons, first-principles calculations change the order of the three families. Tight binding gives $\Delta_{3p} \gtrsim \Delta_{3p+1} > \Delta_{3p+2} = 0$. First-principles calculation gives $\Delta_{3p+1} > \Delta_{3p} > \Delta_{3p+2} \neq 0$, because edge effects open the gap of the $3p + 2$ family (Son, Cohen and Louie). Zigzag ribbons have gaps from magnetic ordering, falling with width as $9.33/(w + 15.0)$.

Lithium in graphite, the anode of every lithium-ion cell, forms LiC_6, with a theoretical capacity of 372 mAh/g. Stage-1 compounds alternate graphite and lithium layers; stage-2 compounds have two graphite layers between lithium layers. As lithium enters, the colour changes from blue to red to gold. Potassium forms KC_8, because the potassium–potassium distance is twice the distance between hexagons (Graphite intercalation compound). Staging is itself a ladder: stage 1 and stage 2 differ by one meeting of graphite layers. Icosahedral carbon cages form a series: Goldberg polyhedra with $T = m^2 + mn + n^2$ have $20T$ vertices, 12 pentagons and $10(T - 1)$ hexagons. The dodecahedron is GP(1,0), and the truncated icosahedron, the shape of C_{60}, is GP(1,1) (Goldberg polyhedron).

Interpretation.

1. *Strand.* I is the path of an electron wave round the circumference of a nanotube. P is the set of lattice sites of graphene, the hexagonal lattice whose points are the Eisenstein integers Z[ω].
2. *Passage and trit.* Writing $a_1 = 1$ and $a_2 = 1 + \omega$ (the 60° neighbour), the chiral vector is $C = (n + m) + m\omega$. Its residue at the prime $\beta = 1 - \omega$ above 3 is $(n + m) + m \equiv n - m \pmod 3$ (Foundations T1, remark on the algebraic home of the trit; T14). The trit of a tube is this residue: 0, 1 or T.
3. *Closure.* The tube is metallic exactly when the residue is 0, that is, when C lies in $\beta\mathbb{Z}[\omega]$ (Calculation 6.10). The electron's wave closes round the circumference through the Dirac point exactly when the trit vanishes. The two semiconducting families, residue 1 and T, are mirror images: the trit mirror of T12.1. The armchair ribbons repeat the split by width mod 3. Their gaps are ordered by family, and the first-principles order shows the residue-2 family is special. Inside the closed class there is a finer split. For an armchair tube (n, n), $C/\beta = n(1 + \omega)$, a whole number times a unit of $\mathbb{Z}[\omega]$. For other closed tubes, C/β has norm 3, 7, 12, 13, 19 and so on (Calculation 6.16). The truly metallic tubes are exactly those whose quotient by β is a real multiple of a unit; the quasi-metallic ones, with small curvature gaps, are the rest. The residue decides closure; the quotient decides whether the closure is exact.
4. *Scale.* Of all tubes with $0 \le m \le n \le 30$, 175 of 495 are metallic, 0.354, near the T6 baseline of 1/3 (Calculation 6.10). Icosahedral cages are $C_{20}T$ with T an Eisenstein norm: C_{20}, C_{60}, C_{80}, C_{140}, C_{180}, C_{240} (Calculation 6.11). C_{60} is $T = 3 = N(\beta)$, the norm of the prime above 3. Every such cage has exactly 12 pentagons, whatever T is. That number is Euler's: for a closed surface of pentagons and hexagons with three bonds per atom, the pentagons must number $6\chi = 12$, with $\chi = 2$ the Euler characteristic of the sphere. Twelve lies on the ladder ($2^2 \cdot 3$), but it comes from topology, not from T15. In LiC_6, two carbons per hexagon means one lithium per three hexagons. Of the four sublattices of index 3 in the hexagonal lattice, only one keeps the 60° symmetry, and it is $\beta\mathbb{Z}[\omega]$ (Calculation 6.12). DNA Borromean rings are three rings linked only as a whole, a ring that does not reduce to meetings.
5. *Test.* The mod-3 rule for tubes is established physics, so it fixes the reading. A test that could fail: the theory reads every rolled or patterned hexagonal lattice through the residue at β. It predicts that any property set by closure of a wave round a hexagonal circuit splits into a residue-0 class and two mirror classes. For armchair ribbons this holds in tight binding, but first principles makes 3p and 3p + 1 differ in size. So the mirror symmetry of the two nonzero classes fails there. In the ribbon the classes are 3p, 3p + 1, 3p + 2 by width, and the residue-2 class plays the role of 0. The test as stated is partly refuted for ribbons: the two "open" families are not mirrors.

Consequences. Nanoscience holds the clearest single instance in this chapter of closure mod 3 deciding a physical property. A nanotube conducts when its chiral vector lies in the ideal $\beta\mathbb{Z}[\omega]$, and the two non-closing residues give the two semiconducting families. This was published in 1992 from band theory, independently of the Foundations. The theory adds the identification of $n - m \bmod 3$ with the residue at the

prime above 3, which is algebra, and a programme: look for residue-mod-β rules in every hexagonal system. The ribbon case shows the cost. The edges of a ribbon break the symmetry that made the two open residues mirror images, and the residue pattern survives only as three families with a non-symmetric gap order. So the reading predicts the split into three, not the order within it. The LiC_6 and C_{60} facts are geometry of Z[ω], and the theory adds nothing to their chemistry. A sorting process that separates tubes by electronic type is, in this language, sorting by trit. The theorem check for nanoscience: T1 and T14 apply exactly to tubes; T6 gives the one-third baseline, met within 0.02 for small tubes; T15 holds for the icosahedral series only through the Eisenstein norms, which include 7, 13 and 19, off the ladder; T13 counts substitution patterns on cages; T19 applies to DNA knots, which topoisomerases tie and untie. T16–T18, T21–T24 have no object at this scale. The register rules that matter here are the residue and the norm, both from Z[ω]. One further test follows from T6. If a growth method picked chiral indices without preference, the metallic share among small tubes would be close to the 0.354 counted for $n \le 30$. A measured share far from that value is a measure of selection by the growth process, not of the register. Stating this before looking at growth statistics keeps the residue reading honest: it predicts the split, not the yield.

Grade. A for nanotube metallicity ($n - m \equiv 0 \bmod 3$; Hamada et al. 1992) and icosahedral cages on Eisenstein norms; B, partly refuted, for mirror symmetry of the two open families in ribbons.

Chapter 7 — Reactions and Heat

This chapter reads exchange in matter: heat and phase equilibria, reaction rates and catalysts, charge transfer at electrodes, and chemical systems far from equilibrium. The strongest link is in the field's own models of chirality. Kagan's ML_2 model of non-linear effects, central to the Nobel Prize in Chemistry of 2026, rests on heterochiral pairs (R, S), which are meetings (x, −x) closed with carry 0 (Foundations T7.1); the theory then predicts where hyperpositive effects should appear, a test that can fail. Excitable media modelled as three-state automata carry winding defects of ±1 that are conserved exactly, the ring of T7 in a chemical wave, and hexagonal patterns are built on resonant triads, the closed window of T4 in the plane. The Debye–Hückel limiting law is the context reading N4 in published form: ideality is the limit of a state function. Elsewhere the theory names what chemistry already knows. Grades: physical chemistry A for N4 as the activity law, B for the lever of re-encounters; kinetics and catalysis A for the ML_2 meeting structure, B for the ML_3 prediction, C for molecularity; electrochemistry A for the index-3 lattice of LiC_6 and the Nernst state function, C otherwise; nonlinear chemistry A for winding defects, resonant triads and the period-doubling ladder, B for spiral-core charge in chemical films, C for the rest.

7.1 Physical Chemistry

History. Physical chemistry joined heat, energy and chemical change. Sadi Carnot's *Réflexions* (1824) bounded the work a heat engine can do. Rudolf Clausius named entropy in 1865. Josiah Willard Gibbs published "On the Equilibrium of Heterogeneous Substances" in parts between 1875 and 1878. It contained the phase rule, $F = C - P + 2$, which counts the degrees of freedom of a system of C components in P phases (Gibbs phase rule). For one component, three phases coexist only at a single temperature and pressure, the triple point; four cannot coexist at all.

Benoît Clapeyron (1834) and Clausius gave the slope of a coexistence line, which links the heat of a phase change to its change of volume. Ludwig Boltzmann tied entropy to the count of microstates in 1877. Johannes van der Waals wrote an equation of state for real gases in 1873. It contained a critical point, where liquid and gas become one phase, and a law of corresponding states: in reduced units, all gases follow one curve. That law is the first statement in chemistry that universality is the limiting case of a state-dependent description, and it holds only approximately. Van 't Hoff, Arrhenius and Ostwald founded the *Zeitschrift für physikalische Chemie* in 1887 and made the field a discipline. Walther Nernst's heat theorem (1906) became the third law. Gilbert Lewis introduced activity in 1907 to keep the form of ideal equations for real solutions. In 1923 Peter Debye and Erich Hückel computed the activity of ions in dilute solution: each ion is surrounded by a cloud of opposite charge, and its activity coefficient falls as the square root of the ionic strength.

Quantum mechanics entered after 1926: the Heitler–London bond (1927), Hückel's π-electron theory (1931), and the statistical mechanics of molecules. Linus Pauling's

count of the configurations of ice in 1935 joined structure to entropy: R ln(3/2) of residual entropy, matching calorimetry (Chapter 6, Section 6.2). James Franck and Eugene Rabinowitch described the cage effect in 1934. Two radicals formed together in a liquid are held by the surrounding solvent molecules and meet again before they can escape. The open questions were how far ideal laws carry, how liquids differ from gases, and how to compute properties from first principles.

Current state. Physical chemistry now runs on computation and high-pressure experiment. Ice is a test case. Twenty-two crystalline phases of ice have been observed. Ice XXI, metastable between ice VI and VII, was reported in 2025, together with a plastic form of ice VII. In 2026 ice XXII was reported, formed from ice X at 298 K and 310 GPa, along with an unnamed hexagonal close-packed phase at 1,800 K and 200 GPa (Phases of ice). Superionic phases, in which oxygen stays fixed while protons flow, were reported as ice XVIII (2019) and ice XX (2021). Seven phases form clear pairs of a proton-disordered structure and its ordered counterpart. The measured residual entropy of ordinary ice is 3.4 ± 0.1 J mol^{-1} K^{-1} = R ln(1.50 ± 0.02), against Pauling's 3.37 (Ice Ih).

Water from air is now a materials problem. The 2025 Nobel Prize for metal–organic frameworks named desert water harvesting and carbon dioxide capture among the uses of porous crystals (Nobel Prize 2025). Their physical chemistry is adsorption: how many molecules a pore holds at a given pressure and temperature.

Equations of state remain an active topic. Tabulated values of the pressure derivative of the bulk modulus, K_0', range from 3.08 for magnesite to 5.59 for sodium iodide; many substances have K' near 3.5, and theory predicts $K' \to 5/3$ in the extreme Thomas–Fermi limit (Murnaghan equation of state). Chapter 6, Section 6.5, shows that the T23 energy density gives $K' = 2 + \gamma$ exactly. The debates are about phases at planetary pressures, about whether the hydrogen order of phases such as ice XIX is settled (it was still debated in 2022), and about the reliability of machine-learned potentials far from their training data.

Interpretation.

1. *Strand.* I is the course of a thermodynamic system through its states, or the path of a molecule through a liquid. P is the set of states (temperature, pressure, composition) or of positions. A loop is a return to a state already visited: a thermodynamic cycle, or a molecule meeting its partner again.
2. *Passage and trit.* For a cycle, a passage through a state compares the system's response with the response at the first passage: the same (1), reversed (T), or different (0). For a geminate pair, a passage is a re-encounter; it ends in reaction or the pair separates again.
3. *Closure.* Three structures in physical chemistry fit the closure rules. First, the triple point is the invariant point of a one-component system: three phases in mutual equilibrium. It is the only closed configuration of phases with no freedom left, a ring of three. For C components the invariant point has C + 2 phases: 3, 4, 5, 6 for C = 1 to 4; 5 is off the ladder (Calculation 7.1). Second, Debye–Hückel. The activity coefficient is a state function of ionic strength, and the ideal law is its limit as ionic

strength goes to zero. That is the context reading N4 word for word: “universality is the empirical limiting case when the state variable does not vary.” At 25 °C, the limiting-law constant computed from fundamental constants is A = 0.5085, and a 1:1 electrolyte at 0.1 mol/kg has γ = 0.69 (Calculation 7.2). Third, the cage effect. A geminate pair in three dimensions meets again with a probability P set by the geometry of escape. On a cubic lattice, P = 0.3405 (Pólya; Watson 1939). The number of encounters V, the first included, is geometric, and its lever is γ = −(1 + P)/P = −3.9365 (T24). A Monte Carlo run of 20,000 walks of 2,000 steps gives a return share of 0.3338 and a lever of −3.933 (Calculation 7.3).

4. *Scale.* The cage effect merges many solvent collisions into one encounter: a block of fine passages becomes one coarse passage (N5 as a reading). The phase rule adds one phase per component; it climbs neither ladder. Pauling’s ice count (Chapter 6) gives 3/2 states per molecule from meetings.
5. *Test.* For the cage effect, T24 makes a claim that can fail. If re-encounters of a geminate pair are memoryless, the number of encounters before escape is geometric and its lever is fixed by the return share alone. Stated before data: in a simulation or single-molecule measurement of geminate pairs in a liquid, compute the share P of pairs that re-encounter at least once and the lever $-\bar{n}\kappa_3/\kappa_2^2$ of the encounter count. If the lever differs from −(1 + P)/P by more than its sampling error, the encounter count is not geometric, and the free-path hypothesis of T24 fails for liquids. For Debye–Hückel the reading cannot fail; it restates the law.

Consequences. Physical chemistry gives the context reading N4 its clearest published form. Lewis introduced activity in 1907 so that the ideal equations would keep their shape, with a correction that depends on the state. Debye and Hückel showed in 1923 how the correction depends on ionic strength, and that it vanishes in the limit. N4 says the same about every constant of the medium. The theory adds nothing to the law; it adds the claim that this is the general pattern, of which T23’s density-dependent stiffness is another case (Chapter 6, Section 6.5). The cage effect gives T24 a physical object. Geminate recombination is the chemical case of the free path, and the lever of its encounter count is a measurable number. That is a real test, cheap with modern simulation. It does not test IX.1, the open question whether the lever equals the γ of the medium; Chapter 6 found that identification fails in sign for ordinary solids. Thermodynamics has its own word for closure. A state function is a quantity whose integral round any closed cycle vanishes: $\oint dU = 0$, $\oint dS = 0$. Heat and work are path functions; their integrals round a cycle need not vanish, and the net work of a Carnot cycle is exactly that remainder. In the language of N2, state functions close, and work is the tension a cycle passes on. That is a translation of the first law into the theory’s terms, not a consequence of the theory. It does show why the theory’s closure rule and thermodynamics fit easily: both are statements about what returns and what does not. The triple point as a ring of three phases is a reading. The phase rule has no reason to prefer three, and C + 2 does not stay on the ladder. The theorem check: T7 applies to the triple point only as a reading; T13, T14, T16–T22 have no object in equilibrium thermodynamics; T23 and N4 apply; T24 applies to the cage effect.

Grade. A for N4 as the activity law (Lewis 1907; Debye and Hückel 1923); B for the lever of geminate re-encounters (T24); C for the triple point as a ring.

7.2 Chemical Kinetics and Catalysis

History. Ludwig Wilhelmy measured the rate of sucrose inversion in 1850, the first rate law. Guldberg and Waage stated the law of mass action in 1864. Van ’t Hoff’s *Études de dynamique chimique* (1884) classified reactions by order. Svante Arrhenius gave the temperature dependence of rate in 1889: $k = A \exp(-E_a/RT)$. Wilhelm Ostwald defined a catalyst as a substance that changes the rate of a reaction without appearing in its products, and received the Nobel Prize in 1909.

Mechanism became the central concept. An elementary step has a molecularity: one, two or three molecules taking part. A unimolecular step is a single molecule rearranging; a bimolecular step is two molecules colliding; a termolecular step needs three reactants to collide at once. Steps of higher molecularity are not known because four molecules almost never meet at once (Molecularity). Lindemann explained unimolecular reactions in 1922 by collisional activation. Michaelis and Menten wrote the rate law of an enzyme in 1913. Langmuir described adsorption on surfaces in 1916. Eyring, Evans and Polanyi built transition-state theory in 1935. Rudolph Marcus began his theory of electron transfer in 1956. It predicted an “inverted region” where very exergonic transfers slow down; Miller, Calcaterra and Closs confirmed it in 1984, and Marcus received the Nobel Prize in 1992 (Marcus theory).

Catalysis became industry with ammonia. Fritz Haber demonstrated ammonia synthesis in 1909 at about 125 mL an hour. Carl Bosch scaled it up at BASF; the first plant at Oppau opened in 1913 and reached 20 tonnes a day in 1914 (Haber process). Haber received the Nobel Prize in 1918 and Bosch in 1931. Paul Sabatier’s principle held that a good catalyst binds reactants neither too strongly nor too weakly. Balandin introduced volcano plots of rate against binding strength (Sabatier principle). Asymmetric catalysis made one mirror form of a product. From 1981, and in a 1986 paper, Kagan and colleagues reported that the enantiomeric excess of the product need not be proportional to that of the chiral catalyst. The questions left were how to design catalysts from first principles and how to control selectivity.

Current state. The 2026 Nobel Prize in Chemistry, announced on 7 October 2026, went to Henri Kagan and Kenso Soai “for the discovery of non-linear effects and autocatalysis in asymmetric organic synthesis” (Khaleej Times). The laureates share 12 million Swedish crowns.

Kagan’s ML_2 model explains the non-linear effect. A metal M binds two chiral ligands. Three complexes form: two homochiral (R, R and S, S) and one heterochiral (R, S). The model has two parameters: β, the amount of heterochiral complex relative to the homochiral ones, and g, the reactivity of the heterochiral complex relative to a homochiral one. When $\beta = 0$ or $g = 1$, the effect vanishes. When the heterochiral complex is less reactive and sits in an inactive reservoir, the product’s excess exceeds the ligand’s: a positive non-linear effect. Kagan’s 1986 case was the Sharpless epoxidation of geraniol with $Ti(O\text{-}i\text{-}Pr)_4$ and diethyl tartrate. In the ML_3 model, with

three ligands, four complexes react, two of them heterochiral; if the heterochiral complexes are more reactive, a hyperpositive effect can occur (Non-linear effects).

Industrial catalysis is being re-engineered for emissions. Ammonia production was 230 million tonnes in 2018. It uses 1–2 % of global energy and accounts for about 3 % of carbon emissions. It runs at 15–25 MPa and 450–550 °C over promoted iron, with about 15 % conversion per pass and 97–98 % overall after recycling (Haber process). Global hydrogen demand was almost 100 million tonnes in 2024; low-emission hydrogen was on track for about 1 million tonnes in 2025, under 1 % of production (IEA Global Hydrogen Review 2025). The debates are about electrified and low-temperature routes to ammonia, about whether machine learning can move the top of the volcano, and about how far non-linear effects can be designed rather than found.

Interpretation.

1. *Strand.* I is the sequence of elementary steps in a mechanism. P is the set of species, including the catalyst's states. A catalytic cycle is a loop: the catalyst returns to the state it started in.
2. *Passage and trit.* For chiral catalysis, each ligand is a trit: R is 1, S is T, an achiral or empty site is 0. A complex of two ligands is a word of two trits; of three, a word of three.
3. *Closure.* In Kagan's ML_2 model the heterochiral complex (R, S) is the word (1, T): a meeting, closed with carry 0 (T7, T7.1). The homochiral complexes (1, 1) and (T, T) are open words with sum ±2: in balanced ternary 1T and T1, a nonzero digit kept and a carry passed on (Calculation 7.4). The positive non-linear effect is the case in which the closed meetings leave the cycle (the reservoir) and only the open words react. With random pairing (equilibrium constant 4) and an inactive heterochiral complex (g = 0), a ligand excess of 0.10 gives a product excess of 0.198 times the maximum, nearly double; 0.50 gives 0.80 (Calculation 7.5). In ML_3, the closed words are different. The homochiral triples (1, 1, 1) and (T, T, T) are rings, closed with carry ±1. The heterochiral triples (1, 1, T) and (1, T, T) are open.
4. *Scale.* The catalytic cycle merges many molecular events into one turnover, a coarse passage. Molecularity has the values 1, 2 and 3 and, in practice, no others. These are the lengths of the three elementary closures of T7: standing, meeting and ring. The source gives the physical reason, the improbability of four molecules meeting at once. The Haber loop climbs the recycle: after 22 passes at 15 % per pass, 97 % has reacted (Calculation 7.6). On surfaces, the two classic mechanisms differ by closure. In a Langmuir–Hinshelwood step both partners are adsorbed and meet on the surface, a meeting of two adsorbed species. In an Eley–Rideal step an adsorbed species reacts with a molecule arriving from the gas. Both are bimolecular; the theory does not prefer either.
5. *Test.* The closure reading says that in ML_2 the closed complex is the heterochiral meeting, and in ML_3 the closed complexes are the homochiral rings. If closed complexes tend to leave the cycle, as in Kagan's reservoir, then ML_2 systems should show positive effects from inactive heterochiral complexes, and ML_3 systems should show effects driven by inactive homochiral rings and reactive heterochiral words: the hyperpositive case. Stated before a survey: among published non-linear effects

classified by the model that fits them, hyperpositive effects should be found mainly in ML_3-type (odd) systems and rarely in ML_2-type systems. A survey that finds hyperpositive effects as common in ML_2 as in ML_3 refutes this reading.

Consequences. Kagan's model contains a structure of the Foundations as published fact: the split of two-ligand complexes into a mirror pair (R, S), which sums to zero, and the two like pairs, which do not. The model's parameter β counts the closed meetings; g measures whether they react. That is in the field, independently of this theory. What the theory adds is a reason to expect different behaviour for three-ligand complexes, where the closed words are the like triples, not the mixed ones. That gives a survey test. If it holds, it gives synthetic chemists a rule of thumb for designing catalysts with amplifying effects: use odd ligand numbers when the like complex should be the reservoir. If it fails, the closure reading of non-linear effects is decoration. A second, weaker consequence concerns reservoirs. Kagan's reservoir effect is a store of closed meetings that exchanges with the active pool. The Foundations say that what closes persists and what does not passes on. The reservoir is the persisting part, the catalysis the passing part. That is a description of the model, not a prediction. The prediction is the ML_3 survey above. Molecularity 1, 2, 3 matches the lengths of T7 but for a reason the field already gives, collision statistics; it adds nothing. The Marcus parabola is a meeting in a different sense: the rate is greatest when the free energy change equals minus the reorganisation energy, symmetric about it, so -0.5 eV and -1.5 eV give the same rate when $\lambda = 1$ eV (Calculation 7.7). The theory does not predict λ. The volcano plot and the Haber recycle have no structure of the Foundations. The theorem check: T7 and T7.1 apply to ML_2 and ML_3; T12 gives the carry of the open pairs; T2–T4 apply to ML_3 words as windows of three; T13–T22 have no object here; T24 applies only through the cage effect of Section 7.1.

Grade. A for the ML_2 split into a closed mirror pair and open like pairs (Kagan; published model); B for the ML_3 prediction about hyperpositive effects; C for molecularity as the lengths of T7 and for the catalytic cycle as a loop.

7.3 Electrochemistry

History. Alessandro Volta built his pile in 1800, a stack of zinc and copper discs separated by brine-soaked card. It gave the first steady current. Within weeks Nicholson and Carlisle used it to split water. Humphry Davy isolated sodium and potassium by electrolysis in 1807. Michael Faraday stated the laws of electrolysis in 1834: the amount deposited is proportional to the charge passed, and equal charges deposit chemical equivalents. He named electrode, anode, cathode, ion and electrolyte. John Daniell's cell (1836) gave a steady voltage; William Grove made the first fuel cell in 1839; Gaston Planté the lead–acid battery in 1859; Georges Leclanché the zinc–carbon cell in 1866.

Theory followed the cells. Walther Nernst related a cell's voltage to the concentrations in it in 1889: the potential shifts by RT ln 10/(nF) per tenfold change of concentration. Julius Tafel found in 1905 that current grows exponentially with overpotential. Butler and Volmer joined these into the rate law of electrode kinetics in the 1920s and 1930s. Faraday's laws had fixed the unit of charge per mole of electrons, now the Faraday

constant, 96,485 C/mol. Svante Arrhenius's theory of ionic dissociation (1884–87) explained why solutions conduct, and Debye and Hückel's theory of 1923 explained why they conduct less than ideal ions would. Jaroslav Heyrovský invented polarography in 1922 and received the Nobel Prize in 1959. Marcus's electron-transfer theory (1956) gave the rate a molecular basis.

The second half of the century belonged to batteries. In the 1970s M. Stanley Whittingham built a lithium battery with a titanium disulphide cathode at just over two volts. In 1980 John Goodenough showed that a cobalt oxide cathode gives about four volts. In 1985 Akira Yoshino paired it with a petroleum-coke anode, avoiding metallic lithium, and made the first commercially viable lithium-ion battery. The three received the 2019 Nobel Prize (Nobel Prize 2019). Graphite became the standard anode. Fuel cells returned with spaceflight in the 1960s. Their reaction takes place only at the triple-phase boundary, where gas, electrolyte and electrode meet; electrode design for fuel cells is largely the design of that boundary. The questions left were energy density, cost, safety, and the supply of lithium, cobalt and nickel.

Current state. Battery prices fell again in 2025. BloombergNEF's survey of 9 December 2025 put the average pack price at $108/kWh, 8 % below 2024. LFP packs averaged $81/kWh and NMC packs $128/kWh. Stationary storage fell 45 % to $70/kWh, the cheapest segment for the first time (BloombergNEF). Global electric car sales topped 17 million in 2024, over 20 % of car sales, and were expected to exceed 20 million in 2025 (IEA Global EV Outlook 2025).

Sodium-ion cells are entering the market. In April 2025 CATL launched its Naxtra sodium-ion brand at 175 Wh/kg, against 185 Wh/kg for LFP, and it passed China's GB 38031-2025 certification in September 2025. Natron Energy ceased operations on 3 September 2025. In June 2026 GM and Peak Energy announced a partnership for grid-scale sodium-ion storage (Sodium-ion battery). Sodium-ion packs remained about 30 % more expensive than LFP in 2025. An IRENA report of 2025 suggests sodium-ion cell costs could fall to $40/kWh against $70/kWh for LFP. CATL claims 5C charging, over 10,000 cycles and 93 % capacity retention at −30 °C for Naxtra. Alsym reported 18 GWh of announced deployment agreements by mid-2026, and in August 2026 a sodium-ion mining truck was delivered to a Chinese mine (Sodium-ion battery).

Electrolysis is the other frontier. Installed water-electrolysis capacity reached 2 GW in 2024, with over 1 GW more added through July 2025. China holds 65 % of installed and committed capacity and about 20 GW a year of manufacturing capacity, far above demand. Electrolysers cost USD 2,000–2,600/kW outside China and USD 600–1,200/kW in China (IEA Global Hydrogen Review 2025). Graphite's LiC_6 remains the reference anode, with a theoretical capacity of 372 mAh/g (Graphite intercalation compound). The debates are about solid electrolytes, about sodium versus lithium for storage, and about the cost of green hydrogen.

Interpretation.

1. *Strand*. I is the path of charge round a circuit: through the external wire, across an electrode interface, through the electrolyte, and back across the other interface. P

is the set of places charge can sit: electrode, interface, solution. A working cell is a loop.

2. *Passage and trit.* At an electrode, a passage of charge is oxidation (1), reduction (T) or no transfer (0). A battery cycle is a meeting: charge goes in and comes back (x, −x), closed with carry 0 in the ideal case. Capacity fade is the part that does not come back, an open remainder. At an electrode in equilibrium, the anodic and cathodic currents are equal and opposite: the exchange current flows both ways and the net current is zero. That is a dynamic meeting, closed with carry 0. Overpotential opens it: one direction wins, and the Butler–Volmer law gives the net current as the difference of two exponentials. In the symmetric case the transfer coefficient is 1/2, the meeting split evenly.
3. *Closure.* The Nernst equation makes the electrode potential a state function of concentration: 59.16 mV per decade at 25 °C for one electron, 29.58 for two, 14.79 for four (Calculation 7.8). That is N4 again: the potential is not a constant but depends on the state, and the standard potential is the reference state. In graphite, LiC_6 places one lithium per three hexagons, since each hexagon has two carbons. The only sublattice of index 3 that keeps the hexagonal symmetry is βZ[ω], the ideal of the prime above 3 (Chapter 6, Calculation 6.12). The theoretical capacity follows: F divided by the mass of six carbons gives 371.9 mAh/g, matching the published 372 (Calculation 7.8). Staging (stage 1, stage 2) adds graphite layers between lithium layers one meeting at a time.
4. *Scale.* Water splitting takes two electrons per H_2 and four per O_2: 192,971 C per mole of hydrogen (Calculation 7.8). Electron counts of electrode reactions are 1, 2 and 4 in the common cases, on the meeting ladder. The three-electrode cell (working, reference, counter) is the standard measuring set-up; its three are roles, not a closure.
5. *Test.* A test that could fail concerns the lithium ordering in LiC_6. If the lithium layer keeps the sixfold symmetry of graphite, it must sit on βZ[ω]: a $\sqrt{3} \times \sqrt{3}$ superlattice rotated by 30°. Any ordered LiC_6 layer with sixfold symmetry on another index-3 sublattice would refute the reading. None can exist, because the arithmetic forces it (Chapter 6, Calculation 6.12). So the statement fixes the reading. For electrode kinetics the theory states no test.

Consequences. Electrochemistry has two structures of the kind the Foundations name. The Nernst potential is a state function with a reference state, as N4 says of every constant of the medium; the field has written it so since 1889. The graphite anode stores lithium on an index-3 sublattice of the hexagonal lattice, the ideal of the prime above 3. Its theoretical capacity, 372 mAh/g, follows from that count. Neither is new. The theory adds a programme: in any hexagonal host, symmetric ordering of guests at one per three sites must take the βZ[ω] form. For sodium, which does not form a stage-1 compound in graphite in the same way, the theory says nothing. Battery cycling as meetings, and capacity fade as the open remainder, is a reading with no number attached. The triple-phase boundary is the one place where electrochemistry requires three things to meet for a closure: gas, ion conductor and electron conductor. It is a ring of three phases in the plain sense that remove any one and the reaction

stops. The theory has no number for it. A test would need a claim about how the active length of the boundary scales with structure, and the theory gives none. What electrochemistry offers the theory in return is a clean example of N4 at work in a device. The open-circuit voltage of a cell is not a constant of the chemistry. It moves with state of charge, temperature and concentration, and battery management systems read the state of charge from it. The standard potential is the value at the reference state, and the Nernst term is the state-dependent correction, written as a logarithm of activities. The structure is the one T23 writes for stiffness: a reference value times a correction that depends on the departure from the reference. The theory adds no new term to it. The costs of batteries and electrolysers belong to engineering (Chapter 9). The theorem check: T14 applies to LiC_6; N4 applies to Nernst; T7 applies as a reading of cycling; T15 holds for electron counts 1, 2, 4 but they are small numbers; T2–T6, T11, T13, T16–T24 have no object.

Grade. A for the index-3 lithium lattice of LiC_6 as $\beta Z[\omega]$ (count and capacity) and for the Nernst potential as a state function (N4); C for cycling as meetings and for the three-electrode cell.

7.4 Nonlinear Chemistry

History. Chemistry far from equilibrium began with oscillations nobody believed. Alfred Lotka wrote a model of oscillating reactions in 1910 and 1920. William Bray reported an oscillating iodate–peroxide reaction in 1921. Most chemists took such reports as errors, because they thought oscillations would violate the second law. In 1951 Boris Belousov, seeking an inorganic analogue of the Krebs cycle, saw the cerium(IV)/cerium(III) ratio oscillate, the solution turning yellow and colourless in turn. Two journals rejected his paper. It appeared in 1959 in a little-known, unreviewed collection (Belousov–Zhabotinsky reaction). Anatol Zhabotinsky, a graduate student of Simon Shnoll, took up the reaction in 1961. It became known in the West at a conference in Prague in 1968. Field, Kőrös and Noyes analysed its mechanism in 1972; its mechanism involves around 18 steps.

The theory came from several directions. Alan Turing showed in 1952 that diffusion can make a uniform mixture form stripes and spots (Turing pattern). Prigogine and Lefever proposed the Brusselator in 1968, a minimal model of a chemical oscillator. Prigogine received the Nobel Prize in 1977. Edward Lorenz found deterministic chaos in three equations in 1963. The Poincaré–Bendixson theorem, completed by Bendixson in 1901, implies that a two-dimensional continuous system cannot have a strange attractor (Poincaré–Bendixson theorem). Sharkovskii (1964) and Li and Yorke (1975) showed that a map of the line with a point of period 3 has points of every period (Sharkovskii’s theorem). Mitchell Feigenbaum found in 1978 that period doubling has a universal rate.

In 1982 Simoyi, Wolf and Swinney ran the Belousov–Zhabotinsky reaction in a stirred flow reactor and found “one-dimensional dynamics”: a period-doubling sequence leading to chaos and periodic states, ordered as the theory of one-dimensional maps predicts (Simoyi, Wolf and Swinney 1982). Greenberg and Hastings (1978) modelled

excitable media as cellular automata with three states: resting, excited and refractory. The question left was how far simple maps and automata capture real chemistry.

Current state. Nonlinear chemistry received its first Nobel Prize since Prigogine in 2026. Kenso Soai's reaction, reported in *Nature* in 1995, is an asymmetric autocatalysis: the product catalyses its own formation with the same handedness, so a small enantiomeric excess grows into a large one (Soai reaction). It is the leading laboratory model for how one hand of a molecule can take over a system. The prize, shared with Kagan, cites "non-linear effects and autocatalysis in asymmetric organic synthesis" (Khaleej Times).

The same symmetry question has become a matter of policy. In December 2024, 38 scientists warned in *Science* against creating mirror-image bacteria. In 2026 the UN Scientific Advisory Board called for multilateral action, and Chinese and US policy groups issued a joint statement calling for national and international frameworks (Mirror life).

The theoretical tools are settled. Period doubling, the Feigenbaum constants and the order of periodic windows are textbook results; the period-3 window of the logistic map opens at $r = 1 + \sqrt{8}$. Turing patterns are studied in chemistry and biology, with hexagons and stripes as typical solutions of reaction–diffusion equations (Turing pattern). The Belousov–Zhabotinsky reaction remains the model system; the source describes it as staying far from equilibrium for a long time and evolving chaotically. The debates are about the origin of biological homochirality, about whether Soai-type amplification has any prebiotic analogue, and about how to control spiral waves in excitable media, including the heart (Chapter 17).

Interpretation.

1. *Strand.* I is time in a reactor or the sequence of cells in an excitable medium. P is the set of chemical states: concentrations, or the three states of an excitable cell.
2. *Passage and trit.* In an excitable medium, each cell cycles resting (0) → excited (1) → refractory (2 ≡ T) → resting. Going round the cycle once is a ring: three steps of +1, sum 3, closed with carry 1 (T7.1). Between two neighbouring cells, the difference of states mod 3 is a trit. Round a small square of four cells, the four trits sum to 3w, where w is the winding number: 0, +1 or −1.
3. *Closure.* A square with w = 0 is closed. A square with w = ±1 is a defect: the core of a spiral wave. In random initial states of the three-state automaton, the share of defects is 24/81 = 0.296; a 120 × 120 field gives 4,268 defects (0.296), with net winding 0 (Calculation 7.9). After 300 steps the count is unchanged: 4,268, net 0, in each of three runs. A single excitation without a defect dies out. A broken wave, with one defect of each sign, keeps 7,200 cells active after 300 steps. The winding is the conserved charge of the medium. Hexagonal patterns give a second closure. A hexagonal Turing or convection pattern is built from three plane waves whose wave vectors k_1, k_2, k_3 lie at 120° and sum to zero: a resonant triad. Because $k_1 + k_2 = -k_3$, the product of two modes feeds the third, and a quadratic term in the amplitude equations favours hexagons (Cross and Hohenberg 1993). The sum of the three cosines is invariant under a 60° turn. Four modes at 90°, the square pattern,

contain no triad that sums to zero (Calculation 7.13). The triad is the cube roots of unity, $1 + \omega + \omega^2 = 0$, the closed heterogeneous window of T4 in the plane. It is a discrete analogue of T22, an integer that continuous rewriting cannot change. In the flow reactor, the route to chaos goes by period doubling: periods 1, 2, 4, 8, … The superstable parameters of the logistic map give Feigenbaum ratios of 4.709, 4.681, 4.663, 4.668, 4.669 (Calculation 7.10).

4. *Scale.* Period doubling climbs the meeting ladder, 2^k. The period-3 window and its doublings, $3 \cdot 2^k$, lie on the ladder too. But the windows of a unimodal map include every period, 5 and 7 among them, which are off the ladder. Sharkovskii's order runs 3, 5, 7, …, 2·3, 2·5, …, with the powers of two last. In the Brusselator, the steady state loses stability at $B = 1 + A^2$: at $A = 1$, $B = 1.8$ gives no oscillation, $B = 2.2$ and 3.0 give limit cycles (Calculation 7.11). Soai amplification takes a minute excess to one hand: in a minimal model with autocatalysis, mutual inhibition and outflow, an excess of 2×10^{-6} becomes 0.044 by time 20 and 1.000 by time 40 (Calculation 7.12).
5. *Test.* For excitable media, the reading predicts that persistent activity requires nonzero winding, and that defects are created and destroyed only in pairs of opposite sign. In the automaton both hold (Calculation 7.9). The test in chemistry, stated before data: in Belousov–Zhabotinsky films imaged over time, the net topological charge of spiral cores in a closed domain stays constant unless a core crosses the boundary. A single spiral core created alone, without a partner of opposite sign or a boundary crossing, would refute the reading. For period doubling the reading is the field's own result, so it fixes the interpretation.

Consequences. Nonlinear chemistry contains two structures of the Foundations. Period doubling is the meeting ladder in time, observed in a chemical reactor in 1982. Spiral cores carry a conserved winding number of ±1, and the three-state model of an excitable cell is a ring of three. Both were known before this theory. The theory adds a unified name, and one test: conservation of topological charge in real chemical media, which experiments on spiral waves can check. It also marks a limit. The ladder does not select periodic windows: the logistic map has windows of every period, and periods 5 and 7 are as real as 3 and 6. The ladder describes the cascade, not the windows. For homochirality, the theory has a reading and a link. The reading: the racemate is a meeting $(x, -x)$, closed with carry 0; amplification breaks the meeting and carries the excess. The link: Kagan's heterochiral reservoir (Section 7.2) and Soai's autocatalysis both turn on the inactivity of the mirror pair. The theory does not say which hand wins; the models say chance decides, and Calculation 7.12 shows how quickly a tiny excess takes over. The Brusselator's oscillation, and Turing patterns, add nothing beyond their own mathematics. The theorem check: T7 and T7.1 apply to the excitable cycle; T22 applies by analogy to winding conservation; T15 applies to the cascade; T6 gives the defect baseline 24/81; T16–T18 do not apply, since the cascade does not mix ×2 and ×3; T10's rewrite R is a different automaton (averaging, not excitation), so its fixed points have no direct counterpart here.

Grade. A for the period-doubling cascade as the meeting ladder (Simoyi, Wolf and Swinney 1982), for conserved ±1 winding in three-state excitable media, and for

hexagonal patterns as resonant triads ($k_1 + k_2 + k_3 = 0$); B for charge conservation of spiral cores in chemical films; C for homochirality and for Turing patterns beyond the triad.

Chapter 8 — The Earth

This chapter reads the Earth as a set of cycles: plates that open and close oceans, tides and currents, the circulation of the air, the glacial rhythm and the water cycle. Three published structures fit the Foundations. A triple junction of plates exists only if the three relative velocities close to zero, as a ring of three does, and four-plate junctions survive only for an instant (McKenzie and Morgan 1969). Late-Pleistocene ice ages ended every second or third obliquity cycle (Huybers and Wunsch 2005), both rungs of the ladder $2^{a}\cdot 3^{b}$. Climate sensitivity depends on the background state, the context reading N4 as measured fact. Against these stand clear misses: the QBO is not locked to the year, the number of Hadley-type cells changes with a planet's rotation, and the Gutenberg–Richter factor of 10 is off the ladder. Grades: geology A for triple junctions, C for seismicity; oceanography A for amphidromic winding and topological equatorial waves, C otherwise; atmospheric science A for the trefoil orbit of the Lorenz system and for hexagonal convection as triads, B refuted for ladder locking of the QBO, C for the cell count, which N4 explains better; climate science A for skip-beat terminations and state-dependent sensitivity, B for integer locking in an independent age model; hydrology A for the meeting ladder of stream order and for the closed global balance, C for the rest.

8.1 Geology and Plate Tectonics

History. James Hutton argued in the 1780s that the Earth's rocks record cycles of erosion, deposition and uplift with "no vestige of a beginning, no prospect of an end". Charles Lyell's *Principles of Geology* (1830–33) made gradual change the rule. William Smith's geological map of England (1815) used fossils to order strata. By 1900 geologists could read the order of rocks but not why continents and oceans sat where they did. Seismology then looked inside. Andrija Mohorovičić found the base of the crust in 1909, and Inge Lehmann the solid inner core in 1936. The Earth became a layered body: crust, mantle, outer core, inner core.

Alfred Wegener described continental drift in 1912 and expanded it in *The Origin of Continents and Oceans* (1915). Arthur Holmes proposed in 1928 that mantle convection might move the plates. Most geologists rejected drift for want of a mechanism. After the Second World War, marine geophysics supplied one. Robert Dietz (1961) and Harry Hess (1962) proposed seafloor spreading. In 1963 Vine, Matthews and Morley linked the magnetic stripes of the seafloor to spreading and to reversals of the geomagnetic field. Tuzo Wilson added transform faults in 1965. In 1966 he asked whether the Atlantic had closed and opened again; the cycle of an ocean opening and closing now bears his name (Wilson cycle). W. Jason Morgan proposed in 1967 that the surface consists of twelve rigid plates; McKenzie and Parker gave an independent model; Xavier Le Pichon's six-plate model of 1968 completed the acceptance of plate tectonics (Plate tectonics).

In 1969 Dan McKenzie and Jason Morgan published “Evolution of triple junctions” in *Nature*. Where three plates meet, the relative velocities must close: AvB + BvC + CvA = 0, a velocity triangle (Triple junction). They classified junctions by the boundaries that meet (ridge R, trench T, fault F), found 16 possible types and 14 stable ones, and showed that meeting points of four plates exist only instantaneously.

Seismology measured the plates at work. Fusakichi Omori described the decay of aftershocks in 1894. Charles Richter defined his magnitude in 1935. Beno Gutenberg and Richter found in 1944 that the number of earthquakes falls tenfold for each unit of magnitude. The questions left were what drives the plates, how subduction starts, and whether large earthquakes can be forecast.

Current state. On 30 July 2025 at 11:24 local time, an earthquake of magnitude 8.8 struck off Kamchatka. It was a megathrust on the Kuril–Kamchatka Trench, where the Pacific plate subducts beneath the Okhotsk plate at 76–90 mm a year. Its hypocentre was about 35 km deep. The tsunami was weaker than feared across the Pacific, mostly 1 m or less, but ran up 33.1 m in a narrow valley near Vestnik Bay. It was the strongest earthquake since Tōhoku in 2011, tied as the sixth-strongest ever recorded (2025 Kamchatka earthquake).

Global seismicity follows a steady statistical law. From 2000 to 2021 the world had 0 to 4 earthquakes of magnitude 8 or more a year, 6 to 23 of magnitude 7–7.9, 104 to 185 of magnitude 6–6.9, and 1,201 to 2,276 of magnitude 5–5.9 (USGS statistics). The Gutenberg–Richter b-value is commonly close to 1.0 in active regions, ranging from about 0.5 to 2 and reaching 2.5 in swarms (Gutenberg–Richter law).

Plates move from zero to about 10 cm a year: 10–40 mm a year at the Mid-Atlantic Ridge, about 160 mm a year for the Nazca plate. There are usually seven or eight major plates, depending on whether the Indo-Australian plate is split (Plate tectonics). Active RRR junctions include Afar (the only one above sea level), Rodrigues, Galápagos and the Azores. McKenzie and Morgan classed FFF and RRF junctions as unstable; York later showed RRF can be stable under some conditions (Triple junction). The debates are about when plate tectonics began, whether slabs or ridges drive the plates, and how far earthquake forecasting can go.

Interpretation.

1. *Strand.* I is geological time. P is the set of configurations of plates and boundaries: which plates meet where. On a smaller scale, I is the path of a point round a junction, crossing from plate to plate.
2. *Passage and trit.* Round a triple junction, a path crosses three boundaries and returns. Each crossing changes the plate. With the relative velocities read as vectors, the three steps of the circuit add to zero. In the symmetric case of three ridges spreading at equal rates and 120°, the three velocities are proportional to 1, ω and ω^2, the cube roots of unity, and $1 + \omega + \omega^2 = 0$ (Calculation 8.1).
3. *Closure.* The closure condition of a triple junction is the field’s own equation: AvB + BvC + CvA = 0. It is a ring of three that closes, the heterogeneous closed window of T4 in vector form. An RRR junction is always stable, because the perpendicular bisectors of any triangle meet in a point. A junction of four plates exists only for an

instant; it splits into two triple junctions. That is T7 at the scale of plates: closures reduce to rings of three and, here, nothing longer persists. A boundary that does not close (a velocity mismatch) passes its tension on as deformation, the rule of N2 read physically. The velocity triangle is the small-time form of a statement about rotations. On a sphere every rigid plate motion is a rotation about an Euler pole, by Euler's fixed-point theorem. The rotation of B relative to A, followed by C relative to B, followed by A relative to C, must compose to no rotation at all. Plate reconstructions use this closure round every triple junction to check their poles; a circuit that fails to close by more than the data error marks a missing plate boundary or internal deformation. That is a closure test the field runs routinely.

4. *Scale.* The Wilson cycle opens an ocean and closes it again: a meeting (x, −x) at the scale of ocean basins, hundreds of millions of years long. The 16 junction types of McKenzie and Morgan exceed the 10 unordered triples of the three boundary types (Calculation 8.1); the extra six come from the polarity of trenches and faults. The source itself also mentions "ten possible types", so the count depends on what is distinguished. Seismicity does not climb the ladder. A b-value of 1 means a factor of 10 per magnitude unit, which is not $2^a \cdot 3^b$, and the b-value varies from 0.5 to 2.5 (Calculation 8.2). Energy grows by 10^1.5 ≈ 31.6 per unit.
5. *Test.* The reading claims that persistent plate junctions are triple. Stated before data: in a global plate model, every junction that persists over the model's time span is a junction of three plates; any junction of four or more plates that persists for more than one time step of the model refutes the reading. McKenzie and Morgan proved this geometrically for rigid plates; for deforming plates it is an open empirical question. For seismicity no test is stated.

Consequences. Plate tectonics contains one closure of the Foundations as published geometry. The velocity triangle at a triple junction is a ring that sums to zero, and four-plate junctions do not persist. The theory adds the claim that this is the same rule as the window of the register. That is a statement about the form of closure, not about plate dynamics. It predicts nothing about where junctions form or how fast they migrate. The Wilson cycle as a meeting is a reading. Seismicity is a clear negative: the frequency–magnitude law has a factor of 10, its exponent varies by region and by time, and nothing in it lies on the ladder. That variation is consistent with N4: the b-value is a state function of the fault zone, not a universal constant. The theory adds no forecast. One thing the reading does point to is where a test of N4 could be run in seismology. If b is a state function, it should vary with a measured state variable of the fault, such as differential stress or pore pressure, and return to its regional value when the state does. Seismologists already study b-value changes before large earthquakes; the theory adds no new variable to that work. The theorem check: T4 and T7 apply to triple junctions; T15 fails for seismicity; T13 does not count junction types, which are not rotation classes; T23 and N4 apply in the weak sense that b varies with state; T11–T12, T14, T16–T22 and T24 have no object here.

Grade. A for the triple junction as a closed ring of three and the instability of four-plate junctions (McKenzie and Morgan 1969); C for the Wilson cycle as a meeting and for seismicity, where the ladder fails.

8.2 Oceanography

History. Edmond Halley mapped the trade winds in 1686 and the magnetic variation of the Atlantic in 1701. Benjamin Franklin printed a chart of the Gulf Stream in the 1770s. Matthew Fontaine Maury compiled ships' logs into *The Physical Geography of the Sea* (1855). The Challenger expedition of 1872–76 made oceanography a science: it sounded, trawled and sampled the deep sea worldwide.

Tides were the first ocean phenomenon to be predicted. Newton explained them as the pull of Moon and Sun. Laplace gave the first dynamic theory of the ocean tides in 1775 and his tidal equations in 1776 (Theory of tides). William Whewell drew cotidal lines and proposed points of zero tide; Captain William Hewett confirmed one in the North Sea in 1840 (Tide). Lord Kelvin built a tide-predicting machine in the 1870s. Arthur Doodson published the harmonic development of the tide-generating potential in 1921, with 388 tidal frequencies. He labelled each constituent by a Doodson number: small integer multipliers of six astronomical arguments, each except the first shifted up by 5 so that no digit is negative.

Ocean dynamics followed. Vagn Walfrid Ekman explained in 1905 why wind-driven surface water moves at an angle to the wind. Harald Sverdrup, Walter Munk and Henry Stommel built the theory of wind-driven gyres between 1947 and 1950. In 1961 Stommel showed that a two-box model of the thermohaline circulation can have two stable states. Jacob Bjerknes identified the coupling of ocean temperature and trade winds in the Pacific in 1969, the core of El Niño–Southern Oscillation (ENSO). Walter Munk's "Abyssal recipes" (1966) balanced deep upwelling against downward mixing of heat and set the scale of ocean mixing for decades. William Thomson (Lord Kelvin) described the waves trapped against a coast by rotation in 1879; their equatorial form is the Kelvin wave of El Niño. The questions left were how stable the overturning circulation is, and how the ocean stores heat.

Current state. The ocean is now observed continuously. The Argo programme began deploying floats in 2000. It collected its millionth profile in November 2012, its two millionth in November 2018 and its three millionth in July 2024; close to 4,000 floats are active (Argo). Satellite altimetry shows global mean sea level 97.8 mm above 1993, with the annual rate rising from about 2.0 mm a year in 1993 to 4.4 mm a year (NASA).

The Atlantic Meridional Overturning Circulation is the main debate. The RAPID array at 26° N has observed it directly since 2004. A 2024 study in *Nature Geoscience* estimated a slowdown of 0.46 Sv per decade since 1950. A February 2025 *Nature* study across 34 models found the circulation resilient to extreme forcing and collapse unlikely this century; a January 2026 preprint using high-resolution modelling suggested collapse may be more likely than earlier simulations indicated. The IPCC judges a weakening "very likely" this century and an abrupt collapse "not expected before 2100", with medium confidence (AMOC).

The Pacific is in a strong El Niño. NOAA's Climate Prediction Center issued an El Niño Advisory on 8 October 2026. It gave a chance greater than 83 % of a strong-to-very-strong El Niño through January–March 2027, and an 83 % chance for October–

December that the three-month relative index reaches +2.5 °C or more, beyond any El Niño since 1950 (NOAA CPC).

Data now reach users fast. In real time, 90 % of Argo profiles reach users through two global data centres within 12 hours, and delayed-mode quality control has been applied to 75 % of eligible profiles. The programme needs 500 to 600 new floats a year to stay at strength, and its target rises from 3,000 to 3,800 floats when seasonal sea-ice zones and marginal seas are included (Argo). Theory has found topology in the ocean. Delplace, Marston and Venaille showed in 2017 that Earth's rotation gives the bulk wave modes a first Chern number of 2, and that this invariant guarantees the existence of the equatorially trapped Kelvin and Yanai waves (arXiv 1702.07583).

Interpretation.

1. *Strand.* I is time at a fixed place, or a path round a region of ocean. P is the set of states of the sea surface: height and phase of a tidal constituent, or temperature anomaly.
2. *Passage and trit.* For a tidal constituent, the phase at a place returns once per period. Round a closed path in the ocean, the phase can return to itself after winding once forward (+1), once backward (−1) or not at all (0).
3. *Closure.* An amphidromic point is a place where a constituent's amplitude is zero and the cotidal lines meet. The tide rotates round it, anticlockwise in the Northern Hemisphere and clockwise in the Southern (Amphidromic point). Round such a point the phase winds by one full turn: winding +1 or −1. A path that encloses no amphidrome has winding 0, a closed window. An amphidrome is the ocean's version of the winding defect in an excitable medium (Chapter 7, Section 7.4). The equatorial Kelvin and Yanai waves are protected by a Chern number of 2. In both cases an integer invariant fixes a feature that smooth changes cannot remove, the logic of T22 (charge conserved without Noether), on a different space.
4. *Scale.* The spring–neap cycle is the beat of the lunar (M2, 12.4206 h) and solar (S2, 12.00 h) semidiurnal tides: 14.765 days (Calculation 8.3). Doodson's multipliers are small signed integers. He shifted them by 5 to avoid minus signs. In balanced ternary the multipliers need no shift: M2 is (1T, 0, 0, 0, 0, 0), K1 is (1, 1, 0, 0, 0, 0) and O1 is (1, T, 0, 0, 0, 0) (Calculation 8.3). That is fact (a) of Foundations III.2, sign without a minus. Stommel's two-box model has two stable states of the overturning, a strong one and a weak or reversed one, and a range of freshwater forcing in which both exist. Two closed states with a hysteresis loop between them is a meeting in the sense of N5: going round the loop and back returns the system, but only through two jumps. ENSO recurs every two to seven years (ENSO); of the integers 2 to 7, 5 and 7 are off the ladder (Calculation 8.4).
5. *Test.* For amphidromes, the winding round any closed curve equals the sum of the windings of the amphidromes inside it. That is the argument principle for a smooth field and fixes the reading. A test that could fail: if the theory is right that a closed ocean basin is a closed window at the scale of the basin, then the net winding of M2 amphidromes inside an enclosed basin should equal the winding of the coastal phase. Stated before data: in a global tidal atlas, count amphidromes with sign in each semi-enclosed basin and compare with the phase winding along its coast.

Degenerate amphidromes, whose centres lie inland, must be counted. A mismatch after counting them would refute it.

Consequences. Oceanography has two published topological closures: the amphidromic winding of tides and the Chern number of equatorial waves. Both were found by the field, the second in 2017 with an explicit claim of topological protection. The theory adds a common name, winding as closure, and the link to T22. It does not predict where amphidromes lie; that needs the basin geometry and Laplace's equations. The Doodson notation is a small, real example of why balanced notation matters: Doodson needed an offset of 5 because his digits could be negative. The theory adds nothing to tidal prediction. For the AMOC the theory has a reading: Stommel's two stable states are two closed configurations, and a collapse would be the loss of one closure, its tension passed to other scales. That is not a forecast. It adds no number to a debate between a 2025 Nature study and a 2026 preprint. ENSO's 2–7 year period is not on the ladder as a set. Kelvin waves link the two halves of this section. The equatorial Kelvin wave that carries warm water east at the start of an El Niño is one of the two modes whose existence the Chern number guarantees. The topology says the wave must exist; it says nothing about its amplitude, which decides how strong the 2026 event becomes. The theorem check: T22's logic applies to amphidromes and Chern numbers; T12 and III.2 apply to Doodson numbers; T15 fails for ENSO periods; T7, T11, T13–T14, T16–T18, T23–T24 have no clear object; N4 applies to the state dependence of ocean mixing, which the field already models.

Grade. A for amphidromic winding and the Chern number of equatorial waves (Delplace, Marston and Venaille 2017) as integer closures; C for Doodson notation as balanced ternary (notation only), for the AMOC and for ENSO.

8.3 Atmospheric Science

History. Halley explained the trade winds in 1686 by solar heating. George Hadley added the Earth's rotation in 1735, in "On the Cause of the General Trade Winds" (Hadley cell). William Ferrel proposed a three-celled circulation in 1860. By the early twentieth century meteorologists accepted three cells per hemisphere: the tropical Hadley cell, the mid-latitude Ferrel cell and the polar cell. The Bergen school of Vilhelm Bjerknes described fronts and cyclones around 1919. Lewis Fry Richardson attempted a numerical forecast by hand in 1922. Carl-Gustaf Rossby described planetary waves in 1939. In 1950 Jule Charney and colleagues made the first computer forecast on ENIAC.

Edward Lorenz reduced a model of convection, derived from Barry Saltzman's work, to three equations in 1963 (Lorenz system). With $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$, the solutions never repeat and depend sensitively on the start: deterministic chaos. Henri Bénard had seen hexagonal convection cells in a heated liquid layer in 1900. Lord Rayleigh analysed buoyant convection in 1916, and John Pearson showed in 1958 that Bénard's cells were driven by surface tension (Rayleigh–Bénard convection).

The stratosphere held a surprise. Researchers at the UK Meteorological Office found in the 1950s that equatorial stratospheric winds reverse direction with a period of 28 to

29 months, the quasi-biennial oscillation. Lindzen and Holton explained it as driven by waves rising from the tropical troposphere (Quasi-biennial oscillation). Weather satellites flew from 1960. Gaspard-Gustave de Coriolis had described the deflecting force of rotation in 1835; Rossby and his successors made it the organising principle of large-scale flow. Lorenz's discovery set a limit: a forecast can only be as good as the knowledge of the starting state allows, and errors grow exponentially. The questions left were the limits of predictability, the role of waves and convection, and how the circulation responds to warming.

Current state. The Hadley circulation is widening. Since the 1980s it has expanded poleward by about 0.1° to 0.5° of latitude per decade, according to the IPCC's Sixth Assessment (Hadley cell). The dominant boreal-summer cell extended from 13° S to 31° N in 1979–2001 reanalysis data. Other planets show that the number and size of cells depend on rotation. Venus, rotating slowly, may have Hadley cells reaching high latitudes; Mars has a single cell most of the year; Titan may have a single pole-to-pole cell at solstice and two cells near equinox. Faster rotation produces more contracted cells.

The QBO broke its pattern in February 2016, when the expected change to easterly winds was disrupted by an unexpected band of westerlies. Its phase is not tied to the annual cycle (Quasi-biennial oscillation).

Convection theory is settled at its onset: instability begins at a critical Rayleigh number of 1,708, and the cells tend to be hexagonal prisms when the flow is not turbulent (Rayleigh–Bénard convection). The Lorenz attractor was proved to be a strange attractor by Warwick Tucker in 2002, answering Smale's 14th problem. For ρ = 99.96 the Lorenz system has a periodic orbit that is a T(3,2) torus knot (Lorenz system). The year 2026 opened under a developing El Niño, which by October had become strong (Section 8.2). The 2016 disruption made the oscillation less predictable than its record had suggested. Radiosonde soundings show that, unlike many stratospheric features, the QBO's phase is not tied to the annual cycle. The debates are about the response of jet streams and storm tracks to warming, about the predictability of the QBO, and about machine-learned forecast models.

Interpretation.

1. *Strand.* I is the path of an air parcel or of the state of a model through its phase space. P is the set of positions or states. A loop is a parcel returning to a latitude, or the state returning near a former state.
2. *Passage and trit.* In the meridional circulation, a parcel rises (1), sinks (T) or moves horizontally (0). For an orbit in phase space, the crossings of its projection are over, under or none, the trits of the embedding hypothesis of Foundations Part VII. The Lorenz flow needs this hypothesis: its orbits live in three dimensions, and only there can a closed orbit be knotted.
3. *Closure.* The Lorenz system has three variables. By the Poincaré–Bendixson theorem, a continuous system with two variables cannot have a strange attractor; three is the least that can. Its periodic orbit at ρ = 99.96 is the torus knot T(3,2), which is the trefoil: the first knot of T19, chiral by T20, appearing as a closed orbit

of the convection model. Bénard's hexagonal cells, like hexagonal Turing patterns, are built on resonant triads of wave vectors that sum to zero (Chapter 7, Calculation 7.13): the heterogeneous closed window of T4 in the plane. The Lorenz attractor's sensitivity can be computed: with Lorenz's parameters the largest Lyapunov exponent is 0.904, so errors grow by a factor e in about 1.1 time units (Calculation 8.9). That number belongs to the model, not to the theory.

4. *Scale.* Earth has three cells per hemisphere, six in all, both on the ladder (Calculation 8.4). But the number is not fixed. Mars has one cell much of the year, Titan one or two, and faster rotation gives more. The cell count is a state function of rotation rate and the size of the planet: N4, not T15. The QBO period of 28–29 months is 2.33–2.42 years (Calculation 8.4). It sits between the ladder's 2 and 3 and is not phase-locked to the annual cycle. It is a natural oscillation that does not close on the year. The three cells can also be read as a word. The Hadley and polar cells are thermally direct, warm air rising and cold air sinking (1); the Ferrel cell between them is thermally indirect (T), driven by the eddies of the mid-latitudes. The word (1, T, 1) has two equal trits and one different. By T4 it is an open window: it does not close, and its sum, 1, is kept as a nonzero digit. The register then says the three-cell circulation passes tension on. The field says the same in its own terms: the Ferrel cell is not a closed thermal circuit but the mean of eddy fluxes, and the jet streams carry the momentum. That is a reading, but a consistent one.
5. *Test.* The ladder reading would predict that an oscillation driven within the annual cycle locks to it at a ratio $2^a \cdot 3^b$. Such ratios are dense (T16), so the claim must be bounded before data. With exponents of size at most 3, the ratios between 2 and 3 years are 2, 9/4, 8/3 and 3: periods of 24, 27, 32 and 36 months (Calculation 8.8). Stated before looking: a QBO locked on the ladder has a long-run mean period of 27 or 32 months. The observed mean of 28 to 29 months is 1 to 2 months from 27 and not on any of them, and the phase is not tied to the season. The QBO refutes ladder locking for the stratosphere, at least with small exponents.

Consequences. Atmospheric science gives the theory two published structures and two clear failures. The trefoil orbit of the Lorenz system is a fact of the field's model, found by its own methods; the Foundations name it the first knot. The hexagonal convection cell is the triad. Neither adds to forecasting. The failures are informative. The three-cell circulation looks like a ring of three, but other planets show that the count depends on rotation: the number three is a context value (N4), not a closure law. The QBO is not locked to the year, and the bounded ladder test fails on it. The theory gains from saying so plainly. N4 earns its place here: universality, three cells everywhere, is the limiting case of a state-dependent count. The theory adds nothing to the debate on jet streams. A further point concerns the Lorenz knots. Birman and Williams showed in 1983 that the periodic orbits of the Lorenz flow form a special family of knots, the Lorenz knots. The trefoil at $\rho = 99.96$ is the simplest member found in practice. The Foundations prove that the trefoil is the first knot (T19); the Lorenz flow produces it as a closed orbit of a convection model. The link is real and published on both sides, but it does not say why the atmosphere should have Lorenz-type dynamics; Lorenz's three modes are a severe truncation. The theorem check: T19–T20 apply to the Lorenz trefoil; T4 to triads; T15 fails for cell counts and the QBO; T7 does

not apply to the cells as closures; T16–T18 do not apply, since no near-commensurability of ×2 and ×3 is involved; T23–T24 have no stated object.

Grade. A for the trefoil periodic orbit of the Lorenz system (T19) and for hexagonal convection as resonant triads; B, refuted, for ladder locking of the QBO (27 or 32 months predicted, 28–29 observed); C for the three-cell circulation, which N4 explains better than T15.

8.4 Climate Science

History. Joseph Fourier argued in 1824 that the atmosphere keeps the Earth warmer than it would otherwise be. John Tyndall measured the absorption of heat by water vapour and carbon dioxide in 1859. Svante Arrhenius calculated in 1896 how much a change in carbon dioxide would change the temperature. Guy Callendar linked rising carbon dioxide to measured warming in 1938. Charles Keeling began measuring carbon dioxide at Mauna Loa in 1958. Manabe and Wetherald's model of 1967 gave the first modern estimate of climate sensitivity. The Charney report of 1979 put equilibrium sensitivity at 1.5 to 4.5 °C per doubling.

The ice ages got their pacemaker from astronomy. James Croll proposed orbital causes in the nineteenth century. Milutin Milanković computed the insolation at 65° N from the orbital elements in the 1920s and 1930s, and predicted 41,000-year ice-age cycles from obliquity (Milankovitch cycles). Hays, Imbrie and Shackleton found the orbital periods in deep-sea sediments in 1976. But the last million years showed a dominant cycle near 100,000 years, matching eccentricity, whose forcing is weak: the 100,000-year problem. The Mid-Pleistocene Transition, roughly 1.25 to 0.7 million years ago, separates the 41,000-year world from the 100,000-year world.

In 2005 Peter Huybers and Carl Wunsch tested glacial terminations against each orbital cycle. They rejected at the 5 % level the hypothesis that terminations are independent of obliquity, but not for eccentricity or precession, and inferred that "the ice sheets terminated every second or third obliquity cycle at times of high obliquity" (Huybers and Wunsch 2005). Huybers extended this in 2007 with a depth-derived age model, built to avoid tuning ages to the orbit. A Rayleigh test rejected random phasing relative to obliquity at the 1 % level, with R = 0.7 against a critical value of 0.47 for the early Pleistocene and R = 0.8 against 0.52 for the late Pleistocene; precession and eccentricity showed no significant phase stability. In that model 33 of 36 deglaciations over two million years occurred when obliquity was anomalously large; early-Pleistocene terminations came nearly every obliquity cycle and late ones mostly every two (about 80,000 years) or three (about 120,000 years) (Huybers 2007). Lenton and colleagues defined tipping elements of the climate system in 2008: subsystems, such as the Greenland ice sheet and the AMOC, that can pass a threshold into another state. The IPCC was founded in 1988. The questions were the size of climate sensitivity, the cause of the 100,000-year cycle, and the risk of abrupt change.

Current state. Berkeley Earth puts 2025 at 1.44 ± 0.09 °C above the 1850–1900 average, the third-warmest year since 1850 after 2024 and 2023. Its 30-year smoothed trend has passed 1.4 °C and may reach 1.5 °C in less than five years; the report states

that “the 1.5 °C goal will not be achieved” (Berkeley Earth). The WMO put 2024 at 1.55 °C above pre-industrial levels, the hottest year in 175 years of observation (WMO water report). Carbon dioxide at Mauna Loa averaged 426.07 ppm in September 2026, against 424.37 ppm in September 2025 (NOAA GML).

The IPCC’s Sixth Assessment gives equilibrium climate sensitivity a best estimate of 3 °C, likely range 2.5 to 4 °C, and puts 2011–2020 at 1.09 °C above 1850–1900 (IPCC AR6 SPM). Sensitivity depends on the climate state: studies of the last 800,000 years find it greater in glacial than in interglacial periods, the PETM may have had above-average sensitivity, and a second doubling of CO_2 may warm by a different amount than the first (Climate sensitivity).

In October 2026 the Pacific is in a strong El Niño, with an 83 % chance of a historic event by late 2026 (Section 8.2). The AMOC debate is open. The IPCC’s figure of 1.09 °C for 2011–2020 and Berkeley Earth’s 1.44 °C for 2025 differ because one is a decade mean and the other a single year near an El Niño; the long-term trend lies between them. The Mid-Pleistocene Transition is still unexplained; proposed causes include ocean circulation, carbon dioxide and ice-sheet feedbacks (Milankovitch cycles). The debates are about the pace of warming in the 2020s, about tipping points, and about how much of the recent jump is El Niño.

Interpretation.

1. *Strand.* I is time. P is the space of climate states: ice volume, temperature, carbon dioxide.
2. *Passage and trit.* A termination is a return of the climate to an interglacial state. Measured in obliquity cycles since the previous termination, a return is either on a cycle (an integer number of cycles, closed) or off it (open). Read as trits per obliquity cycle: the ice sheet grows (1), terminates (T) or holds (0).
3. *Closure.* A glacial cycle closes when the termination falls on a high-obliquity phase. Huybers and Wunsch found this phase locking statistically. The skip-beat structure, every second or third obliquity cycle, uses the two rungs of the ladder: 2 and 3. A mean of 100,000 years from gaps of 82 and 123 kyr needs about 44 % three-cycle gaps (Calculation 8.5). Sensitivity is a state function: higher in glacial states. That is N4 as a measured fact, and T23’s form, a reference value corrected by the departure of the state. Written as $S = S_0(1 + \gamma\delta)$, with δ the departure of a state variable such as global ice volume from its reference, the published finding is that γ has the sign that makes S larger when ice is larger. The ice–albedo feedback, which grows with ice cover, is the mechanism the field gives. The theory adds the form, not the value of γ.
4. *Scale.* Before the Mid-Pleistocene Transition, terminations came every obliquity cycle (ratio 1); after it, every two or three. The transition moved the closure from 1 to the products of the ladder. In the LR04 stack, the nine intervals between the starts of interglacial stages 1 to 21 are 116, 113, 94, 87, 100, 97, 91, 78 and 76 kyr: 2.83, 2.76, 2.29, 2.12, 2.44, 2.37, 2.22, 1.90 and 1.85 obliquity cycles (Calculation 8.5). Rounded, seven are 2 and two are 3. Only four of the nine lie within 0.2 cycles of an integer, against 40 % expected by chance. A crude Rayleigh test of the

termination phases with a fixed 41-kyr period gives p ≈ 0.05. But LR04 ages are partly tuned to orbital forcing, so this check is not independent.

5. *Test.* The ladder reading says that late-Pleistocene glacial cycles close on whole numbers of obliquity cycles, two or three, and never on non-integer numbers. Stated before data: in an age model built without orbital tuning (Huybers's depth-derived model, or radiometric dates on speleothems), the intervals between terminations, divided by the local obliquity period, cluster within ±0.2 of 2 or 3 more often than the 40 % expected by chance. If they are spread uniformly between 1.8 and 3.2 cycles, the integer-locking reading fails, and only the weaker published phase locking remains.

Consequences. Climate science holds two structures of the Foundations as published fact. The skip-beat terminations, every second or third obliquity cycle, are rungs 2 and 3 of the ladder; Huybers and Wunsch found them with their own statistics. State-dependent sensitivity is N4: no single sensitivity holds everywhere, and the IPCC's 3 °C is a value at a reference state. The theory adds a sharper claim, that the gaps are integers, and a test against an untuned age model. The crude LR04 check is weak: four of nine intervals near integers is chance level. So the sharper claim is not yet supported, and may fail. For present warming, the theory adds nothing to the numbers. Reading tipping points as the loss of a closure, with tension passed to the next scale, is a reading, not a forecast. The comma (T18) does not appear in the orbital ratios: 100/41 = 2.44 and 41/23 = 1.78 are not near ladder ratios with small residues. There is one place where the reading makes a definite claim about the future. If terminations close on two or three obliquity cycles, the present interglacial, which began about 11,700 years ago, should end near a high-obliquity phase two or three cycles later, tens of thousands of years from now, unless the carbon added since 1850 removes the closure altogether. That is no help to anyone alive, and it is not testable on a human time scale. The theorem check: T15 applies to the skip-beat rule; N4 and T23 to sensitivity; T16–T18 do not apply; T7 applies only as a reading; T19–T22 and T24 have no object.

Grade. A for skip-beat terminations on rungs 2 and 3 (Huybers and Wunsch 2005; Huybers 2007) and for state-dependent sensitivity as N4; B for integer locking of interglacial intervals in an untuned age model (weak in LR04); C for tipping as loss of closure.

8.5 Hydrology

History. Hydrology became quantitative when it closed a balance. In 1674 Pierre Perrault measured rainfall over the upper Seine basin and showed that it exceeded the river's flow by a factor of about six. Edme Mariotte confirmed the result. Edmond Halley estimated evaporation from the Mediterranean in 1687 and found it enough to feed the rivers. Together they ended the old idea that rivers come from underground seas: precipitation, evaporation and runoff make one cycle.

Henry Darcy's law of flow through sand (1856), from his work on the fountains of Dijon, founded groundwater hydrology. Leroy Sherman introduced the unit hydrograph in 1932. Robert Horton described infiltration in 1933 and, in 1945, laws of stream

networks: numbers of streams fall and lengths rise geometrically with order (Strahler number). Arthur Strahler simplified the ordering in 1952 and 1957. In his scheme a stream with no tributaries has order 1; where two streams of equal order i meet, the result has order i + 1; where unequal orders meet, the higher order continues. Ronald Shreve noted in the 1960s that Horton's laws should be expected from any topologically random network.

Harold Edwin Hurst studied the Nile to size the reservoirs of Aswan. In 1951 he found that the range of cumulative flows grows faster with time than independent years would allow (Hurst exponent). Wet years cluster with wet years, dry with dry: long memory. Howard Penman combined the energy balance and the aerodynamics of evaporation in 1948, still the basis of evaporation estimates. Hurst's finding was later called the Hurst phenomenon, and Benoit Mandelbrot linked it to fractional Brownian motion in the 1960s. The questions left were how to predict floods and droughts in a changing climate, and how to manage groundwater that is being used faster than it refills.

Current state. The global water cycle moves about 505,000 km³ a year. About 434,000 km³ evaporates from the oceans, 86 % of the total, and about 398,000 km³ falls back on them. Rain on land delivers about 107,000 km³ a year. Water stays about 9 days in the atmosphere and about 3,200 years in the ocean. Of about 1,386 million km³ of water, the oceans hold about 97 %; ice caps, glaciers and permanent snow hold 1.7 % of all water but 68.7 % of fresh water (Hydrological cycle).

The WMO's *State of Global Water Resources 2024*, published on 18 September 2025, found that "only one third of river basins had normal conditions in 2024". All glacier regions reported losses for the third year running (WMO). The strong El Niño of 2026 is expected to shift rainfall patterns again (Section 8.2).

Stream-network statistics remain a field of their own. Most British rivers have a bifurcation ratio, the ratio of the numbers of streams of successive orders, between 3 and 5 (Strahler number). Long memory in river flows, Hurst's discovery, is measured by the Hurst exponent H, between 0 and 1; H = 0.5 is short memory, higher values indicate persistence (Hurst exponent). The water cycle and the energy cycle are one: the 434,000 km³ evaporated each year carry latent heat from the ocean to the atmosphere, and the 2024 heat record coincided with the abnormal basins of the WMO report. The debates are about attribution of droughts and floods, groundwater depletion, and how far long memory in flows is real or an artefact of trends.

Interpretation.

1. *Strand.* I is the path of a water parcel through the cycle: sea, air, land, river, sea. P is the set of reservoirs. In a river network, I is the path downstream and P the set of junctions.
2. *Passage and trit.* A parcel's step is up (evaporation, 1), down (precipitation, T) or along (flow within a reservoir, 0). At a river junction, two streams meet. A dry year after a dry year continues the first passage's direction (1); a wet year after a dry one reverses it (T); a normal year is 0. Hurst's persistence means that runs of equal trits are longer than independence allows.

3. *Closure.* The global water balance closes. Over the oceans, precipitation minus evaporation is −36,000 km³ a year; over land it is +36,000 km³; the sum is zero (Calculation 8.6). The ocean's deficit returns as river runoff. That is closure in the sense of mass conservation, which the field has used since Perrault. In a river network, Strahler's rule is a merging rule: two streams of equal order meet and become one passage of the next order. That is a meeting merged into a coarse passage, the ×2 rung of T15. A stream of order k therefore needs at least 2^(k−1) sources (Calculation 8.6). A junction of unequal orders does not raise the order; the smaller stream is absorbed. In the language of T11, a seam between unequal blocks carries no closure.
4. *Scale.* The order k measures depth on the meeting ladder, as the number of digits measures depth on the ring ladder in T12.3. The bifurcation ratios of real networks, 3 to 5, include 3 and 4 (on the ladder) and 5 (off it). The residence times differ by a factor of about 130,000 between atmosphere and ocean (Calculation 8.6): the cycle runs at very different speeds in its parts. Hurst's long memory is N3 as a reading: the river carries its history in its present state.
5. *Test.* For stream order the ladder is built into the definition, so the minimum 2^(k−1) fixes the reading. A test that could fail concerns real networks. If the meeting rule describes how basins grow, then for networks extracted at fine resolution, the bifurcation ratio should cluster near 4, the value of a network where each order is formed by exactly two of the next lower order on average and side tributaries are rare. Stated before data: across a large sample of basins, the median bifurcation ratio lies within 3.5–4.5. A median near 5 or above would refute the reading as a description of growth. Topologically random networks are known to obey Horton's laws, so the test can only separate this reading from others if the spread, not only the median, is compared.

Consequences. Hydrology contains the meeting ladder in the field's own definition: Strahler order rises by one only where two equal orders meet. The minimum count 2^(k−1) follows. The theory names it as T15's ×2 rung and adds a test on bifurcation ratios. If the test holds, it adds little the field did not know from Horton's laws and random-network theory. If it fails, the reading of basin growth as merging meetings is wrong. The closed global balance is mass conservation; the theory renames it. Hurst's long memory is the strongest link to N3, the reading that the net carries its history as its shape. But N3 says nothing about the size of H, and a test would need a predicted value. The theory gives none. Another link is to T6, the closure probability of a window with a biased law. If annual flows are coded as trits (dry, normal, wet) with probabilities p_{-}, p_{0}, p_{+}, the share of three-year windows that close (all equal or all different) is (1/3) [1 + $\hat{p}(\omega)^3$ + $\hat{p}(\omega^2)^3$]. For independent years with equal chances the share is 1/3; even a strongly biased law such as (0.3, 0.2, 0.5) gives only 0.34. Persistence changes it much more: if each year repeats the last with probability 0.5, the share is 0.375; at 0.7 it is 0.535; at 0.9 it is 0.815 (Calculation 8.10). A test follows, stated before data. Code a long annual flow record, such as the Nile, into terciles. If the share of closed three-year windows exceeds T6's value for the observed law by more than sampling error, the record has memory at the scale of three years. That is a new statistic, not a new law: it measures what Hurst's exponent measures, at one window length. It is useful only

because it is the register’s own measure. The WMO’s finding that only a third of basins were normal in 2024 is a measured fact; the theory reads it as widespread open windows and adds nothing quantitative. The theorem check: T15 applies to Strahler order; T11 to unequal junctions as a reading; T12.3’s depth count has an analogue in order; T6, T13–T14, T16–T24 have no object; N3 applies to long memory as a reading.

Grade. A for Strahler order as the meeting ladder (Strahler 1952, 1957) and for the closed global balance; B for bifurcation ratios clustering near 4 and for an excess of closed three-year windows in long flow records (T6); C for long memory as N3.

Chapter 9 — Building and Making

This chapter reads the practical sciences of matter, where something does not close and engineers make it close. The richest field for the Foundations is the mechanics of frames and fixtures. Maxwell's count of $2j - 3$ bars makes the triangle the only rigid polygon, a ring of three; but a rigid frame need not contain a triangle, as the six-joint frame $K_{3,3}$ shows, and the chapter says so. Exact constraint locates a body with six contacts, which Kelvin split as 3 + 2 + 1, the lengths of the three elementary closures of T7; and three equally spaced planet gears fit exactly when the sun and planet tooth counts sum to 0 mod 3. Geodesic domes, like virus shells, are built on the Eisenstein norms of T14. Chemical and environmental engineering contain little of the theory's structure beyond what Chapter 7 found. Grades: civil engineering A for Maxwell counting and the dome series, B for the triangle claim, refuted as a general rule by $K_{3,3}$; mechanical engineering A for exact constraint and the planetary mod-3 condition, A for Brocot's convergents, C for the Wankel engine; chemical engineering B for reactor dynamics, C otherwise; materials engineering A for Maxwell lattices and the $\Sigma 3^n$ series, C for batteries and photovoltaics; environmental engineering C.

9.1 Civil Engineering

History. Builders knew long before theory that triangles hold their shape. Roman roof trusses and medieval timber frames braced rectangles with diagonals. Galileo's *Two New Sciences* (1638) began the theory of beams. Robert Hooke stated his law of springs in 1678. Claude-Louis Navier gave the theory of elastic beams in 1826. The nineteenth century built iron and then steel: trusses for railway bridges, analysed joint by joint. Gustave Eiffel's tower of 1889 is a lattice of wrought-iron trusses, triangulated at every scale from the main legs down to the small members.

In 1864 James Clerk Maxwell published "On the calculation of the equilibrium and stiffness of frames". He counted: a plane frame of j joints has 2j coordinates, of which 3 are rigid motion of the whole; each bar fixes one distance. So a frame needs at least $2j - 3$ bars to be rigid, and in space $3j - 6$ (Structural rigidity). Luigi Cremona's reciprocal diagrams (1872) turned the forces in a truss into a drawing. Lebrecht Henneberg characterised minimally rigid plane frames in 1911: start from one bar and repeat two moves, adding a joint attached by two bars, or splitting a bar and attaching the new joint to a third. Hilda Geiringer gave the exact combinatorial condition in 1927, and Gerard Laman rediscovered it in 1970: $2j - 3$ bars in all, and at most $2k - 3$ among any k joints (Laman graph).

Materials and failures shaped practice. Joseph Aspdin patented Portland cement in 1824; Joseph Monier patented reinforced concrete in 1867. The Tacoma Narrows Bridge failed in 1940 by wind-driven oscillation and made aerodynamics part of bridge design. The finite element method grew from structural analysis in the 1950s and 1960s. Domes became a field of their own. Walther Bauersfeld of Carl Zeiss Jena designed the first geodesic dome to house a planetarium projector; the dome opened

on 18 July 1926. Buckminster Fuller patented the geodesic dome on 29 June 1954 (Geodesic dome). Frei Otto's cable nets and membranes for the Munich Olympic stadium (1972) carried the tensile structure to a large scale. The questions left were how to build with less carbon, how to maintain ageing infrastructure, and how to make structures resilient to extremes.

Current state. The American Society of Civil Engineers' 2025 Report Card gave US infrastructure an overall grade of C, with an investment gap of $3.6 trillion over ten years. Eight of 18 categories improved, and for the first time since 1998 no category received a D− (ASCE Report Card).

Cement is civil engineering's largest climate problem. Heating limestone releases carbon dioxide from the rock itself, before any fuel is burned. On 18 June 2025 Heidelberg Materials inaugurated Brevik CCS in Norway, the first industrial-scale carbon capture at a cement plant. It is to capture about 400,000 tonnes of CO_2 a year, half of the plant's emissions, and to supply "evoZero", described as the world's first carbon-captured cement (Heidelberg Materials).

Rigidity theory is now a field of mathematics and engineering together. Laman graphs are the bases of the two-dimensional rigidity matroid, and $K_{3,3}$, the complete bipartite graph on two sets of three joints, is a Laman graph that is not planar (Laman graph). Rigidity percolation, the point at which a random network becomes rigid, has been measured numerically: for bonds on the triangular lattice the threshold is p_c = 0.6602 (Rigidity percolation). In three dimensions no simple counting rule like Laman's is known to be sufficient. Tensegrity structures, from Kenneth Snelson's sculptures and Fuller's designs, stand with fewer bars than Maxwell's count requires. Christopher Calladine explained them in 1978 by extending Maxwell's rule: the bar count minus 3j − 6 equals the number of states of self-stress minus the number of mechanisms. A tensegrity has mechanisms, but prestress in its cables stiffens them. Calcination is the reason capture is needed even with clean fuel: turning limestone into lime releases 0.785 tonnes of CO_2 for every tonne of lime, from the rock alone (Calculation 9.3). The debates are about low-carbon cements, about the cost of capture, and about how to inspect and prioritise ageing bridges.

Interpretation.

1. *Strand.* I is the sequence of members in a frame, traced from joint to joint. P is the set of joints. A loop is a closed circuit of members.
2. *Passage and trit.* Each member carries tension (1), compression (T) or no force (0). A circuit of members returns to the joint it left.
3. *Closure.* A polygon of j bars has j − 3 internal degrees of freedom. Only the triangle, j = 3, is rigid (Calculation 9.1). The triangle is the least closed circuit that cannot be deformed: a ring of three, the shortest closure with winding ±1 in T7. A quadrilateral has one free mode and becomes rigid with one diagonal, which splits it into two triangles. A strip of n triangles has n + 2 joints and 2n + 1 bars, exactly 2j − 3: minimally rigid. Henneberg's first move, a new joint attached by two bars, is a meeting added to the frame; each move keeps the count at 2j − 3.

4. *Scale.* The minimal mean number of bars per joint tends to 4 in the plane and 6 in space as frames grow (Calculation 9.1). Geodesic domes are built on the icosahedron subdivided by a pair (h, k): the frequency is $T = h^2 + hk + k^2 = 1, 3, 4, 7, 9, 12, 13, \ldots$, the norms of the Eisenstein integers of T14 (Calculation 9.2). A dome of frequency T has $10T + 2$ vertices, of which exactly 12 have five neighbours, whatever T is; Euler's formula forces the twelve. These are the same numbers as the triangulation numbers of virus capsids (Chapter 10) and the icosahedral carbon cages of Chapter 6. The engineer's frequency choice is thus a choice of Eisenstein integer $h + k\omega$, and the dome's struts lie along the hexagonal lattice of $Z[\omega]$ folded onto the icosahedron. A frequency of $T = 3$, the norm of the prime above 3, gives the dual of C_{60}'s cage.
5. *Test.* The reading "the ring of three is the unit of rigidity" can be stated as a claim that fails: every minimally rigid plane frame contains a triangle. It is false. $K_{3,3}$ at generic positions has 9 bars, rank 9 and no internal freedom, and contains no triangle (Calculation 9.1). So the triangle is the least rigid polygon, but rigidity is a global count, not a sum of triangles. The test was stated before the computation and the reading lost.

Consequences. Civil engineering holds two structures of the Foundations as published fact. Maxwell's count and its exact form by Geiringer and Laman make the triangle the only rigid polygon. The geodesic dome uses the Eisenstein norms for its frequencies, as Caspar and Klug later did for viruses. Neither is new, and neither needs the theory. What the theory adds is a test, and it fails: rigid frames without triangles exist. The ring of three is the shortest rigid circuit, as T7 makes it the shortest closure that goes round, but closure of a frame is not built from rings. That is a useful correction. It keeps the theory from claiming triangulation as its own. Tensegrity gives the theory a better image than the triangle. Its frames do not close by count; they have open mechanisms. Tension, carried by cables, is what makes them stand. In the words of N2, what does not close passes on as tension, and here the tension is literal and holds the structure up. Calladine's extension of Maxwell's rule is the field's exact statement of it. The theory adds the word, not the rule. Joining two rigid bodies in the plane is a seam: they have three degrees of freedom relative to each other and need three bars, not all parallel and not all meeting in one point, to close it (Calculation 9.9). Two bars leave the seam open, a hinge. For cement the theory has nothing to say: the 0.785 tonnes of CO_2 per tonne of lime from calcination is stoichiometry (Calculation 9.3). Brevik's half-capture implies plant emissions of about 800,000 tonnes a year. The theorem check: T7 applies to the triangle as the shortest ring; T14 to dome frequencies; T15 to the mean connectivity 4 and 6, which are on the ladder, but trivially; T11 has an object in the joint between two rigid sub-frames, which needs three bars to close, a seam that carries tension unless closed; T13, T16–T24 have no object.

Grade. A for Maxwell counting with the triangle as the only rigid polygon (Maxwell 1864; Laman 1970) and for dome frequencies on Eisenstein norms; B for "every minimally rigid frame contains a triangle", refuted by $K_{3,3}$; C for cement.

9.2 Mechanical Engineering

History. Mechanical engineering began with engines and the problem of making motion exact. Thomas Newcomen's engine of 1712 and James Watt's separate condenser of 1769 gave the machine age its power. Watt's parallel motion (1784) made a straight line from rotating links. Sadi Carnot bounded the efficiency of heat engines in 1824. Robert Willis's *Principles of Mechanism* (1841) and Franz Reuleaux's *Kinematics of Machinery* (1875) made mechanisms a science of pairs and chains of links.

Precision required a theory of support. Maxwell published the principles of kinematic coupling in 1871: three V-grooves pointing to the centre, each holding a curved surface at two points, six contacts in all. Lord Kelvin's coupling of 1868–71 rests three balls in a tetrahedral dent (3 contacts), a V-groove (2 contacts) and a flat (1 contact). In exact-constraint design, the number of contacts equals the number of degrees of freedom to be removed. Too few lets the part move; too many forces it to deform, for example under thermal expansion (Kinematic coupling).

Gearing was the other precision art. Achille Brocot, a Paris clockmaker, published in 1861 a method for choosing gear trains whose ratio is close to a desired value, using ratios of numbers with small factors. The same tree of fractions was introduced independently by the number theorist Moritz Stern in 1858 (Stern–Brocot tree). Epicyclic gearing shares the load among several planets and so raises torque capacity (Epicyclic gearing).

Felix Wankel received his first rotary engine patent in 1934, joined NSU in 1951, and his DKM 54 first ran on 1 February 1957. Its rotor, shaped like a Reuleaux triangle, turns in an epitrochoidal housing; the eccentric shaft turns three times for each turn of the rotor. The NSU Spider of 1964 was the first Wankel car sold; Mazda ended rotary production in 2012 (Wankel engine). The first industrial robot was installed in 1961. The questions left were efficiency, automation and the precision of motion at ever smaller scales.

Current state. Industrial robots spread fast. The International Federation of Robotics counted 542,000 industrial robots installed in 2024, 54 % of them, 295,000 units, in China. The operational stock reached 4,664,000, up 9 % in a year, with more than 2 million in China (IFR, 25 September 2025).

Precision mechanics keeps the old principles. Kinematic couplings in optical mounts, wafer handling and metrology still use the Maxwell and Kelvin patterns, because six well-placed contacts locate a part repeatably without overconstraint. The source notes that the Maxwell coupling is thermally stable, since the curved surfaces expand or contract together in their grooves, while the Kelvin coupling has contact-stress problems under high loads (Kinematic coupling).

Planetary gearboxes dominate wind turbines, vehicle transmissions and robot joints, where load sharing among planets gives high torque density. Electric drive is changing the field's centre of gravity. With 17 million electric cars sold in 2024 (IEA), the reciprocating engine is losing ground, and the rotary engine survives mainly as a range extender. Robot density, not only numbers, now measures industrial strength; the IFR

reports that the global installation rate has more than doubled in ten years. The debates are about humanoid and collaborative robots, about the efficiency of electric drivetrains, and about additive manufacturing of parts that cannot be cast or machined.

Interpretation.

1. *Strand.* I is the sequence of contacts or meshes in a mechanism. P is the set of parts. A loop is a closed kinematic chain: motion that returns to its start.
2. *Passage and trit.* A contact constrains a direction (1), leaves it free (0), or overconstrains it against another contact (T, a conflict that must deform).
3. *Closure.* A rigid body in space has six degrees of freedom. Exact constraint closes them with six contacts. Maxwell's coupling uses three pairs, 3 × 2: three meetings. Kelvin's uses 3 + 2 + 1: a three-point seat, a two-point groove and a single flat (Calculation 9.4). Read with T7, Kelvin's coupling is a ring, a meeting and a standing: the three elementary closures, one each, of lengths 3, 2 and 1. A three-legged stool stands on any floor because three points always lie in a plane; four random feet do not, and the stool rocks (Calculation 9.4). For planetary gears, the sun and ring mesh positions repeat every 360/(N_s + N_r) degrees of carrier rotation. So n equally spaced planets fit only if n divides N_s + N_r. Since N_r = N_s + 2N_p for coaxial gears of one module, three planets fit exactly when N_s + N_p ≡ 0 (mod 3) (Calculation 9.4). Among sun and planet tooth counts from 12 to 60, one third of pairs allow three planets: 0.334, the T6 baseline.
4. *Scale.* The Wankel engine is a ring in two senses. Its rotor has three faces; each face completes the four strokes once per rotor turn; the shaft turns three times per rotor turn. So there is one power stroke per shaft turn (Calculation 9.4). The phasing gear inside it has a ratio of 3 to 2: the rotor's internal gear has one and a half times as many teeth as the fixed gear. Three to two is a ring against a meeting, the first comparison of the two ladders (T17's closure depth 1, the fifth of music). It is fixed by the geometry of a three-flanked rotor in a two-lobed housing. Brocot's method is the machinery of T17. For the ratio of the lunar to the solar semidiurnal tide, M2/S2 = 12.4206/12 = 1.03505, the convergents are 1, 29/28, 30/29, 59/57, 443/428, … (Calculation 9.5). The best fraction with denominator up to 200 is 59/57, error 3.6×10^{-5}. Restricted to numbers with factors 2, 3, 5 and 7, as gear trains prefer, the best is 28/27, error 1.9×10^{-3}. Restricted to the ladder $2^a \cdot 3^b$ alone, nothing better than 1/1 exists in that range.
5. *Test.* The planetary condition is arithmetic, so it fixes the reading. A test that could fail: if exact constraint is the closure of T7, then fixtures that use more than six contacts (overconstrained) must show internal stress that varies with temperature, and fixtures with exactly six must not. That is the engineering principle itself; the theory adds nothing testable to it.

Consequences. Mechanical engineering contains several structures of the Foundations as published practice. Exact constraint counts six contacts for six freedoms, and Kelvin's split 3 + 2 + 1 uses each elementary length once. Three planets fit when a sum vanishes mod 3, a closure in the plain sense of T4, and one third of tooth combinations satisfy it, the baseline of T6. Brocot's gear method is the Stern–

Brocot tree, whose best approximations are the convergents of T17. The theory adds names, and one observation that matters for practice: the ladder alone ($2^a \cdot 3^b$) approximates small ratios poorly, so gear trains need more primes. That is T16 in the workshop: the ladders ×2 and ×3 never meet, and between them there is room only for commas. The identification of Kelvin's 3 + 2 + 1 with ring, meeting and standing is a reading; Maxwell's 3 × 2 is equally exact and uses only meetings. The Wankel engine's three faces are geometry, not closure. Its 3:2 gear and its three-to-one shaft ratio follow from the epitrochoid, and its weaknesses, sealing the rotor tips and burning fuel completely in a long, moving chamber, have nothing to do with the register. Kelvin's and Maxwell's couplings show a design choice that does connect: the Maxwell pattern of three meetings is symmetric and thermally stable; the Kelvin pattern of ring, meeting and standing has a fixed centre of rotation at the tetrahedral seat but concentrates load there. Engineers choose between them by load and temperature, not by any count, since both close exactly. One theorem of the Foundations has a mechanical object that this section does not analyse: T21, the return of an attached strand after 4π and not 2π. A cable or hose fixed at both ends and passed round a rotating part tangles after one turn and can be untangled after two; this is the belt trick, and some rotating connections use it to avoid slip rings. It is a real application of the double cover, but no design data were checked for this chapter. The theorem check: T4 and T6 apply to planetary assembly; T7 to exact constraint as a reading; T16–T17 to gear ratios; T15 fails as a gear rule; T21 (the 4π return) has an object in mechanisms that pass a cable or belt through a rotating joint, Dirac's belt trick, but none is analysed here; T23–T24 have no object.

Grade. A for exact constraint (6 = 3 × 2 = 3 + 2 + 1; Maxwell 1871, Kelvin) and for three-planet assembly as closure mod 3 (derived here; one third of tooth pairs qualify); A for Brocot's gear method as the convergents of T17; C for Kelvin's split as T7 and for the Wankel engine.

9.3 Chemical Engineering

History. Chemical engineering grew from the need to make chemicals at scale. George Davis lectured on it in Manchester in 1887, and MIT began its Course X in 1888. The Leblanc and Solvay soda processes, sulphuric acid and dyestuffs were the first industries. The Haber–Bosch process made ammonia from the air: Haber's laboratory demonstration of 1909, Bosch's plant at Oppau in 1913 (Haber process). Its central device is the recycle: a pass converts only about 15 %, so the unreacted gas is cooled, the ammonia removed, and the rest sent back.

Arthur D. Little named the "unit operations" in 1915: distillation, absorption, filtration, drying and the rest, each studied once and used everywhere. McCabe and Thiele's graphical method for distillation (1925) counted ideal stages on a diagram. Fluid catalytic cracking came in 1942. Danckwerts described residence-time distributions in 1953. Bird, Stewart and Lightfoot's *Transport Phenomena* (1960) unified momentum, heat and mass transfer. Rutherford Aris and Neal Amundson made reactor analysis mathematical in the 1950s and 1960s, studying multiple steady states and their stability. The engineer's question became not only what a reactor produces but

whether it settles, oscillates or runs away. Reactor engineering learned that a stirred tank can have several steady states and can oscillate. In 1982 Simoyi, Wolf and Swinney ran the Belousov–Zhabotinsky reaction in a continuous stirred-tank reactor and found period doubling and periodic windows in the order of one-dimensional maps (Simoyi, Wolf and Swinney 1982). Ternary phase diagrams became standard design tools for extraction and crystallisation. Process control and safety became disciplines after the accidents at Flixborough (1974) and Bhopal (1984). The questions left were energy efficiency, safety, and the replacement of fossil feedstocks and heat. Each was posed as a question about loops: heat recovered and returned, unreacted feed recycled, solvent regenerated and reused.

Current state. Chemical engineering is being electrified. On 17 April 2024 BASF, SABIC and Linde inaugurated the first demonstration plant for electrically heated steam cracking at Ludwigshafen. Its two furnaces process about 4 tonnes of hydrocarbon feed an hour using 6 MW of renewable electricity, and the technology could cut CO_2 emissions by “at least 90 %” against current crackers (BASF). That is 1.5 MWh of electricity per tonne of feed (Calculation 9.6).

Ammonia remains the largest single process. Production was 230 million tonnes in 2018. It consumes 1–2 % of world energy and 3–5 % of natural gas and accounts for about 3 % of carbon emissions. It runs at 15–25 MPa and 450–550 °C, with about 15 % conversion per pass and 97–98 % overall after recycling (Haber process). Twenty-two passes at 15 % reach 97 % (Calculation 9.6). Each tonne of ammonia thus sees its reactor many times; the recycle compressor and the separation of ammonia by cooling are a large part of the plant’s energy use.

Hydrogen is the test of the transition. Global demand was almost 100 million tonnes in 2024. Low-emission hydrogen production grew 10 % in 2024 and is on track for about 1 million tonnes in 2025, under 1 % of the total. Projects operating or past final investment decision could reach 4.2 million tonnes a year by 2030, while the announced pipeline for 2030 fell from 49 to 37 million tonnes a year (IEA Global Hydrogen Review 2025). Electrolysers, the units that would make hydrogen from water, cost USD 2,000–2,600 per kW outside China in 2024 against USD 600–1,200 per kW for systems made and installed in China, and China held 65 % of installed and committed capacity (IEA Global Hydrogen Review 2025). Installed capacity reached 2 GW in 2024, against Chinese manufacturing capacity of about 20 GW a year. The debates are about the cost of electrified heat, the role of carbon capture, and whether green hydrogen can scale before 2030.

Interpretation.

1. *Strand.* I is the path of material through a plant: feed, reactor, separator, recycle. P is the set of units and streams. A recycle is a loop.
2. *Passage and trit.* A molecule passing through a reactor reacts (1), passes unchanged (0), or reverts (T) where the reaction is reversible. Each pass round the recycle is a return to the reactor inlet.
3. *Closure.* A process closes when its mass balance closes: what enters equals what leaves plus what accumulates. That is conservation, not the closure rule of N2. The

recycle makes a low-conversion reaction close at the scale of the plant: 15 % per pass becomes 97 % in 22 passes. In the language of N5 read as T7.1, what does not close in one pass is carried to the next. A ternary phase diagram is a closed simplex: three mole fractions summing to one, two free. Its three binaries are its edges. That is a ring of three components in a plain geometric sense. The recycle gives T24 an object outside physics. A molecule that does not react returns to the reactor with probability P, the recycled share. Its number of passes, the first included, is geometric if each return is independent of the last, and its lever is then $-(1 + P)/P$. For Haber–Bosch, with $P = 0.85$, the mean is 6.67 passes and the lever -2.176; a simulation of 200,000 molecules gives -2.171 (Calculation 9.9). A plant with a dead zone, where some molecules return more often than others, has a different lever: a mixture of 20 % returning with $P = 0.95$ and 80 % with $P = 0.80$ gives -2.87, against -2.14 for a memoryless plant with the same mean.

4. *Scale.* A stirred-tank reactor carrying an oscillating reaction can double its period: 1, 2, 4, 8, the ×2 ladder; its periodic windows include period 3, at $r = 1 + \sqrt{8}$ in the logistic map (Chapter 7, Calculation 7.10). The same windows include periods 5 and 7, off the ladder.
5. *Test.* As in Chapter 7: a reactor in which period 3 appeared outside the window order of one-dimensional maps would break the reading of reactor dynamics as the ladder of the logistic family. Stated before data: in any continuous stirred-tank reactor showing period doubling, the periodic windows appear in the order of the logistic family (in the logistic map, period 6 at $r = 3.6275$, then 7, 5 and 3 at $r = 3.8285$; Calculation 9.8), within experimental resolution. A different order refutes the reading for that reactor. A second test uses the lever. Stated before data: in a recycle process traced by a pulse of tracer, the distribution of the number of passes through the reactor has the lever $-(1 + P)/P$, with P taken from the mean. A lever more negative than that by more than its error indicates a dead zone or bypass, and refutes the memoryless reading of the loop. The test is a diagnostic: it uses the third cumulant, which residence-time methods do not usually report.

Consequences. Chemical engineering has one structure that the Foundations share: the dynamics of stirred reactors follow the universal sequence of one-dimensional maps, including the period-doubling ladder. That sequence belongs to dynamical systems, not to this theory, and the theory shares it rather than explaining it. The recycle as carried remainder and the ternary diagram as a ring are readings. The electrification of crackers and the scale-up of hydrogen are questions of energy prices and policy, and the theory adds nothing to them. The lever test is the one place where the theory gives chemical engineering a new number to compute. It is cheap: a tracer experiment already gives the distribution, and the third cumulant is one more line of arithmetic. Whether it adds anything over standard residence-time analysis, which fits models of tanks in series and dispersion, is an open question; it may only restate their skewness in other units. Process chemistry also inherits the Kagan and Soai effects of Chapter 7. A pharmaceutical process that uses a chiral catalyst of modest purity can still make a product of high purity if a non-linear effect works in its favour. If the ML_3 prediction of Chapter 7 holds, it is a design hint for such processes. If it fails, nothing here depends on it. Beyond that, the honest conclusion is a short one: chemical

engineering is not a field where the Foundations add structure. The theorem check: T15 applies to period doubling; T7.1 to the recycle as a reading; T4 and T6 have no natural window; T11–T14, T16–T24 have no object.

Grade. B for the window order of reactor dynamics (Simoyi, Wolf and Swinney 1982), shared with dynamical systems, and for the lever of recycle pass counts as a test of memorylessness (T24); C for the recycle as carry and for the ternary diagram.

9.4 Materials Engineering

History. Materials engineering turned materials science into products. Henry Bessemer's converter (1856) and the Siemens–Martin open hearth made steel cheap. Alfred Wilm discovered age hardening in aluminium alloys in 1906, which made Duralumin and the all-metal aircraft possible. Harry Brearley's stainless steel (1913) resisted corrosion. A.A. Griffith explained in 1921 why brittle materials break far below their ideal strength: cracks concentrate stress. Hall and Petch showed in the early 1950s that smaller grains make stronger metals.

The second half of the century added new classes. Polymers became engineering materials; composites of glass or carbon fibre followed. Nickel superalloys made jet engines possible. Cellular materials got their theory from Lorna Gibson and Michael Ashby: the stiffness of a foam scales with its relative density to the first power if its cell walls stretch, and to the second power if they bend (*Cellular Solids*, 1988, 2nd ed. 1997). Deshpande, Ashby and Fleck tied stretching to Maxwell's count in 2001: a lattice whose joints have enough struts to be rigid deforms by stretching, and is far stiffer at low density than one that bends.

Energy materials became a central concern. The lithium-ion battery (Whittingham, Goodenough, Yoshino) won the 2019 Nobel Prize (Nobel Prize 2019). Silicon photovoltaics went from satellites to rooftops. Grain-boundary engineering raised the share of low-energy boundaries in metals by repeated straining and annealing, chiefly by multiplying twin ($\Sigma 3$) boundaries. Victor Goldschmidt gave in 1926 a tolerance factor that predicts from ionic radii whether a compound ABX_3 takes the perovskite structure; it is still the first screen for perovskite photovoltaic materials. The questions left were cost, durability, and the supply of critical elements.

Current state. Battery materials are in a price war. BloombergNEF's 2025 survey put the average pack at $108/kWh, LFP at $81/kWh, NMC at $128/kWh and stationary storage at $70/kWh (BloombergNEF). The cheaper iron-phosphate chemistry is displacing nickel-rich cathodes for many uses. Sodium-ion cells entered production in 2025, at 175 Wh/kg for CATL's Naxtra against 185 Wh/kg for LFP, still about 30 % more expensive per pack (Sodium-ion battery).

Perovskite photovoltaics are the main new material in solar energy. Single-junction perovskite cells reached 27 % in 2025, and perovskite–silicon tandems 34.85 % in the NREL chart cited by the source. Stability remains the obstacle: moisture sensitivity, phase changes in $CsPbI_3$, lead toxicity, and studies mostly limited to under 1,000 hours (Perovskite solar cell).

Grain-boundary design continues. Coherent twin boundaries (Σ3) are the main "special" boundaries with low energy, but the source warns that no simple geometric criterion reliably predicts low energy (Grain boundary). Additive manufacturing now makes the lattice materials that Gibson, Ashby and Deshpande described, at scales from micrometres to metres. Discovery is being automated as well. Google DeepMind's GNoME predicted 2.2 million new crystals in November 2023, 380,000 of them reported as the most stable, and a robotic laboratory at Berkeley reported more than 41 new materials made from the predictions (Google DeepMind). How many of these reach engineering use, and how many are truly new, is debated. The scale of demand sets the pace. Electric car sales topped 17 million in 2024, more than 20 % of all cars sold, and battery pack prices fell about 30 % in China that year against 10–15 % in Europe and the United States (IEA). An IRENA report of 2025 suggests sodium-ion cells could reach $40/kWh against $70/kWh for LFP (Sodium-ion battery). The debates are about solid-state batteries, the lifetime of perovskite modules, and whether AI-designed materials reach production.

Interpretation.

1. *Strand.* I is a path through a material: across grains, along struts of a lattice, through the layers of a cell. P is the set of grains, nodes or layers.
2. *Passage and trit.* Crossing a grain boundary is a step whose type is its Σ value. Along a strut of a lattice, the strut is in tension (1), compression (T) or unloaded (0).
3. *Closure.* Lattice materials obey Maxwell's count. In the plane, a large frame needs an average of 4 struts per joint to be rigid; in space, 6 (Calculation 9.1). Lattices above the count stretch; below it, they bend. The triangulated plane lattice has 6 struts per joint and the three-dimensional octet truss 12; both are on the ladder, both well above the minimum. Gibson and Ashby's exponents, stiffness proportional to density for stretching and to its square for bending, are the field's own measure of what closes and what does not. A twin boundary is Σ3; two twins about different axes give Σ9 and three give Σ27; two twins about the same axis cancel to Σ1 (Chapter 6, Calculation 6.9). The twin is a meeting under repetition and a ring ladder across axes. In the language of T7.1, a twin operation about one axis done twice closes with carry 0 (a meeting); twins about three different axes compose to Σ27, a ring whose result is not undone at its own scale but carried into a new boundary type.
4. *Scale.* The $\Sigma 3^n$ series is T15's ring ladder, 3^n. Grain-boundary engineering raises the share of Σ3 and its products; a junction of three boundaries in such a network combines like Σ3, Σ3, Σ9. Stiffness as a function of relative density is a state function, T23's form again (Chapter 6, Section 6.5). The Gibson–Ashby exponents, 1 for stretching and 2 for bending, differ by a factor of two, and the strength of bending-dominated foams scales with an exponent of 3/2. These are consequences of beam mechanics, not of the ladder; that they are small ratios of 2 and 3 is what beams do. For perovskites, the formula ABX_3 has three sites, but the count of three X per B comes from octahedra sharing corners: each octahedron has six X, each X shared by two B. That is 6/2 = 3, a meeting dividing a ring of the ladder; a reading.

5. *Test.* Deshpande, Ashby and Fleck predicted the stretch–bend split from Maxwell's count, and experiments confirmed it; that fixes the reading. A test the theory adds: in grain-boundary-engineered metals, the share of triple junctions of type (Σ3, Σ3, Σ9) relative to (Σ3, Σ9, Σ27) should follow from the share of Σ3 boundaries alone if boundaries combine as independent ring steps. Stated before data: with a Σ3 share s and random combination, the ratio of the two junction types depends only on s. A measured ratio that departs from that prediction by more than its error means the boundaries do not combine as independent steps.

Consequences. Materials engineering contains two structures of the Foundations as published fact. Lattice materials are designed with Maxwell's count, and the stretch–bend split is its measured consequence. Twin-related boundaries form the series $\Sigma 3^n$, the ring ladder. Both were known without the theory. The theory adds one test for boundary networks, which is close to what CSL theory already gives. Grain-boundary engineering is the practical side of the $\Sigma 3^n$ ladder. Its aim, a network in which crack paths along random boundaries are interrupted by twins, is a percolation question: the share of special boundaries must exceed a threshold before random boundaries stop forming a connected path. The theory has no threshold to offer. Rigidity percolation, which Section 9.1 cites at 0.6602 for bonds on the triangular lattice, is the analogue for frames. For batteries and photovoltaics the theory adds nothing: their progress is chemistry, cost and durability. This is worth stating because both are often described in the language of cycles and states that the theory also uses. A battery's charge–discharge cycle is a meeting in the plain sense, and its capacity fade is what does not come back. That language does not predict a cycle life. The decline of battery prices by 8 % in 2025 and the rise of sodium-ion cells are economic and chemical facts. The LiC_6 anode is the one battery fact with structure (Chapter 7, Section 7.3), and it belongs to electrochemistry. The lattice-material result is the strongest here because it is designed, not found. An engineer who builds an octet truss with 12 struts per joint is choosing a closed frame on purpose, and the stiffness it gains at low density is the measured price of closure. A bending-dominated foam is an open frame, and it pays in stiffness: at a relative density of 1 %, stiffness falls to roughly 1 % of the solid if the frame stretches, but to roughly 0.01 % if it bends, by the Gibson–Ashby exponents. The theorem check: T15 applies to $\Sigma 3^n$ and to the connectivities 4, 6 and 12; T7 to twins as meetings; T23 to density-dependent stiffness; T14 to hexagonal hosts (Chapter 7); T11–T13, T16–T22, T24 have no object.

Grade. A for Maxwell-count lattices (Deshpande, Ashby and Fleck 2001) and for the $\Sigma 3^n$ series; B for junction statistics in engineered boundary networks; C for batteries and photovoltaics.

9.5 Environmental Engineering

History. Environmental engineering began with water and disease. John Snow traced a cholera outbreak in London to the Broad Street pump in 1854. Sand filtration and, from 1908, chlorination made piped water safe. In 1913 Edward Ardern and W.T. Lockett, at Davyhulme Sewage Works in Manchester, found that aerating sewage with returned sludge purified it fast; they published in 1914 and called it activated sludge. The first

full-scale continuous-flow plant was built at Worcester about two years later (Activated sludge).

Nitrogen became the next problem. Nitrification oxidises ammonium to nitrite and then nitrate, using oxygen; denitrification reduces nitrate to nitrogen gas without it. The two have conflicting needs, so plants alternate aerobic and anoxic zones. In 1995 Mulder and colleagues in Delft reported anaerobic ammonium oxidation, anammox: bacteria that combine ammonium and nitrite directly into nitrogen gas. It now runs at full scale.

London's sewers, built by Joseph Bazalgette after the "Great Stink" of 1858, had earlier shown that engineering could cut waterborne disease at city scale. Rachel Carson's *Silent Spring* (1962) made chemical pollution a public cause. Regulation shaped the field after 1970. The US Safe Drinking Water Act of 1974 gave the EPA power to set limits for contaminants in public water, the power it used for PFAS fifty years later. The US Environmental Protection Agency was founded in 1970 and the Clean Water Act passed in 1972. The Montreal Protocol of 1987 phased out ozone-depleting substances. Air-pollution control, landfill engineering and remediation of contaminated land followed. The Clean Air Act amendments of 1970 set national air quality standards, and catalytic converters followed in 1975. Landfills became engineered cells with liners and gas collection. Per- and polyfluoroalkyl substances (PFAS), whose carbon–fluorine bonds resist almost every natural process, became the defining contaminant of the 2020s. The questions left were energy use in treatment, persistent chemicals, and resilience to drought and flood.

Current state. On 18 May 2026 the US EPA proposed two rules for drinking water. One keeps the limits for PFOA and PFOS at 4.0 parts per trillion each, with maximum contaminant level goals of zero, and would let water systems request two more years, to 2031, to comply. The other would rescind the limits for PFHxS, PFNA and HFPO-DA (10 ppt each) and the hazard index for mixtures. Under the 2024 rule, treatment had been due by 2029 (US EPA). A limit of 4.0 ng/L is one part in 2.5×10^{11} by mass (Calculation 9.7).

Wastewater treatment is an energy problem. Aeration is the largest operating cost of activated sludge, put at about 90 % of total treatment cost in the source. Waste activated sludge is typically about 80 g per cubic metre of wastewater treated, with primary sludge adding 110–170 g/m³. Aerobic granular sludge (Nereda) cut the sludge volume index from 200–300 to about 40 mL/g; by 2015 about 30 Nereda plants were operating, under construction or in design (Activated sludge).

Water resources are under strain. The WMO found that only one third of river basins had normal conditions in 2024, and all glacier regions reported losses for the third year running (WMO). Carbon capture at cement plants began at industrial scale in 2025 (Section 9.1). The ozone layer shows that a treaty can close a loop: under the Montreal Protocol of 1987 the production of the main ozone-depleting substances was phased out, and their atmospheric concentrations have been falling since. PFAS show the opposite case. They are made in large volumes and almost never break down; regulation can lower their concentration in drinking water to a few parts per trillion, but treatment mostly moves them from water to activated carbon, resin or sludge

rather than destroying them. The debates are about PFAS destruction rather than transfer, about energy-positive wastewater plants, and about the cost of the 2026 PFAS rollback for public health.

Interpretation.

1. *Strand.* I is the path of a substance through a treatment train or an environmental cycle. P is the set of chemical forms and compartments: water, air, soil, sludge, organisms. A loop is a substance returning to a form it has had, as nitrogen returns to N_2 or water to the sea.
2. *Passage and trit.* For nitrogen, the strand runs through oxidation states: ammonium −3, nitrogen gas 0, nitrite +3, nitrate +5. A step oxidises (1), reduces (T) or transfers without change (0). A treatment train applies steps in sequence: first oxidation of ammonium in aerated tanks, then reduction of nitrate in anoxic tanks. Read as trits, the conventional route is (1, 1, T): two oxidations and a reduction, two equal trits and one different, an open window by T4. Anammox is one step that joins the two ends.
3. *Closure.* Anammox combines ammonium (−3) and nitrite (+3) into nitrogen gas (0): (−3) + (+3) = 0, a meeting of opposite states, closed with nothing left over (Calculation 9.7). In balanced ternary −3 and +3 are T0 and 10: each is closed at its own scale and carries ∓1 to the next, and the two carries meet. That is T7.1 read in redox. Conventional nitrification and denitrification take the long way: 6 electrons from ammonium to nitrite, 2 more to nitrate, then 5 back to nitrogen. 5 is off the ladder. The whole nitrogen cycle is a loop: fixation brings N_2 (0) to ammonium (−3), nitrification carries it to nitrate (+5), denitrification returns it to N_2. Anammox is a chord across the loop: it closes a short circuit between two points of the cycle without passing through nitrate. The plant engineer's gain is the shortcut: no aeration to make nitrate, no organic carbon to reduce it again.
4. *Scale.* Treatment trains are staged: primary, secondary, tertiary. The count is a design convention. The scales of the problem are wide: a PFAS limit of 4.0 ng/L is one part in 2.5×10^{11}, while activated sludge plants produce about 80 g of waste sludge per cubic metre treated, a ratio of more than ten orders of magnitude between what is regulated and what is handled. Mass balance closes by conservation, not by the closure rule. PFAS are a case of a cycle that does not close: the compounds are made, used and released, and almost nothing in nature breaks them down. In the language of N2 they are pure tension: they pass on from water to soil to bodies without returning to a closed form. Destruction technologies, by heat, plasma or electrochemistry, try to force the closure that nature lacks.
5. *Test.* No test stated. The anammox reading names a fact of redox chemistry; it predicts no rate, yield or design. A test would need the theory to say which microbial processes are favoured, and the Foundations contain no rule about free energy or growth. The theorems that might seem to apply, T6's closure probability and T24's lever, need a random process with a stated law. Treatment plants are designed, not random; but their residence statistics are, and the lever test of Section 9.3 applies here too: a tracer test in an aeration tank gives a pass-count lever that measures short-circuiting.

Consequences. Environmental engineering is where the Foundations add least. The anammox reaction is a meeting of opposite oxidation states, and it is a better process than the long route because it skips the oxygen and carbon that nitrification and denitrification need. The theory names the shortcut; the microbiology found it. The PFAS limits, the energy cost of aeration and the state of the world's rivers are measured facts with no structure of the theory in them. The only number in this section that the theory touches is the redox balance of anammox, and that is chemistry. A test could be built on it only by claiming that comproportionation reactions, which join opposite oxidation states, are generally favoured by microbes over the long route. That claim is too vague to fail and is not made here. There is also a practical reading that is worth no more than it claims. Where a cycle does not close, as with PFAS, engineering must close it by force, and the cost of closing it is the cost of destruction. Where a cycle has a shortcut, as with anammox, engineering can save the energy of the long way. Both statements are true and both are known to environmental engineers without the theory. Saying so is the useful result: a field can be important without containing the register's structure, and forcing a reading on it would be the re-description the rules forbid. The theorem check: T7 and T7.1 apply to anammox as a reading; T15 fails for electron counts; all other theorems have no object here.

Grade. C. Anammox as a meeting of oxidation states, and PFAS as an open cycle, are readings; no test is stated.

Part IV — Life

Chapter 10 — The Code and the Cell

Life is the level where the window of three stops being a reading and becomes a published fact. The genetic code is read in non-overlapping triplets, frameshifts cancel exactly when their sum is 0 mod 3, and the largest comma-free code of triplets has 20 words, which is the number of aperiodic rotation classes of codons (T13, Calculation 10.1). Virus shells are built on the Eisenstein lattice, and their triangulation numbers are exactly the norms of Z[ω]. Molecular biology, genetics and genomics, structural biology and virology therefore earn grade A for specific, named structures; cell biology earns A for the ring–switch split of gene circuits and for the published parity rule of microtubule seams, and C for reading that seam as T11. Several applications of the theorem list fail and are reported as failures: the T14 rest/turn split is not a property of a capsid (Calculation 10.9), and most helix and lattice counts lie off the ladder 2^a·3^b (Calculation 10.7).

10.1 Molecular biology

History. Molecular biology began as a question about information. In 1944 Oswald Avery, Colin MacLeod and Maclyn McCarty showed that the transforming principle of pneumococcus was DNA. In 1953 James Watson and Francis Crick proposed the double helix in *Nature*, using X-ray data from Rosalind Franklin and Maurice Wilkins. The structure suggested at once how DNA copies itself, but not how it specifies a protein. Four bases had to specify twenty amino acids. Two bases give only $4^2 = 16$ words, too few; three give $4^3 = 64$, enough (Calculation 10.1). George Gamow proposed overlapping triplet codes in 1954. Sequence data on proteins soon ruled most overlapping codes out.

In 1957 Crick, John Griffith and Leslie Orgel asked a sharper question. If a cell reads a string of bases with no punctuation, how does it find the frame? They proposed a “code without commas”: a set of triplets such that no word can be read out of phase across the junction of two code words. They proved that such a code has at most 20 words, and that a code of 20 exists. The coincidence with twenty amino acids made the idea famous. It was wrong as biology: Marshall Nirenberg and Heinrich Matthaei showed in 1961 that poly-U codes for phenylalanine, and UUU cannot belong to a comma-free code. But the bound of 20 is still correct mathematics, and it returns below.

The decisive experiment came the same year. Crick, Leslie Barnett, Sydney Brenner and Richard Watts-Tobin used proflavin to insert or delete single bases in the rIIB gene of phage T4. A single insertion or deletion destroyed the gene. An insertion combined with a nearby deletion restored it. Three insertions together, or three deletions together, also restored it. Their paper in *Nature* (vol. 192, 1961) concluded that the code is read in triplets from a fixed starting point. The National Library of Medicine’s Crick papers page summarises the finding: the frame returned only after “a sequence of three such additions or deletions”. By 1966 the whole table of 64 codons was read, by Nirenberg, Har Gobind Khorana and others.

The decades that followed turned the code into a mechanism. Messenger RNA (1961), the ribosome, transfer RNA, the operon of François Jacob and Jacques Monod (1961), restriction enzymes, recombinant DNA (1973), and Sanger sequencing (1977) built the field. Two later discoveries matter for this chapter. Reading frames can be changed on purpose: programmed ribosomal frameshifting, found in retroviruses in the 1980s, lets one RNA encode two proteins. And gene regulation can be wired into circuits. In January 2000 the same issue of *Nature* (vol. 403) carried two synthetic circuits: the repressilator of Michael Elowitz and Stanislas Leibler, a ring of three repressors, and the toggle switch of Timothy Gardner, Charles Cantor and James Collins, a pair of mutually repressing genes. The questions left open are how far the triplet logic extends beyond coding, and which circuit topologies cells use and why.

Current state. Molecular biology in October 2026 is dominated by sequence models and genome writing. Evo 2, a DNA language model from the Arc Institute, appeared in *Nature* on 4 March 2026 after a preprint in February 2025 (Arc Institute). It was trained on more than 9.3 trillion nucleotides from over 128,000 whole genomes and reads up to one million nucleotides at once; it classifies BRCA1 variants as benign or pathogenic with over 90% accuracy (Astrobiology, secondary report). After fine-tuning on phage genomes related to ΦX174, 16 of 285 designed genomes produced viable phages (Arc Institute).

Genome writing has reached the code itself. In 2025 Jason Chin’s group at the MRC Laboratory of Molecular Biology reported Syn57, an *E. coli* whose 4 Mb genome was rebuilt from 38 fragments with more than 101,000 codons replaced by synonyms. Seven codons were removed: four of the six serine codons, two of the four alanine codons and one stop codon (MRC LMB; *Science* 390, eady4368). The organism runs on 57 codons. Freed codons can be reassigned to non-standard amino acids.

Regulation by small RNAs was recognised by the 2024 Nobel Prize in Physiology or Medicine to Victor Ambros and Gary Ruvkun, “for the discovery of microRNA and its role in post-transcriptional gene regulation” (Nobel Prize). The 2024 chemistry prize went to David Baker for computational protein design and to Demis Hassabis and John Jumper for protein structure prediction (Nobel Prize). On the circuit side, the repressilator of 2000 oscillated with a period of about 150 minutes, roughly three times the cell-division time of 50 to 70 minutes, and at least 40% of cells oscillated in each recording (Elowitz and Leibler 2000, PDF). Natural clocks contain the same motif: the model of the *Arabidopsis* clock by Pokhilko and colleagues includes “a three-component repressilator circuit” (Warwick repository).

Open debates include how much of the genome is functional, how far language models generalise beyond their training data, and whether recoded organisms can be kept contained.

Interpretation.

1. *Strand.* I is the position along an RNA or DNA molecule, read 5′ to 3′. P is the set of codon positions; σ assigns each base to its place in the reading frame. The ribosome is the passage.

2. *Passage and trit.* A frameshift adds a base (+1) or removes one (−1); no shift is 0. By T1 the sum of the shifts is a digit and a carry. The digit is the residue the frame keeps; the carry is the number of whole codons added or removed (Calculation 10.2).
3. *Closure.* The frame closes when the digit is 0. Crick's results are exactly the reduction of T7: a single shift is open; a pair (+1, −1) closes as a meeting with carry 0; three equal shifts close as a ring with carry ±1, that is, one whole codon added or removed (T7.1). Of 2, 4, 8 and 16 combinations of one to four shifts, 0, 2, 2 and 6 restore the frame (Calculation 10.2). The comma-free bound is T13 at alphabet size four: the 64 codons fall into $g_4(3) = 24$ rotation classes, 4 constant and 20 aperiodic, and a comma-free code can take at most one word from each aperiodic class and none from a constant class. Crick, Griffith and Orgel's 20 is $(64 - 4)/3$ (Calculation 10.1). The theory supplies the count of closed configurations read without a start; the biology uses exactly that count.
4. *Scale.* A ring of three shifts becomes one codon of the next scale, as T7.1 states: three bases become one amino acid. The rung above is the circuit. A ring of three repressors oscillates; rings of 2, 4, 6 and 8 settle into alternating states; rings of 5 and 7 also oscillate (Calculation 10.6). So the published rule is parity: odd rings oscillate, even rings switch. A meeting (two repressors) is a switch, the toggle of Gardner and colleagues; a ring of three is the least oscillator.
5. *Test.* T7 says every restoring combination of frameshifts reduces to meetings and rings. Stated before data: a reading frame restored by a net shift that is not 0 mod 3, in any organism using the standard triplet ribosome, refutes the interpretation. A second test: in a library of synthetic repressor rings built in one host, an even ring that oscillates stably, without an added activator or delay element, would break the meeting–ring split.

Theorems checked and not used: T8 and T9 (words and rings that close in every window) would require every window of three bases to close, which has no meaning for a four-letter alphabet; they apply only to the frame digit, which is trivially periodic. T12 (unique address) adds nothing to the codon table. T14–T18 have no counterpart in coding. T22 (an integer charge that changes only by cutting) has a counterpart in DNA topology, treated in Section 10.4. T23 and T24 need an energy density or a free path that molecular biology does not supply.

Consequences. If the interpretation holds, molecular biology already uses three results of the Foundations without naming them. The frame rule is T1's residue. Frameshift suppression is T7's reduction, and T7.1 adds a bookkeeping the field did not state: the carry counts the whole codons gained or lost, which is the length change of the protein. The comma-free bound is T13's count of aperiodic necklaces. None of this changes a laboratory protocol. What it adds is a reason why the same numbers recur. The bound 20 is not a coincidence with the amino acids; it is the number of closed configurations of three letters from four that have no rotational symmetry. The circuit result adds a design rule that is already known, odd rings oscillate, and gives it a place: a meeting closes with carry 0 and stands still as a switch; a ring closes with carry and moves. The cost is that the theory must be read at the right scale. Inside a codon the

three bases are not trits, and nothing in the theory says why the base alphabet has four letters. The theory adds nothing to the chemistry of base pairing.

Two current developments fit the reading without forcing it. Syn57 shows that the code's redundancy can be removed codon by codon while the frame arithmetic stays untouched: 101,000 synonymous swaps change no digit, because each swap replaces a word of length three by another of length three. And programmed frameshifting shows that cells and viruses use an open digit on purpose, to make a second protein from one message. In the theory's terms such a shift is a deliberate open window, kept by the structure of the RNA rather than removed by selection.

Grade. A for the reading frame, frameshift suppression (Crick et al. 1961) and the comma-free bound 20 = (64 − 4)/3 (Crick, Griffith and Orgel 1957); A for the ring–switch split of repressor circuits (Elowitz and Leibler 2000; Gardner et al. 2000); B for the stated frameshift test.

10.2 Genetics and genomics

History. Gregor Mendel read "Versuche über Pflanzenhybriden" to the Natural History Society of Brno on 8 February and 8 March 1865; it was printed in 1866. His pea crosses gave ratios near 3 to 1. In his first experiment he counted 5,474 round and 1,850 wrinkled seeds; in the second, 6,022 yellow and 2,001 green. The ratios are 2.959 and 3.009, and both fit 3:1 well ($\chi^2 = 0.26$ and 0.015; Calculation 10.4). Mendel also wrote the counting rule for many characters. In the English translation at the Electronic Scholarly Publishing archive: "If n represent the number of the differentiating characters in the two original stocks, 3n gives the number of terms of the combination series, 4n the number of individuals which belong to the series, and 2n the number of unions which remain constant." The exponents are 3^n, 4^n and 2^n. Three genotype classes per locus is Mendel's own arithmetic.

The paper was ignored until 1900, when Hugo de Vries, Carl Correns and Erich von Tschermak rediscovered it. William Bateson named genetics in 1905. In 1908 G. H. Hardy (*Science* 28) and Wilhelm Weinberg independently showed that random mating keeps genotype frequencies at p^2, 2pq and q^2. Thomas Hunt Morgan's fly room linked genes to chromosomes from 1910, and Alfred Sturtevant drew the first linkage map in 1913. Ronald Fisher, J. B. S. Haldane and Sewall Wright built population genetics between 1918 and 1932. Fisher's 1918 paper reconciled Mendelian loci with continuous variation by summing many small effects; the textbook form, due to Douglas Falconer, gives the three genotypes at a locus the values +a, d and −a, measured from the midpoint of the two homozygotes.

Genomics began with Sanger's sequencing of ΦX174 in 1977 and became industrial with the Human Genome Project, whose draft appeared in 2001 and whose finished sequence followed in 2003. Two observations from the early genome era matter here. James Fickett (1982) and later S. Tiwari and colleagues (1997) found that coding DNA carries a strong signal at period three. Tiwari's group called the peak at frequency 1/3 "the major signal in coding regions" and used it to find genes without training data (IAS repository). And in 1988 Anthony Monaco and colleagues explained the two forms

of dystrophin deficiency: deletions that keep the reading frame give the milder Becker muscular dystrophy, deletions that shift it give Duchenne. Genomics left two open questions: how variation maps to traits, and how much of the non-coding genome carries function.

Current state. The scale of human genomics is now set by biobanks and pangenomes. UK Biobank has whole genomes for 490,640 participants, sequenced at an average depth of 32.5×, with about 1.5 billion variants identified (Inside Precision Medicine); the main analysis was published in *Nature* on 18 September 2025 (Illumina). The Human Pangenome Reference Consortium released its second data set on 12 May 2025, with genomes from 232 individuals (HPRC). The microbial side has grown faster still: the Genome Taxonomy Database release 226 classifies 732,475 genomes into 143,614 species clusters (GTDB).

The reading-frame rule remains the backbone of clinical interpretation. For Duchenne and Becker muscular dystrophy, exceptions to the rule occur in about 10% of patients (Ikeda et al. 2019, *Human Genome Variation*, citing Aartsma-Rus et al. 2006). The exceptions have known causes: very large in-frame deletions that remove essential domains, and alternative translation starts near the 5′ end. In population data, short coding indels show the frame rule as a statistical fact. Lin and colleagues (2017) found 3,362 non-frameshift and 3,775 frameshift indels among 7,137 unique coding indels in human genomes (*Scientific Reports* 7, 9313). That is 47.1% in frame, against 33.3% if indel length were blind to the frame; the excess is far beyond chance (binomial $p \approx 10^{-127}$; Calculation 10.5).

Sequence models now enter variant interpretation directly. Evo 2's accuracy of over 90% on BRCA1 variants (Astrobiology, secondary report) comes from a model trained without labels, on raw sequence; whatever it learned about frames, it learned from the statistics of genomes. The same is true of the period-3 signal: it can be found with a Fourier transform and no annotation at all. Repair of the frame is now a therapy. Antisense oligonucleotides that make the splicing machinery skip one more exon turn an out-of-frame Duchenne deletion into an in-frame Becker-like one; the exceptions to the frame rule tell clinicians where this will not work.

The debates of 2026 concern fine-mapping of non-coding variants, the clinical use of polygenic scores, and the transfer of results across ancestries, which the pangenome is meant to address.

Interpretation.

1. *Strand.* For genomics, I is the position along a chromosome and P the three codon positions. For classical genetics, I is the sequence of loci and P the genotype states at each locus.
2. *Passage and trit.* At a diploid locus the additive score is a trit: AA = +1, Aa = 0, aa = −1, which is Falconer's +a, d, −a with d = 0 for a purely additive locus. Mirroring the trit (T12.1) swaps the two homozygotes. In coding DNA, the position mod 3 is a trit fixed by the first codon, as the trit rule fixes direction by the first passage.
3. *Closure.* An indel closes when its length is 0 mod 3; it leaves the frame open otherwise. T6 gives the baseline for blind lengths: with a uniform law, one third

close. The observed 47.1% is the signature of selection for closure (Calculation 10.5). The period-3 signal is closure measured directly: in the *Arabidopsis* chloroplast genome, the spectral peak at 1/3 has a median height of 7.26 times the mean in 45 coding genes and 0.85 in 52 non-coding stretches; 73% of coding genes exceed 4, and none of the non-coding stretches do (Calculation 10.3). The Duchenne–Becker rule is T1 in a clinic: digit 0 gives the milder disease.

4. *Scale.* Mendel's 3^n is the count of open words of length n (T13), with the trit as the genotype at each locus. His 2^n "constant unions" are the homozygotes, words without the digit 0. His 4^n counts ordered gamete pairs, the meetings that make each individual. The F2 law at one locus is (Aa, AA, aa) = (1/2, 1/4, 1/4); by T6 three unlinked loci then have additive score 0 mod 3 with probability 0.34375 (Calculation 10.4). That number has no known biological meaning, and the interpretation does not claim one.
5. *Test.* Stated before data: across species with standard codes, the in-frame share of coding indels must exceed the T6 baseline of 1/3 wherever purifying selection acts, and the excess must vanish in pseudogenes and non-coding DNA. A data set in which coding indels show no excess over 1/3, or non-coding indels show one, refutes the closure reading of the frame rule.

Theorems checked: T1 and T2 apply to the frame (above). T3–T5 apply to the genotype trit only as counting. T13 applies twice: Mendel's 3^n is the count of open words, and the circular genomes of organelles and phages are closed configurations read without a start; their lengths are not constrained mod 3 (154,478 ≡ 2 and 5,386 ≡ 1; Calculation 10.11), so closure is a property of reading frames, not of whole genomes. T15 does not apply: genome sizes and chromosome numbers are not formed by equal merges. T24 does not apply to genetic drift in one dimension: a one-dimensional walk returns with probability 1 (Pólya 1921), so the lever $-(1 + P)/P$ is not defined there.

Consequences. Genetics gains a vocabulary for something it already counts. Mendel's 3^n, the three genotype values of quantitative genetics and the reading-frame rule all use three values, but for different reasons. The genotype trit comes from diploidy plus two alleles; the frame trit comes from triplet reading. The theory unites them only as digits. It does not explain why loci are diploid, and closure mod 3 has no meaning for a sum of genotype scores. The frame rule is a different matter. There the theory's closure is the biology's survival: what keeps digit 0 is retained by selection, what does not is removed or causes disease. The period-3 test and the indel test are cheap. They could be run on every new genome as a check that annotation and selection behave as closure predicts. The cost is honesty about the limit: about 10% of dystrophin cases break the rule, and their causes lie at the scale of protein domains, not codons.

The theory does suggest one shift of emphasis. Clinical genetics treats the frame rule as a rule about one gene. The closure reading treats it as a rule about every coding sequence, with the T6 baseline as the null model. That turns the frame rule into a measurable quantity, the excess of in-frame indels over one third, which can be compared between genes, tissues and species. Genes under weak constraint should sit close to 1/3; essential genes far above it. That is not new biology, but it is a clean statistic that the theory supplies and that the field does not routinely report.

Grade. A for the reading-frame rule (Monaco et al. 1988), the period-3 signal of coding DNA (Fickett 1982; Tiwari et al. 1997) and Mendel's 3^n genotype classes (Mendel 1866); B for the indel test; C for any closure reading of genotype sums.

10.3 Cell biology

History. Robert Hooke named the cell in *Micrographia* (1665). Matthias Schleiden (1838) and Theodor Schwann (1839) proposed that all plants and animals are made of cells, and Rudolf Virchow added in 1855 that every cell comes from a cell. Walther Flemming described chromosomes and mitosis around 1880. Edouard van Beneden saw centrosomes in 1876, and Theodor Boveri named the centriole in 1895. The electron microscope opened the interior after 1945: Keith Porter, George Palade and Albert Claude mapped the endoplasmic reticulum, ribosomes and mitochondria, and Palade's pulse-chase work traced secretion.

The cytoskeleton came into view in the 1960s and 1970s. Microtubules are hollow tubes built from α/β-tubulin dimers. Most have 13 protofilaments. In 1986 Eva-Maria Mandelkow and colleagues showed that the tubulin monomers lie on a left-handed 3-start helix whether a microtubule has 13 or 14 protofilaments, and that the dimers pack in a "B-lattice", which implies at least one helical discontinuity (*J. Cell Biol.* 102, 1067–1073). That discontinuity is the seam. In 1984 Tim Mitchison and Marc Kirschner discovered dynamic instability: microtubules switch between growth and rapid shrinkage. Cilia and flagella were shown to have nine doublet microtubules around a central pair, the 9 + 2 axoneme, and the basal body at their root was shown to have nine triplets.

The geometry of the microtubule raised a puzzle that took two decades to settle. A tube of 13 protofilaments whose dimers are 8 nm long, with neighbouring protofilaments offset by a fraction of a monomer, cannot close as a perfect helix of dimers. Either the lattice is an "A-lattice", with α next to β across every lateral contact, or a "B-lattice", with α next to α, and then it must break somewhere. Mandelkow's freeze-fracture images and later cryo-EM decided for the B-lattice with a single seam. Cryo-electron microscopy in 1992 showed that the protofilament number varies: microtubules from purified tubulin mostly have 14 protofilaments, those assembled in *Xenopus* egg extracts mostly 13, and the number can change abruptly along a single tube, usually by one and sometimes by two, at what the authors read as dislocation-like defects (*J. Cell Biol.* 117, 1031, abstract). The cell therefore builds a tube whose closure rule is not fixed by the subunit alone.

The cell cycle became a molecular object in the 1980s. Paul Nurse, Lee Hartwell and Tim Hunt identified cyclins and cyclin-dependent kinases; they shared the 2001 Nobel Prize. Cell biology then moved to systems: fluorescent proteins (GFP, 1994), live imaging, and the synthetic circuits of 2000. Its open questions concern how cells measure size and time, how organelles count themselves, and how mechanical structure and signalling interlock.

Current state. The field is now organised around atlases. The Human Cell Atlas lists more than 3,900 members at more than 1,700 institutes in over 100 countries, and

published a collection of more than 45 papers in *Nature* journals in November 2024 (Human Cell Atlas). Imaging has reached the lattice. Recent cryo-EM work describes the seam as the boundary where the tube closes unlike the other inter-protofilament contacts; there α subunits meet β subunits laterally, "resulting in an offset at the seam of 1.5 dimers for a 13-PF microtubule," and the seam "is generally believed to be a weak point of the microtubule structure" (Goodson and Jonasson, NSF repository). Sui and Downing (2010) stated the rule that decides whether a seam exists: "The seam arises because MTs with an odd start number assemble so as to produce a discontinuity," while 2-start and 4-start lattices close without one (*Structure* 18, 1022–1031). A 2026 model of microtubule fracture places the seam at the closing boundary of the 3-start lattice and lets defects move it (Zablotsky et al., arXiv).

Cilia keep their nine-fold plan: the axoneme has nine doublets, each an A-tubule of 13 protofilaments and a B-tubule of 10, rooted in a basal body of nine triplets (Lodish, *Molecular Cell Biology*). The nine-fold plan is not universal: *Drosophila* centrioles carry nine doublets and those of *C. elegans* nine singlets (Wikipedia, "Centriole", secondary). So the triplet is a variable unit while the nine-fold ring is conserved.

Cell biology's most visible prize of 2026 used a cellular tool rather than a cellular discovery: the Nobel Prize in Physiology or Medicine went on 5 October 2026 to Karl Deisseroth, Peter Hegemann and Georg Nagel "for discoveries concerning light-gated ion channels and optogenetics" (Khaleej Times). Light-gated channels let an experimenter switch single cells on and off, which turns cell biology's circuits into things that can be driven from outside. The 2025 prize, to Mary Brunkow, Fred Ramsdell and Shimon Sakaguchi "for their discoveries concerning peripheral immune tolerance" (Nobel Prize), concerns a cell type and is treated in Chapter 11. Synthetic cell biology continues to use the two founding circuits as standard parts: a pair of repressors for memory, a ring of three for a clock.

Interpretation.

1. *Strand.* For the lattice, I is the path of lateral contacts around the tube and P is the set of protofilaments. For a gene circuit, I is time and P the set of expression states.
2. *Passage and trit.* Around a microtubule the strand returns to its starting protofilament after 13 lateral steps, shifted by the helix start number S in monomers. The lateral contact is like-to-like (α–α) or unlike (α–β). In a circuit, each gene is high (+1), low (−1) or in transition (0).
3. *Closure.* The seam is a published case of T11. A closed window joined to a different closed window carries tension at the joint; here the helical lattice closes around the tube only by joining α to β. The rule is parity: odd S leaves a seam, even S closes without one (Calculation 10.7). That is the ×2 side of the register: a meeting closes, a half-dimer offset does not. The 3 in "3-start" does not enter as a window of three; it enters only through its parity.
4. *Scale.* Ladder check (T15, Calculation 10.7): 9 triplets and 9 doublets lie on the ladder (3^2); 13, 14 and 10 protofilaments do not. The ladder does not organise protofilament number. In circuits, the meeting–ring split holds across scale: two repressors make a switch, three make a clock, and the rule extends by parity to 4 through 8 (Calculation 10.6).

5. *Test.* Stated before data: if the seam is T11 tension, then in lattices assembled with even start numbers (seamless 4-start tubes, which exist in vitro) the measured lateral bond energy along every contact should equal the bulk value, and fracture or depolymerisation should show no preferred longitudinal line. A seamless lattice that still splits preferentially along one line refutes the reading.

Theorems checked: T4 and T7 apply to the circuit states (above). T11 applies to the seam as a reading; the seam rule itself is parity, which the theory's register does not derive (the register has three values, the seam rule two). T15 applies to counts and mostly fails (above). T16–T18 have no counterpart. T21, the 4π return of an attached rotor, has no established counterpart in cilia or centrioles, and no claim is made. T24 could apply to the free diffusion of a molecule in the three-dimensional cytoplasm, with lever -3.9365 (Calculation 11.6), but nothing in cell biology measures a lever of visit counts, so it is left unused.

Consequences. The microtubule seam gives cell biology a concrete instance of a seam in the sense of T11: two closed patterns that cannot be joined without a defect line. The field already knows this geometry and its parity rule; the theory adds no new number. What it adds is the expectation that the seam is the line where tension collects, which the field already "believes" but has not fully measured. If the test holds, the seam becomes a predictable site of failure and regulation, a target for drugs that stabilise or cut it. The nine-fold symmetry of centrioles sits on the ladder, but so would many other numbers; a single count on the ladder is no evidence. The circuit split is stronger, because it is a rule over a whole family: every even ring settles, every odd ring above one oscillates. The theory names that split as the difference between closure with carry 0 and closure with carry. It costs nothing and adds a name.

One caution follows from the calculation. The split in Calculation 10.6 is parity, not a window of three: rings of 5 and 7 oscillate as well as rings of 3. The theory predicts that 3 is the shortest oscillating ring, which is correct, but it does not predict that 5 oscillates. A reader who wants the theory to explain all odd rings must use the meeting side of the register rather than the window of three. An even ring of repressors can settle in the alternating state high, low, high, low, which reduces completely to meetings $(x, -x)$ in the sense of T7; an odd ring cannot alternate all the way round and is left with an unpaired element, so it keeps moving. That is consistent with T7, but it is a weaker claim than the codon rule of Section 10.1. Cell biology therefore supports the meeting side of the register firmly and the ring side only at its minimum.

Grade. A for the ring–switch split of gene circuits (Elowitz and Leibler 2000; Gardner et al. 2000); A for the parity rule of microtubule seams as published fact (Sui and Downing 2010), C for reading it as T11; B for the seam-fracture test; C for ladder readings of organelle counts.

10.4 Structural biology

History. Structural biology began with fibres. William Astbury's X-ray patterns of wool and silk in the 1930s showed regular repeats. In 1951 Linus Pauling, Robert Corey and Herman Branson proposed the α-helix and the β-sheet from bond geometry alone

(*PNAS* 37, 205–211). The α-helix has 3.6 residues per turn and a rise of 1.5 Å per residue. In 1953 the DNA double helix followed. Max Perutz and John Kendrew solved haemoglobin and myoglobin by X-ray crystallography between 1958 and 1960. The Protein Data Bank was founded in 1971 with a handful of structures. G. N. Ramachandran's group in Madras proposed the collagen triple helix in 1954–55: three chains, each with glycine at every third residue, wound around a common axis.

Crystallography of large assemblies followed. Aaron Klug developed image analysis of electron micrographs in the 1960s and 1970s; Donald Caspar and Klug published the rules of virus shells in 1962 (Section 10.5). James C. Wang measured the helical repeat of DNA in solution in 1979: about 10.5 base pairs per turn, rather than the 10 of the fibre model (*PNAS* 76, 200–203). Knotted proteins were proposed by Marc Mansfield in 1994 and found in real structures soon after. In 1997 Hiroyuki Noji and colleagues filmed F1-ATPase rotating, and Paul Boyer and John Walker received the Nobel Prize for the mechanism of ATP synthesis (Chapter 11).

DNA topology became part of structural biology in the same years. A closed circular DNA has a linking number Lk, the number of times one strand winds around the other. James White proved in 1969 that Lk splits into twist and writhe, Lk = Tw + Wr, and that Lk cannot change while both strands stay intact. Enzymes that change it were found in the 1970s: topoisomerases. In 1980 Leroy Liu, Chung-Cheng Liu and Bruce Alberts showed that the T4 type II topoisomerase can knot and unknot DNA by passing one duplex through a transient double-strand break, and that it changes Lk only in steps of two (*Cell* 19, 697–707; TMU record). Type I enzymes, which cut one strand, change Lk in steps of one.

Cryo-electron microscopy changed the field after 2013, when direct electron detectors made near-atomic resolution routine; Jacques Dubochet, Joachim Frank and Richard Henderson shared the 2017 chemistry prize. In 2020 AlphaFold 2 predicted single-chain structures at near-experimental accuracy. The open questions shifted from "what is the structure" to "how do structures move, assemble and fail". Prediction also exposed what is not yet understood: why some chains fold into knots at all, when threading a chain through its own loop seems kinetically costly, and how three-chain folds such as collagen choose their register.

Current state. The Protein Data Bank passed 250,441 entries in 2026, and listed 261,139 entries on 21 October 2026 (PDBj). The AlphaFold Protein Structure Database holds over 214 million predicted structures (Varadi et al. 2024, *Nucleic Acids Research*). The 2024 Nobel Prize in Chemistry went half to David Baker "for computational protein design" and half to Demis Hassabis and John Jumper "for protein structure prediction" (Nobel Prize). The 2026 chemistry prize, to Henri Kagan and Kenso Soai "for the discovery of non-linear effects and autocatalysis in asymmetric organic synthesis", announced on 7 October 2026 (Khaleej Times), bears on the handedness of biological molecules.

Topology has become a structural property. Knotted proteins make up about one percent of the PDB; most of their knots are deep trefoils, 3_1, and AlphaFold analyses also find mostly trefoils among predicted knots; a 5_2 knot occurs in human ubiquitin

hydrolase, and a 7_1 knot has been reported in an 89-residue protein (Wikipedia, "Knotted protein", secondary). Collagen remains the clearest periodic fold: three chains with repeated Gly-X-Y sequences "associate with a one-residue shift (stagger, register) to fit glycine residues in the inner core" (Boudko and Bächinger 2016, *Scientific Reports* 6, 37831).

Interpretation.

1. *Strand.* I is the residue index along a chain; P is the set of positions in space. A helix is a strand that returns to the same angular place after a fixed number of residues.
2. *Passage and trit.* In collagen the position mod 3 along each chain is a trit: glycine occupies one residue class. In a knot diagram, each crossing is over (+1), under (−1) or absent (0), the embedding hypothesis of Part VII.
3. *Closure.* Collagen is a published heterogeneous closed window (T4). The three chains are staggered by one residue, so at each level of the triple helix the three chains present residue phases 0, 1 and 2: all different, sum 0 mod 3, one glycine per level in the core (Calculation 10.7). This is the only arrangement of three chains in which every level has exactly one glycine, which is what the packing requires. The trefoil is T19 in a protein: the simplest knot that cannot be undone, and the most common protein knot.
4. *Scale.* Ladder check (T15, Calculation 10.7). The α-helix repeat is 3.6 = 18/5: 18 residues in 5 turns; $18 = 2 \cdot 3^2$ lies on the ladder. B-DNA in solution, 10.5 = 21/2: 21 base pairs in 2 turns; $21 = 3 \cdot 7$ does not. One helix fits, one does not. The theory makes no claim about helix repeats, and this check confirms none. DNA topology adds a published counterpart of T22. Lk is an integer invariant of a closed duplex; it changes only when a strand is cut, which is what topoisomerases do. T22 states the same for the Hopf charge: an integer that can change only through cutting, leaving the target space or crossing a boundary. The analogy is structural, not an identity: Lk is a linking number of two closed curves, the Hopf charge a linking number of two preimage fibres (T22, proof). The step of two for type II enzymes is a meeting: one duplex passes through another, and the crossing changes sign, which changes Lk by 2.
5. *Test.* Stated before data: T19 and T20 say the trefoil is the least knot and is chiral. If protein knots are closures of the chain, then among knotted protein families the trefoil must remain the most frequent type in any larger census, and both handedness classes need not be equal; a census in which an achiral knot (4_1) outnumbers trefoils refutes the reading. A second test concerns collagen: any natural triple helix with two chains in the same register (a homogeneous rather than heterogeneous window) and stable glycine packing would refute the T4 reading.

Theorems checked: T4 (collagen), T15 (helix repeats), T19–T20 (protein knots) and T22 (linking number) are used above. T21, the 4π return of an attached rotor, is a property of rotation groups, and rotary enzymes such as F1-ATPase turn through 360° in three 120° steps (Chapter 11); no measurement distinguishes 2π from 4π in a protein, so no claim is made. T14 is examined for capsids in Section 10.5 and fails there. T23 and T24 have no counterpart in static structures.

Consequences. Structural biology contains two clean structures of the Foundations. Collagen is a closed window of the heterogeneous kind: three different phases at each level. Proteins knot mostly as trefoils, the first knot of T19. Both are published facts that predate the theory. The theory adds a reason to expect them: a three-chain fold needs one closing residue per level, and the simplest persistent entanglement is the trefoil. It does not predict which proteins knot, nor the 3.6-residue helix, nor DNA's 10.5 base pairs per turn. The new structure databases make the knot test cheap: AlphaFold's 214 million models can be scanned for knot types and handedness. Kagan and Soai's prize work on non-linear effects in asymmetric synthesis touches the same question from chemistry: how one handedness takes over. The theory's chirality result (T20) is about knots, not molecules; linking the two would need a stated mechanism, which the theory does not provide.

DNA topology is where the theory and the field agree most cleanly on a conservation law. Cell biologists know that a closed plasmid keeps its linking number until an enzyme cuts it; the theory states the same kind of law for its own charge and proves it without a symmetry argument. The agreement costs the theory nothing and adds to the field only a classification: Lk is one more integer invariant that changes by cutting, in the family to which the Hopf charge belongs. The step of two for type II topoisomerases is a published fact, and it is the meeting of T7 at the scale of the double helix: a strand passage that changes a crossing from over to under, +1 to −1, a difference of 2.

Grade. A for collagen's one-residue stagger as a heterogeneous closed window (Ramachandran; Boudko and Bächinger 2016), for the trefoil as the dominant protein knot (Mansfield 1994 and later censuses), and for the linking number as an integer changed only by cutting, in steps of two for type II topoisomerases (White 1969; Liu, Liu and Alberts 1980); B for the knot-census and collagen-register tests; C for helix repeats (one of two on the ladder, no claim made).

10.5 Virology

History. Dmitri Ivanovsky (1892) and Martinus Beijerinck (1898) showed that tobacco mosaic disease passes through filters that stop bacteria; Beijerinck called the agent a *contagium vivum fluidum*. Frederick Twort (1915) and Félix d'Hérelle (1917) discovered bacteriophages. Wendell Stanley crystallised tobacco mosaic virus in 1935, and Helmut Ruska saw viruses in the electron microscope in 1939. The phage group around Max Delbrück, Salvador Luria and Alfred Hershey made phages the model organisms of molecular biology; the Hershey–Chase experiment of 1952 confirmed DNA as the genetic material.

In 1956 Crick and Watson argued that small viruses must build their shells from many copies of one protein, arranged with cubic symmetry. Caspar and Klug (1962) solved the geometry. A shell can be folded from a sheet of hexagons with twelve pentagons inserted. The positions of the pentagons are fixed by two whole numbers h and k, and the triangulation number $T = h^2 + hk + k^2$ counts how many subunits sit in each of the 60 symmetric units. The shell has 60T subunits, arranged as 12 pentamers and $10(T - 1)$ hexamers. The rule held for virus after virus. In 1977 Sanger's group sequenced ΦX174, 5,386 nucleotides, the first DNA genome, and found genes that overlap in

different reading frames. Retroviruses, discovered through reverse transcriptase by Howard Temin and David Baltimore in 1970, revealed programmed −1 frameshifting.

The open questions after 1980 were how shells assemble so reliably, why some viruses break the Caspar–Klug rules, and how genomes pack more information than their length suggests. Large DNA viruses sharpened the first question. Their shells, at T in the hundreds, are built from trisymmetrons and pentasymmetrons, patches of capsomers that tile the faces and vertices, and Sinkovits and Baker (2010) derived rules for how such patches fit any (h, k) lattice. The giant mimivirus, discovered in 2003, would sit near T ≈ 1,000 if its lattice were complete, which shows how far the same norm rule stretches.

Current state. The International Committee on Taxonomy of Viruses released MSL41 on 20 March 2026: 10 realms, 11 kingdoms, 22 phyla, 105 orders, 427 families, 4,149 genera and 17,554 species (ICTV), up from 7 realms and 16,215 species in MSL40 a year earlier (ICTV). The largest solved icosahedral shells reach T = 219 for the algal virus PpV01 (h = 7, k = 10) and T = 169 for PBCV-1 and frog virus 3 (h = 7, k = 8) (Sinkovits and Baker 2010, *J. Struct. Biol.* 170, 109–116, PDF). ViralZone lists factsheets for T = 1, 3, 4, 7, 9, 13, 16, 21, 25, 27, 28 and 31, and a T = 75 shell for a bacterial carboxysome (ViralZone).

Zoonotic influenza remains the main watch. The US Centers for Disease Control and Prevention reports 72 human cases of A(H5) influenza in the United States since February 2024, in its summary updated on 15 October 2026 (CDC). For SARS-CoV-2, programmed ribosomal frameshifting is “the key event” in translating the genome; a cryo-EM structure shows a viral RNA pseudoknot at the ribosome’s mRNA entry channel creating the tension that drives the shift, and a compound that reduces frameshifting inhibited replication in cells (NCCR RNA & Disease). AI-designed phage genomes are now viable (Section 10.1).

Interpretation.

1. *Strand.* For the shell, I is a path on the hexagonal lattice from one pentamer to the next, and P is the set of lattice points, the Eisenstein integers Z[ω]. For the genome, I is the position along the RNA or DNA and P the reading frames.
2. *Passage and trit.* The path from one pentamer to the next is h steps along one lattice direction and k along the next, the element $z = h - k\omega$ of Z[ω]. Its balanced residue modulo $\beta = 1 - \omega$ is a trit (T1, remark), equal to $h - k \bmod 3$. In the genome, the relative frame of two overlapping genes is a trit: 0, +1 or −1.
3. *Closure.* The allowed T are exactly the norms of Z[ω]: every T up to 50 from $h^2 + hk + k^2$ is a norm $a^2 - ab + b^2$, and conversely (Calculation 10.8). The shell closes on the lattice in which the trit is a residue. T is divisible by 3 exactly when z has residue 0, that is, when $h \equiv k \bmod 3$; this splits the shells into the classes with $T \equiv 0$ (3, 9, 21, 27, 147, 219) and $T \equiv 1$ (1, 4, 7, 13, 16, 25, 31, 169) mod 3 (Calculation 10.8). In ΦX174, the overlapping gene pairs sit at relative frames +1 and −1, and the one pair at frame 0 is gene A with its internal restart A* (Calculation 10.10). Overlap uses the two open digits; frame 0 is the same protein.

4. *Scale.* A shell holds 60T subunits. Ladder check (T15): of the T values above, 1, 3, 4, 9, 16 and 27 lie on the ladder; 7, 13, 21, 25, 31, 147, 169 and 219 do not (Calculation 10.8). The ladder does not organise capsid size. The T14 split was tested directly: whether the digit orbit of z ends at rest (0) or in the turn ±(1 + ω). The answer depends on which of the 12 equivalent elements (unit multiples and conjugates) describes the same lattice. For T = 3, 8 associates rest and 4 turn; for T = 169, 6 and 6 (Calculation 10.9). The rest/turn label is not a property of a capsid.
5. *Test.* Stated before data: every closed icosahedral shell built from one protein on a hexagonal lattice has T equal to a norm of Z[ω]. A quasi-equivalent shell with T outside the norms (for example T = 2, 5, 6 or 8 in the same construction) refutes it. Note: such shells are known only as pseudo-T or "T = 2*" variants built from dimers on a different lattice, which fall outside the hypothesis. The earlier proposal to test T14 against assembly success is withdrawn: Calculation 10.9 shows the T14 label is not lattice-invariant, so it cannot be a property of assembly.

Theorems checked: T1 (residue of z), T13 (frames as closed configurations), T14 (failed), T15 (mostly off the ladder) are used above. T11 might be read into the twelve pentamers, which are the defects where a flat hexagonal sheet is forced to close into a sphere. But the number twelve comes from Euler's formula for a closed surface, not from the seam rule, and the theory adds nothing to it. T19–T20 apply to packaged genomes: DNA confined in phage heads is often knotted, and confinement favours chiral knots over achiral ones in simulations of phage P4 DNA (Micheletti et al. 2008, record); the theory's statement that the first knot is chiral fits, but the field explains the bias by confinement alone. T21–T24 have no counterpart.

Consequences. Virology holds the most precise match between the Foundations and a biological geometry. The Caspar–Klug numbers are the norms of the ring in which the trit is the residue modulo the prime above 3. This is a published structure, independent of the theory, and it explains why T takes values 1, 3, 4, 7, 9, 12, 13 and never 2, 5 or 6. The mod-3 split of T into multiples of 3 and the rest is a property the field uses implicitly through the skew classes. The theory's further result, T14, does not carry over: whether a digit orbit comes to rest is not invariant under the lattice symmetries, and so it says nothing about a shell. That is a useful negative. It limits the claim to the norm, where it is exact. For genomes, frame arithmetic explains why overlapping genes use the two nonzero digits. Programmed −1 frameshifting is an open digit used on purpose; the SARS-CoV-2 pseudoknot creates the tension that moves the ribosome, which is the theory's word for an open window, used here literally. Drugs that block frameshifting push the digit back to 0 and stop the virus.

The growth of the taxonomy does not change this picture; it widens the sample on which the norm rule can be checked. With 17,554 species in ten realms, and many icosahedral families among them, a single shell outside the norms would be noticed. None has been reported under the hypothesis of one protein on one hexagonal lattice. The cost of the reading is that it explains only what geometry already forces. Caspar and Klug did not need Eisenstein integers to derive their rule; the identification is a translation into the theory's ring. Its value is that the same ring carries the trit of the register and the size of the shell, so the two are no longer separate facts.

Grade. A for capsid triangulation numbers as norms of Z[ω] (Caspar and Klug 1962) and for the frame arithmetic of overlapping genes (Sanger et al. 1977; Calculation 10.10); B for the norm test; C (refuted as formulated) for T14 rest/turn applied to capsids; C for ladder readings of capsid size.

Chapter 11 — Metabolism and Interaction

This chapter studies exchange at the level of life: molecules exchanged in metabolism, cells exchanged between hosts and microbes, signals exchanged between immune cells, and species exchanged as food or harm. The meeting (x, −x) and the ring (x, x, x) of T7 appear here as published facts in three places. ATP synthase turns in three 120° steps around three catalytic sites, so ions per ATP equal c/3 for a rotor of c subunits (Calculation 11.1). Random competition tournaments leave only odd-sized sets of coexisting species, and the least such set is a ring of three (Allesina and Levine 2011; Calculation 11.3). Opposite-sign pairs (predator and prey, a meeting) stabilise large communities while same-sign pairs destabilise them (Allesina and Tang 2012; Calculation 11.4). Biochemistry and the ecology of interactions earn A for these structures; microbiology earns A for the rock–paper–scissors ring of *E. coli* strains and B for a stated four-strain test; immunology earns A for the one-in-three productive rearrangement, which equals the closure baseline of T6, and C otherwise. The c-ring sizes 8 to 17 mostly lie off the ladder 2^a·3^b, and the 12/23 spacer rule of V(D)J recombination is not a mod-3 rule; both are reported as non-fits.

11.1 Biochemistry

History. Biochemistry began when Eduard Buchner showed in 1897 that a cell-free yeast extract ferments sugar: the chemistry of life could run outside the cell. Enzymes became the field's central object. Leonor Michaelis and Maud Menten wrote their rate law in 1913. James Sumner crystallised urease in 1926 and showed that an enzyme is a protein. In the 1930s Gustav Embden, Otto Meyerhof and Jakub Parnas worked out glycolysis. In 1937 Hans Krebs and William Arthur Johnson at Sheffield described the citric acid cycle; Krebs shared the 1953 Nobel Prize in Physiology or Medicine "for his discovery of the citric acid cycle" with Fritz Lipmann, who found coenzyme A (Nobel Prize). The cycle was the first metabolic ring: oxaloacetate is regenerated at the end of each turn, and each turn yields three NADH, one $FADH_2$, one GTP and two CO_2 (Wikipedia, "Citric acid cycle", secondary).

Photosynthesis gave a second ring. Using carbon-14, Melvin Calvin, Andrew Benson and James Bassham at Berkeley traced the path of carbon in algae around 1950. Calvin received the 1961 chemistry prize "for his research on the carbon dioxide assimilation in plants" (Nobel Prize). The cycle fixes one CO_2 per turn; three turns make one surplus three-carbon sugar, glyceraldehyde-3-phosphate (G3P), using nine ATP and six NADPH (Wikipedia, "Calvin cycle", secondary).

Energy coupling was the hardest problem. Peter Mitchell proposed in 1961 that a proton gradient across a membrane drives ATP synthesis, the chemiosmotic theory, against strong resistance; he received the 1978 chemistry prize. Paul Boyer proposed the binding-change mechanism: the three catalytic sites of ATP synthase pass in turn through three states. John Walker's crystal structure of 1994 showed the asymmetric rotor inside the ring of three α and three β subunits. In 1997 Hiroyuki Noji, Ryohei

Yasuda, Masasuke Yoshida and Kazuhiko Kinosita attached an actin filament to the γ-subunit and filmed it turning, more than 100 times, with torque above 40 pN·nm (*Nature* 386, 299–302; RePEc record). Boyer and Walker shared the 1997 chemistry prize "for their elucidation of the enzymatic mechanism underlying the synthesis of adenosine triphosphate (ATP)" (Nobel Prize).

A theoretical line ran alongside. Manfred Eigen (1971, *Naturwissenschaften* 58, 465–523) and later Eigen and Peter Schuster (1979) proposed the hypercycle: a ring of self-replicating molecules, each catalysing the next. The open question biochemistry carried into the 21st century was how such cycles arose, and how a chemistry that uses both mirror forms of molecules became one that uses only one.

Current state. The rotor of ATP synthase is now known in many species. A review by Nirody, Budin and Rangamani lists c-ring sizes of 8 (bovine mitochondria), 9 (*Mycobacterium phlei*), 10 (*E. coli*, yeast, human), 11 (*Ilyobacter tartaricus* and other Na^+-driven enzymes), 12 (*Paracoccus*), 13 (*Bacillus pseudofirmus*, *Chlamydomonas* chloroplast), 14 (spinach chloroplast and cyanobacteria), 15 (*Spirulina*, *Gloeobacter*) and 17 (*Burkholderia pseudomallei*) (arXiv:2006.09357). The review states the rule: each c-subunit carries one ion per revolution, the F1 head has three catalytic β-subunits, so the cost is c/3 ions per ATP; because c is rarely a multiple of 3, the rotary substates are often not spaced exactly 120°, and in the autoinhibited spinach enzyme they lie at 103°, 112° and 145° (same source). A 2024 algorithm predicted rotor stoichiometries for about 1,571 sequence clusters; it matched experiment with a correlation of 0.94, predicted values from 4 to 27, and placed 508 sequences above the experimental maximum of 17 (bioRxiv 2024.02.27.582367).

The chemistry of handedness received the 2026 Nobel Prize in Chemistry. Henri Kagan and Kenso Soai were honoured on 7 October 2026 "for the discovery of non-linear effects and autocatalysis in asymmetric organic synthesis" (Khaleej Times). Soai's reaction amplifies a tiny excess of one mirror form into near-complete dominance: a candidate route to biological homochirality. Protein design, honoured in 2024, now produces enzymes from scratch, and sequence models (Chapter 10) propose new ones. The open debates concern how much of metabolism can be predicted from genomes, and whether the c-ring sizes are tuned to the energy available to each organism; the cited review links the large rings of chloroplasts to the high proton motive force of photosynthesis.

Interpretation.

1. *Strand.* For ATP synthase, I is time and P is the set of rotor angles; the γ-subunit is the strand passing the three catalytic sites. For a metabolic cycle, I is the sequence of reactions and P the set of intermediates.
2. *Passage and trit.* Each 120° step passes one catalytic site; one full turn passes all three. The c-ring passes c ion sites per turn. By T1 the number c is a digit and a carry: c = 3·carry + digit. The carry is the whole number of ions per site; the digit is the remainder (Calculation 11.1).
3. *Closure.* The F1 head is a published ring of three: three sites, three 120° steps, one closed turn. The ring closes with carry, as T7.1 states: one turn of three steps yields

three ATP, one passage of the next scale, the turn of the rotor. The c-ring closes with the head only when c ≡ 0 mod 3 (c = 9, 12, 15); for c = 8, 10, 11, 13, 14 and 17 the digit is nonzero and the ion steps do not fall evenly on the three sites. The irregular 103°, 112°, 145° substates of the spinach enzyme (c = 14, digit −1) are that open digit seen in a structure. The Calvin cycle is a ring with carry in the literal sense: three CO_2 fixed (sum 3) give digit 0 and carry 1, one G3P exported (Calculation 11.2).

4. *Scale.* Ladder check (T15): of the nine c values, 8, 9 and 12 lie on the ladder 2^a·3^b; 10, 11, 13, 14, 15 and 17 do not (Calculation 11.1). The ladder does not select rotor size. The citric acid cycle turns a two-carbon unit into two CO_2 per turn, a meeting at the scale of carbon; the Calvin cycle needs three turns per exported sugar, a ring. Eigen's hypercycle is a counter-example to any claim that three is the ring that oscillates: Eigen and Schuster's analysis gives a stable equilibrium for up to four members and a stable oscillation only above four (Principia Cybernetica).
5. *Test.* Stated before data: if the open digit of the c-ring is tension in the sense of T1 and T2, rotors with c ≡ 0 mod 3 should show three equally spaced dwell angles (120° apart within measurement error), and rotors with c ≢ 0 mod 3 should not. Single-molecule data on a c = 9, 12 or 15 enzyme showing unequal dwells as large as the spinach enzyme's, or a c = 10 or 14 enzyme showing exactly equal dwells, refutes the reading.

Theorems checked: T21 (4π return) could in principle apply to a rotor attached to a membrane, but no experiment distinguishes 2π from 4π for γ; no claim. T19–T20 (chirality of the first knot) do not bear on molecular handedness without a stated mechanism. T23 (stiffness as a state function of density) has no measured counterpart here. T16–T18 do not apply.

Consequences. Biochemistry contains the clearest ring of three in physiology: the F1 head. Its arithmetic is published and independent of the theory. The theory adds a bookkeeping in which the c-ring number is a digit and a carry against that ring. The digit predicts which enzymes turn in equal steps and which do not, and that prediction can be checked against existing single-molecule recordings. If it holds, the c-ring size is not only an energy budget (c/3 ions per ATP) but also a closure property: enzymes with digit 0 would be mechanically regular, others irregular. If it fails, the dwell angles are set by the protein's elastic coupling alone, and the theory adds nothing beyond the field's own c/3.

The ladder is a clear negative. Rotor sizes run from 8 to 17 with no preference for 2^a·3^b; the prediction algorithm's range of 4 to 27 does not suggest one either. And the hypercycle limit, fixed point up to four members and oscillation above, is a published result that does not match the theory's ring of three; it matches instead the general fact that longer delayed feedback loops oscillate. The Calvin cycle's three turns per sugar is exact but is only carbon counting: a three-carbon product needs three one-carbon inputs. The reading in terms of carry is correct and adds no knowledge. Homochirality remains open; the 2026 prize work gives the chemistry a mechanism of amplification that the theory does not supply. Soai's autocatalysis is, in the theory's

vocabulary, a ring that feeds its own product back into its next turn; but the amplification is a property of reaction kinetics, and the theory has no rate law to offer.

Grade. A for the three-site rotary ring of F1 and the cost c/3 ions per ATP (Boyer; Walker; Noji et al. 1997); B for the dwell-angle test; C for the Calvin carry reading (exact, uninformative) and for ladder readings of c-ring size (six of nine off the ladder).

11.2 Microbiology

History. Antonie van Leeuwenhoek saw bacteria through his single-lens microscopes in the 1670s and reported them to the Royal Society. Two centuries later Louis Pasteur showed that fermentation is the work of living microbes and that they do not arise spontaneously (1859–1862). Robert Koch tied specific bacteria to specific diseases, anthrax in 1876 and tuberculosis in 1882, and with Julius Petri and Fanny Hesse introduced solid media for pure cultures. Sergei Winogradsky and Martinus Beijerinck founded microbial ecology in the 1880s and 1890s by studying soil microbes in their own chemical settings: chemolithotrophy, nitrogen fixation, enrichment culture.

The antibiotic era began with Alexander Fleming's observation of penicillin in 1928 and its development by Howard Florey and Ernst Chain around 1940. In 1943 Salvador Luria and Max Delbrück showed that bacterial resistance to phage arises by random mutation before selection: the number of resistant colonies varied from culture to culture with a variance "considerably greater than the mean", which a model of induced resistance could not explain (*Genetics* 28, 491–511; Wikipedia, "Luria–Delbrück experiment", secondary). They shared the 1969 Nobel Prize, partly for this work.

In 1977 Carl Woese and George Fox used ribosomal RNA to divide life into three domains, recognising the archaea. Molecular microbiology then grew from culture to sequence. In 1988 Richard Lenski founded the long-term evolution experiment (LTEE) with 12 populations of *E. coli* transferred daily into fresh medium. In 1995 Mulder, van de Graaf, Robertson and Kuenen reported anaerobic ammonium oxidation (anammox) from a Delft reactor; the first full-scale anammox reactor followed in the Netherlands in 2002 (Wikipedia, "Anammox", secondary). Two open questions shaped the field: how many microbial species exist, and how microbial communities stay diverse when competition should leave one winner. Bacteriocins such as colicin, toxins made by one strain to kill close relatives, gave microbiologists a natural way to build competitive loops in the laboratory.

In 2002 Benjamin Kerr, Margaret Riley, Marcus Feldman and Brendan Bohannan gave a direct experimental answer to the second question (*Nature* 418, 171–174). They used three *E. coli* strains: C produces the toxin colicin and kills S; S is sensitive but grows faster than R; R is resistant but grows faster than C. "Thus, C kills S, S outgrows R, and R outgrows C" (Kerr et al. 2002, PDF). In a shaken flask and on a mixed plate, S and C fell below detection and R remained. On a static plate, where interaction and dispersal were local, all three persisted at high density through the week of the experiment, about 66 generations (same source).

Current state. The catalogue of microbial diversity has grown by orders of magnitude. The Genome Taxonomy Database release 226 holds 732,475 genomes (715,230 bacterial, 17,245 archaeal) in 143,614 species clusters and 189 phyla (GTDB); archaea make up 4.85% of the species clusters (Calculation 11.6). In the human body, bacterial cells number about 3.8×10^{13} against about 3.0×10^{13} human cells, a ratio near 1.3, replacing the old "10 to 1" (Sender, Fuchs and Milo 2016, *PLoS Biology*; BioNumbers).

The LTEE has passed 80,000 generations. After about 75,000 generations under Lenski at Michigan State University, the experiment reached 82,000 generations under Jeffrey Barrick and returned to Michigan State in 2025, with daily transfers continuing (MSU College of Natural Science, November 2025). A daily 1:100 transfer gives $\log_2 100 = 6.64$ doublings a day, so 82,000 generations correspond to about 34 years of transfers (Calculation 11.6). One population evolved aerobic growth on citrate at about 31,000 generations (same source).

Resistance is the clinical frontier. The GRAM study in *The Lancet* (2024) estimated 1.14 million deaths directly attributable to bacterial antimicrobial resistance in 2021 and 4.71 million associated deaths, and forecast more than 39 million attributable deaths between 2025 and 2050 (IHME). Environmental microbiology is turning to engineered communities for waste and pollutants (Chapter 13). Anammox, once a laboratory curiosity, runs in full-scale wastewater plants: ammonium and nitrite are combined into nitrogen gas, $NH_4^+ + NO_2^- \rightarrow N_2 + 2\ H_2O$, after a partial nitritation step that turns about half of the ammonium into nitrite (Wikipedia, "Anammox", secondary). That is a two-organism partnership in which the product of one group is the substrate of the other: a metabolic meeting between ammonia oxidisers and anammox bacteria.

The debates of 2026 concern how much of microbial "dark matter" can be cultured, whether resistance can be slowed by new antibiotic classes or only by stewardship, and how community structure, rather than single species, decides outcomes in soils, guts and reactors. The Kerr experiment remains a reference point in the last debate, because it shows that the spatial scale of interaction, not the species list alone, decides whether a community keeps its members.

Interpretation.

1. *Strand.* I is time; P is the set of strain states on a patch (which strain holds it).
2. *Passage and trit.* When two strains meet on a patch, one wins (+1), loses (−1) or neither displaces the other (0). In Kerr's system each strain wins against one rival and loses to the other.
3. *Closure.* The three relations C > S, S > R, R > C form a ring of three equal steps (x, x, x): going round the cycle, each step is a win. By T7.1 the ring closes with carry: after three replacements a patch returns to the strain it started with, and the carry is one turn of the cycle at the scale of the whole plate. In a shaken flask there is no local patch, so the cycle cannot close locally; the community collapses to one strain, R. Two strains alone form a meeting at best (each beats the other in some respect), and a pair with a strict winner has no closure at all.
4. *Scale.* Local closure on each patch is what makes coexistence at the scale of the plate possible: the patch-level ring becomes the plate-level standing. This is the T7.1

reading of N5: what closes at one scale is one unit of the next. Luria and Delbrück's jackpot distribution is the opposite case: a mutation early in a culture's growth is multiplied by every later doubling, the ×2 ladder of T15, which is why the variance is so large. Ladder check: generations per transfer, $\log_2 100 = 6.64$, is not an integer and not on the ladder; the experiment's 12 populations ($2^2 \cdot 3$) lie on it but were chosen by design.

5. *Test.* Stated before data: the theory says the ring of three is the least closed cycle and that an even cycle splits into meetings (T7). In a four-strain intransitive system of the Kerr type (A > B > C > D > A, with the two diagonal pairs A–C and B–D each decided one way), local dispersal should maintain at most three strains or an odd-sized subset, not all four. Four-strain coexistence on a static plate, stable over a comparable number of generations, refutes the reading. Allesina and Levine's theory (Section 11.4) makes the same prediction from game theory.

Theorems checked: T6 gives the baseline for random three-strain tournaments, 2 of 8 orientations are rings (Calculation 11.3). T15 applies to doubling (above). T24 does not apply: bacteria in a flask are not a free path with returns. T19–T22 do not apply. T12–T14 do not apply.

Consequences. Microbiology supplies the one laboratory system in which a ring of three species has been built, observed to persist only where it can close locally, and observed to collapse where it cannot. That is the closure rule in its plainest form, and it was published without the theory. The theory adds the four-strain test, which follows from T7 and from tournament theory alike; it would separate a ring-based account of diversity from a general claim that any intransitive loop maintains diversity. The test is cheap: plasmid-borne toxins and resistances can be combined to build four-member loops.

For the larger field the reading costs little and adds little. The species counts of GTDB, the LTEE's fitness trajectories, the cell census and resistance burdens have no relation to the register. Microbial consortia for bioremediation could use the ring as a design rule if the test supports it. If four-strain loops do persist, the design rule is parity-free and the theory's ring loses its special place in ecology.

The LTEE offers a second, slower check. Its twelve populations live in shaken flasks, the environment in which Kerr's ring collapsed. The theory therefore expects no stable three-strain cycle of dominance to arise inside a single LTEE flask; coexisting lineages there, such as the long-lived ecotypes that have been reported in some populations, should rest on niche differences (different food sources, cross-feeding) rather than on a pure cycle of wins. A pure intransitive cycle found persisting in a well-mixed flask would count against the local-closure reading.

Grade. A for the ring of three strains that coexists only under local closure (Kerr et al. 2002); B for the four-strain test; C for the rest of microbiology.

11.3 Immunology

History. Edward Jenner vaccinated James Phipps with cowpox in 1796. Louis Pasteur extended vaccination to chicken cholera, anthrax and rabies in the 1880s. In 1890 Emil

von Behring and Shibasaburo Kitasato showed that serum from immunised animals protects others: antibodies. Elie Metchnikoff described phagocytosis in the 1880s, and Paul Ehrlich proposed his side-chain theory in 1897, in which cells carry receptors that are shed as antibodies. Karl Landsteiner found the ABO blood groups in 1901.

The question of diversity defined the 20th century. How can a body make antibodies against molecules it has never met? Macfarlane Burnet's clonal selection theory (1957) answered: each lymphocyte carries one receptor, and antigen selects the cells that fit. Rodney Porter and Gerald Edelman solved antibody structure in the early 1960s: two heavy and two light chains. Susumu Tonegawa showed in 1976 that antibody genes are assembled by recombining DNA segments: V, D and J for the heavy chain. He received the 1987 Nobel Prize. The joining is imprecise; nucleotides are lost and added at the junctions. As a consequence, "Only one in three rearrangements yields the proper reading frame for the rearranged VDJ sequence," and only one productive rearrangement is permitted per cell (Lodish, *Molecular Cell Biology*).

The rules of recombination are written in signal sequences. Each recombining segment is flanked by a heptamer of 7 base pairs, a spacer of either 12 or 23 base pairs, and a nonamer of 9; the RAG1/RAG2 complex joins a 12-spacer segment only to a 23-spacer segment, the 12/23 rule (Wikipedia, "Recombination signal sequences", secondary). Rolf Zinkernagel and Peter Doherty showed in 1974 that T cells recognise antigen only together with self MHC molecules. The human MHC, the HLA region, carries three classical class I genes, HLA-A, -B and -C, and three class II groups, DP, DQ and DR. Charles Janeway's 1989 proposal of pattern recognition, and the discovery of Toll-like receptors in the 1990s, restored innate immunity to the centre.

Tolerance was the last great theme. In 1995 Shimon Sakaguchi identified regulatory T cells; in 2001 Mary Brunkow and Fred Ramsdell found that mutations in *FOXP3* cause the scurfy disease in mice and IPEX in humans. The open questions concern how tolerance and attack are balanced, and how to steer that balance in autoimmunity, cancer and transplantation. Behind them lies the older question of waste: a system that discards most of the receptors it builds must pay for that in cells and time.

Current state. The 2025 Nobel Prize in Physiology or Medicine went to Brunkow, Ramsdell and Sakaguchi "for their discoveries concerning peripheral immune tolerance" (Nobel Prize). The HLA system is the most polymorphic set of genes known: the IMGT database lists 19,031 class I alleles and 7,183 class II alleles, with roughly 6,000 at HLA-A, 7,200 at HLA-B and 5,800 at HLA-C; because the three class I genes are codominant, a person expresses up to six class I molecules (Wikipedia, "Major histocompatibility complex", secondary). Engineered T cells have become drugs: six CAR-T therapies had FDA approval by 2022, and CD19-targeted CAR-T cells have been used in refractory lupus (Wikipedia, "Chimeric antigen receptor T cell", secondary).

The repertoire that recombination produces can now be sequenced in depth, which turns the textbook one-in-three into a measurable quantity. Repertoire sequencing reads millions of receptor genes from a blood sample, productive and non-productive alike. The non-productive rearrangements, carried silently on the second allele of cells

that succeeded on the first, are a record of what the recombinase produces before any selection on the protein. That is the population on which the test below can be run.

Zoonotic infections keep immunology tied to surveillance: the CDC counts 72 human A(H5) influenza cases in the United States since February 2024 (CDC, 15 October 2026). Resistance to antibiotics (Section 11.2) shifts attention to vaccines and to the immune system itself as therapy. The debates of 2026 concern the durability of tolerance-based therapies, the safety of engineered cells outside cancer, and how much of immune variation is set by HLA type.

Interpretation.

1. *Strand.* I is the position along a rearranged receptor gene; P is the set of codon positions. For recognition, I is the sequence of encounters of a T cell, P the set of peptide–MHC complexes it meets.
2. *Passage and trit.* The net change in length at a V(D)J junction, from deletions and additions, has a residue mod 3: 0 keeps the frame of the downstream segment, ±1 shifts it.
3. *Closure.* A rearrangement is productive when the junction residue is 0. If junction lengths are blind to the frame, T6 gives the closure probability 1/3 for a uniform law (Calculation 11.5). The textbook figure, one in three productive, is that baseline. The second allele gives a second chance: at least one productive rearrangement out of two independent attempts has probability $1 - (2/3)^2 = 5/9$ (Calculation 11.5). The immune system uses the open digit as its normal outcome and selects for the closed one.
4. *Scale.* The 12/23 rule is not a mod-3 rule. The spacers are $12 \equiv 0$ and $23 \equiv 2 \bmod 3$, and they correspond to about 1.14 and 2.19 turns of B-DNA at 10.5 base pairs per turn (Calculation 11.5): the rule pairs one turn with two, a ×2 relation in helical geometry, not a closure of three. The HLA count, three class I genes giving up to six molecules, lies on the ladder (3 and 2·3) but is a count, not a closure. Clonal expansion of a selected lymphocyte runs on the ×2 ladder of cell division.
5. *Test.* Stated before data: if junction lengths are frame-blind, the in-frame fraction of rearrangements sequenced from cells before selection (for example non-productive alleles in B cells, or pre-selection repertoires) should be 1/3 within sampling error. A pre-selection repertoire with an in-frame fraction far from 1/3 refutes the claim that the 1/3 is the T6 baseline; it would mean the junction machinery itself is frame-aware.

Theorems checked: T6 (above). T7 does not add here: junction changes are not composed of unit shifts that cancel. T11 (seams) has no counterpart in antigen recognition. T15 applies only to clonal doubling. T24 could apply to the search of a T cell through a lymph node if it were a free three-dimensional random walk; no study measures the lever of its visit counts, so no claim. T19–T23 do not apply.

Consequences. The productive fraction of V(D)J recombination is the closure probability of a frame-blind window. That is a published fact, and the theory supplies its reason: with three equally likely residues, one closes. The pre-selection test can separate that reason from an alternative in which the recombinase prefers some

junction lengths. If the baseline holds, the immune system's waste, two of three rearrangements discarded, is the cost of drawing a digit at random, and allelic exclusion is the second draw that raises the success rate to five in nine. If it fails, the theory's explanation is wrong and the 1/3 is a coincidence of measurement.

A further, smaller consequence concerns the arithmetic of allelic exclusion. The 5/9 figure assumes that the two alleles are tried independently, each with closure probability 1/3. The light chain adds more draws: in B cells, the κ loci on two chromosomes and then the λ loci can be tried in turn, each a new frame-blind window. The probability that a cell eventually succeeds rises with each draw as $1 - (2/3)^n$, and cells that fail all draws die. This is the T6 baseline compounded, and it gives an upper bound on the loss of developing B cells to frame errors alone. Whether real loss matches that bound is a question for developmental immunology, and the theory can only supply the null model.

Beyond that, immunology contains no structure of the Foundations. The 12/23 rule pairs helical turns, not windows. Tolerance, the subject of the 2025 prize, is a balance between attack and restraint; the theory can call it a meeting of two opposed cell programmes, but that is a reading, not a structure. The cost of the theory to immunology is therefore small and its gain is one number. Overstating it would be the re-description that the method forbids.

One tempting link should be named and set aside. Regulatory T cells restrain effector T cells, and effector cells drive the expansion of regulatory cells; the pair looks like a meeting, two opposite effects that cancel. But the published models of tolerance do not reduce to a sign pair; they involve cytokine competition, antigen dose and tissue context. Calling the balance a meeting would describe it in the theory's words without adding a prediction. The 2025 prize work stands on its own experiments, and the theory has nothing to add to it.

Grade. A for the one-in-three productive rearrangement as the T6 closure baseline (textbook fact; Tonegawa); B for the pre-selection test; C for tolerance, the 12/23 rule and HLA counts.

11.4 Ecology of interactions

History. Ecology's account of interaction began with predators and prey. Alfred Lotka wrote equations for oscillating chemical reactions in 1910 and applied the same form to predators and prey in *Elements of Physical Biology* (1925). Vito Volterra derived them independently in 1926, prompted by Umberto D'Ancona's observation that predatory fish increased in the Adriatic catch during the First World War, when fishing stopped. The Lotka–Volterra model has closed orbits around its equilibrium and a conserved quantity along each orbit (Wikipedia, "Lotka–Volterra equations", secondary). The fur records of the Hudson's Bay Company, showing roughly decade-long swings in lynx and snowshoe hare, became the textbook illustration. Georgy Gause tested competition in protozoa in 1934 and stated the competitive exclusion principle: two species with the same niche cannot coexist.

Symbiosis entered the discussion through lichens. Simon Schwendener proposed in 1867 that lichens are partnerships of fungi and algae, and Anton de Bary coined "symbiosis" in 1879. G. Evelyn Hutchinson's "Homage to Santa Rosalia" (1959) and "The paradox of the plankton" (1961) asked why so many species coexist on few resources. Robert MacArthur's niche models of the 1960s and Robert May's *Stability and Complexity in Model Ecosystems* (1973) set the theoretical agenda. May showed in 1972 that a large random community is stable only if its number of species S, connectance C and interaction strength σ satisfy $\sigma\sqrt{(SC)} < 1$ (in units of self-regulation): complexity destabilises. Ecologists have argued since about why real communities are large and stable.

Cyclic competition offered one answer. In 1996 Barry Sinervo and Curtis Lively described side-blotched lizards whose males come in three throat colours: orange males hold large territories and beat blue males; blue males guard one or two females and fend off yellow "sneaker" males; yellow males steal matings from orange males (*Nature* 380, 240–243). Each morph beats one and loses to the other. The open question was whether such cycles are curiosities or a general mechanism.

Current state. Two theoretical results from 2011–2012 turned the cycle and the sign of interactions into general rules. Stefano Allesina and Jonathan Levine studied tournaments in which every pair of species has a winner. In their basic model, "one can never observe an even number of coexisting species"; "a system with an even number of species will always collapse to a smaller one formed by an odd number of species" (Allesina and Levine 2011, *PNAS* 108, 5638–5642). Allesina and Si Tang then extended May's criterion to structured interaction signs (Allesina and Tang 2012, *Nature*, doi:10.1038/nature10832; arXiv). Predator–prey pairs, with opposite signs, raise the stability threshold to $\sigma\sqrt{(SC)} < d\pi/(\pi - 2) \approx 2.75d$; mutualism and competition, with same-sign pairs, lower it; "Predator-prey interactions enable the stable coexistence of networks as large and complex as those observed empirically" (same source).

The two results have opposite flavours. Allesina and Levine's rule is combinatorial: the optimal mixed strategy of a symmetric zero-sum tournament has odd support, a fact from game theory (Fisher and Reeves 1995). Allesina and Tang's rule is spectral: the eigenvalues of a large random matrix fill an ellipse whose width depends on the correlation between the two entries of each pair, and opposite signs make that correlation negative. Both are general mathematics applied to communities, and both have been tested in simulations rather than in the field. Field ecology confirms the cycles and records their decline. Sinervo's lizard population near Los Banos, California, followed for 22 years, shows the dominant morph changing every four to five years (UC Santa Cruz News). The snowshoe hare cycle has a period of about ten years; experiments at Kluane in the Yukon found that predation is the proximate cause of nearly all adult hare deaths and that removing predators and adding food together gave the largest increase in density (Krebs, Boonstra and Boutin 2018, *J. Anim. Ecol.* 87, 87–100; PDF). WWF's Living Planet Report 2024 found a 73% decline in the average size of monitored vertebrate populations between 1970 and 2020, across almost 35,000 populations of 5,495 species, with an 85% decline in freshwater (WWF).

Interpretation.

1. *Strand.* I is time; P is the set of species states in a community. An interaction between species i and j is a pair of signs (effect of i on j, effect of j on i).
2. *Passage and trit.* Each effect is a trit: +1 (benefit), −1 (harm), 0 (none). A predator–prey pair is (+1, −1); competition (−1, −1); mutualism (+1, +1).
3. *Closure.* A pair closes in the sense of T4 and T7 when it is a meeting, (x, −x): sum 0. Predator–prey pairs are meetings; competitive and mutualistic pairs are open, with sum ±2, digit ∓1 and carry ±1 by the digit rule of T1. The Allesina–Tang result says that meetings stabilise and open pairs destabilise. In a random matrix of 250 species at connectance 0.25, the largest real eigenvalue was 7.79 for random signs (theory 7.91), 2.90 for predator–prey pairs (theory 2.87), 11.43 for competition and 50.24 for mutualism (Calculation 11.4). For tournaments, the coexisting set is always odd: in 2,000 random tournaments for each size from 3 to 20, every coexisting set had odd size (Calculation 11.3). The least coexisting set above one species is a ring of three, as in the lizards and in Kerr's bacteria. Of the 8 orientations of a three-species tournament, 2 are rings (Calculation 11.3).
4. *Scale.* Mutualism at its tightest is a merged unit: the lichen of Schwendener and de Bary is a fungus and an alga that function as one organism. In the digit rule a mutualistic pair (+1, +1) is open, digit −1 with carry +1: it passes one unit to the next scale while keeping a residue at its own. The lichen is that carry made visible, but the reading is a description, not a prediction. The Lotka–Volterra orbit is a closed loop of two species, a meeting that goes round: prey up, predator up, prey down, predator down. The hare cycle of about ten years is off the ladder 2^a·3^b (Calculation 12.2). The lizard cycle, with each morph dominant for four to five years, takes three phases to return.
5. *Test.* Stated before data: the theory reads coexistence as closure. In a mesocosm or microbial system with a controlled interaction structure, the persisting set of species under a tournament should be odd, and should contain at least one ring of three. A stable, persisting even set without niche differences, or an odd set without any three-cycle, refutes the reading. For sign structure: in empirical food webs, the fraction of opposite-sign pairs should correlate positively with measured stability across webs of similar size; no or negative correlation refutes the meeting reading.

Theorems checked: T4 and T7 (above). T6 gives the baseline for random sign pairs: half of random pairs are meetings (Calculation 11.4). T13 counts closed configurations without a start: a cycle of dominance is a rotation class, not a sequence. T24 applies to animal movement only if animals perform free random walks in three dimensions, which most do not; no claim.

Consequences. Ecology holds two published rules that are the meeting and the ring of T7. Opposite-sign pairs stabilise, same-sign pairs do not; and in pure competition, the surviving set is odd and built from cycles. Neither result was derived from the theory. Both are the theory's elementary closures in the language of community matrices and tournaments. The theory adds a single frame for the two: a community persists where its interactions close, as meetings between pairs or as rings among triples, and loses

species where they do not. It adds the specific prediction that even coexistence without niche differences should not persist.

What it costs is precision about scope. Allesina and Levine's odd-number rule is a parity result about tournaments, not a mod-3 rule, and the window of three appears only as the smallest odd set above one. The Allesina–Tang rule concerns the sign of the product of a pair, which is the meeting. Neither involves the carry. The Living Planet decline has many causes, habitat loss above all; reading it as a loss of closure adds no explanation. There is also a point where the two rules meet the digit rule of T1 directly. A competitive pair $(-1, -1)$ has sum -2: digit $+1$ and carry -1. A mutualistic pair $(+1, +1)$ has sum 2: digit -1 and carry $+1$. The open pair leaves a residue at its own scale and passes a unit to the next, which in a community matrix is a positive feedback through the pair. That is the mechanism by which same-sign pairs widen the eigenvalue ellipse. The correspondence is exact as arithmetic, but the field's own explanation, the sign of the pair correlation, says the same thing without the digit.

For conservation, the useful consequence is modest: if cycles and meetings carry stability, protecting predators and the competitive loops of a community protects its persistence, which field ecologists already argue on other grounds.

Grade. A for the stabilising meeting (Allesina and Tang 2012) and for odd coexisting sets built from rings of three (Allesina and Levine 2011; Sinervo and Lively 1996); B for the mesocosm and food-web tests; C for population declines read as lost closure.

Chapter 12 — Evolution and Development

This chapter studies the rings of life: cycles that go round and come back, from cell cycles and segmentation clocks to population cycles and the long history of species. Two published structures of the Foundations appear clearly. Population models move to chaos by period doubling, the ×2 ladder of T15, and a cycle of period three forces cycles of every period (Li and Yorke 1975); Sharkovskii's ordering of periods begins with 3 and ends with the powers of 2, so the ring and the meeting ladder stand at the two ends of one published order (Calculation 12.1). The major transitions of evolution, in which units that replicated alone come to replicate only as parts of a larger whole, are the rule that what closes at one scale becomes one passage of the next (T7.1). Population dynamics earns A; evolutionary biology A for the transitions and C otherwise; the plant clock's repressilator gives botany an A, while phyllotaxis does not fit the ladder (Calculation 12.3). Developmental biology and palaeontology earn C with stated tests. Of 25 observed periods and counts checked against the ladder 2^a·3^b, 11 lie on it and 14 do not (Calculation 12.2); the earlier claim that epidemic periods lie on the ladder is contradicted by the measles bifurcation cascade, which includes 5- and 7-year cycles.

12.1 Evolutionary biology

History. Charles Darwin published *On the Origin of Species* in November 1859, a year after he and Alfred Russel Wallace presented their papers to the Linnean Society. The theory had two parts: common descent and natural selection. Common descent was accepted within two decades; selection was not, because no one knew how variation is inherited. Blending inheritance would dilute any new variant. Mendel's paper of 1866 held the answer, but it lay unread until 1900 (Chapter 10).

The reconciliation took three decades. Ronald Fisher's 1918 paper showed that many Mendelian factors produce continuous variation. Fisher's *Genetical Theory of Natural Selection* (1930), J. B. S. Haldane's *The Causes of Evolution* (1932) and Sewall Wright's work on drift and adaptive landscapes (1931–32) built population genetics. Theodosius Dobzhansky (1937), Ernst Mayr (1942), George Gaylord Simpson (1944) and G. Ledyard Stebbins (1950) joined it to field biology, systematics, palaeontology and botany in the Modern Synthesis. Motoo Kimura's neutral theory (1968) added that most molecular change is drift, not selection.

From the 1960s the question shifted to levels. William Hamilton's kin selection (1964) explained cooperation among relatives. Lynn Margulis argued from 1967 that mitochondria and chloroplasts were once free-living bacteria, now accepted. John Maynard Smith and Eörs Szathmáry's *The Major Transitions in Evolution* (1995) generalised the point. They listed transitions in which entities that could replicate independently before came to replicate only as part of a larger whole: replicating molecules into compartments, genes into chromosomes, prokaryotes into eukaryotic cells, single cells into multicellular organisms, solitary individuals into colonies. Each

transition creates a new level of selection. They also noted a common difficulty: at each transition, selection at the lower level threatens the integrity of the higher one, as cancer cells threaten the body and selfish genetic elements threaten the genome. The new unit persists only if conflicts between its parts are suppressed.

Experiments joined theory. Luria and Delbrück (1943) showed that mutations arise before selection. Richard Lenski's long-term evolution experiment, begun on 24 February 1988 with twelve populations of *E. coli*, made evolution observable in real time (Wikipedia, "E. coli long-term evolution experiment", secondary). Polyploidy, the doubling or tripling of whole genomes, was recognised as a route to new plant species in the early 20th century. The open questions concerned the origin of life itself, the rate and repeatability of evolution, and how new levels of individuality arise. Leigh Van Valen's Red Queen hypothesis (1973) added a reason why extinction rates within a group stay roughly constant: each species must keep evolving to keep its place against others that are evolving too. And the side-blotched lizards of Sinervo and Lively (1996) showed that selection itself can run in a cycle, with three male strategies each favoured when another is common (Chapter 11).

Current state. The root of the tree has been dated. A 2024 analysis by Edmund Moody and colleagues in *Nature Ecology & Evolution* placed the last universal common ancestor (LUCA) about 4.2 billion years ago, roughly 400 million years after Earth formed (Sci.News, secondary). The LTEE has reached 82,000 generations and returned to Michigan State University under Jeffrey Barrick in 2025 (MSU). One population evolved aerobic growth on citrate between about 31,000 and 31,500 generations, and a 2013 analysis by Wiser and colleagues found that fitness gains follow a power law, slowing but without a ceiling (Wikipedia, LTEE, secondary). Six of the twelve populations evolved defects in DNA repair that raised their mutation rates (same source).

Polyploidy is counted as a speciation mechanism: about 15% of angiosperm and 31% of fern speciation events involve an increase in ploidy (Wikipedia, "Polyploidy", secondary). Sequence models now read evolution in reverse. Evo 2, trained on more than 128,000 genomes, scores variants by how evolution would treat them (Astrobiology, secondary). The handedness of life's molecules remains an evolutionary question with chemical roots; the 2026 Nobel Prize in Chemistry to Kagan and Soai for non-linear effects and autocatalysis in asymmetric synthesis (Khaleej Times) gives one route by which a small excess could become total.

Debates in 2026 concern the extended evolutionary synthesis (development, plasticity and niche construction as causes), how predictable evolution is, and how the first replicators formed closed reaction networks. Eigen's hypercycle (Chapter 11) is the classical model of the last, and its stability limit of four members is a known difficulty.

Interpretation.

1. *Strand.* I is the sequence of generations; P is the set of genotypes, or at a higher level the set of individuals and groups.

2. *Passage and trit.* A generation is a passage. At a diploid locus the genotype is a trit (AA, Aa, aa = +1, 0, −1), and under random mating its law is (2pq, p^2, q^2) for (0, +1, −1) (Hardy and Weinberg, 1908).
3. *Closure.* A unit persists when it closes: when its parts replicate together and their conflicts cancel. Maynard Smith and Szathmáry's transitions are the published form. Before a transition, the parts are separate passages; after it, they replicate only as one. That is T7.1 read in biology: a closed configuration becomes one passage of the next scale. The eukaryotic cell is the clearest case: a host and an endosymbiont whose replication became locked. T6 can be applied to Hardy–Weinberg genotypes: three unlinked loci have additive score 0 mod 3 with probability 0.3438 at p = 0.5, 0.3036 at p = 0.7 and 0.5460 at p = 0.9 (Calculation 12.5). Nothing in evolution depends on that number, and it is reported only to show that the theorem applies formally and adds nothing.
4. *Scale.* The transitions form a ladder of levels, but not a ladder of numbers. Genome doubling (tetraploidy) and tripling (hexaploidy, as in bread wheat) are merges of meetings and rings in the sense of T15, but whole-genome duplications are not equal-block merges of a strand, so T15's hypothesis is not met. Triploids are sterile, because three homologues cannot all pair (Chapter 13, Calculation 13.1); persistence of a polyploid line needs an even number of sets, which is the meeting rule.
5. *Test.* Stated before data: if a major transition is closure becoming passage, then every transition must be preceded by a stage in which the partners' replication is coupled in a closed cycle (each depends on the other), and no transition should occur through a one-way dependence alone. A documented transition to a higher-level individual in which one partner's replication never depended on the other's refutes the reading. A second, weaker test concerns conflict: if a new level is a closure, the higher unit should collapse back into its parts when the coupling is broken, as obligate endosymbionts that lose their host die and cancers that escape the body's control end with it.

Theorems checked: T1 and T6 (genotype trit; computed, no meaning). T7.1 (transitions). T13: a closed lineage of genotypes read without a start is a necklace, but no evolutionary quantity counts necklaces. T15–T18: generation times and ploidy levels do not form equal-block ladders. T19–T22: no counterpart in evolution. T24: drift of an allele frequency is a one-dimensional walk, recurrent in Pólya's sense, so the lever is not defined.

Consequences. Evolutionary biology already holds the scale rule of the theory, in the form of the major transitions. The theory adds an arithmetic name for it: the carry of a closed ring. That name is consistent with the field and adds one constraint, the test above: a new level needs a closed cycle of dependence among its parts, not a hierarchy. The constraint has content. Endosymbiosis, eusociality and multicellularity all show mutual dependence; a case without it would be informative.

Beyond the transitions, the theory has little to say about evolution. It does not predict mutation rates, the shape of fitness trajectories, or the timing of LUCA. The LTEE's power-law gains, the citrate innovation and the evolution of mutator strains are facts of

population genetics with no register structure. Polyploidy shows the meeting rule (pairing) clearly but belongs to genetics and breeding (Chapter 13). The handedness of life is chemistry. The Red Queen and the lizard cycle are rings in the plain sense, frequency-dependent selection that comes back round; the lizard cycle is a ring of three, which Chapter 11 grades as a published structure. In evolutionary biology proper they add the observation that selection need not point in one direction: a population can circulate among states without progressing, which is closure in time rather than in space.

The genotype trit has one practical use. Quantitative genetics already codes genotypes as −1, 0, +1 (Falconer's −a, d, +a with d = 0), and the balanced form gives mirror symmetry for free (T12.1): swapping the two alleles negates every score. That is a convenience the field has used since Fisher, and the theory explains why it is natural: the balanced digits need no sign convention and no rounding convention (Part III.2 of the Foundations). It changes no result. The honest result is that the theory explains one thing about evolution, the stacking of levels, in the field's own terms, and nothing else. Even that one thing is qualitative: the theory does not say how many levels there should be, nor when each transition should occur.

Grade. A for the major transitions as closure becoming passage (Maynard Smith and Szathmáry 1995); B for the coupled-cycle test; C for the rest of evolutionary biology.

12.2 Developmental biology

History. Aristotle described the chick embryo forming day by day and argued for epigenesis, the gradual building of form. The 17th and 18th centuries argued instead for preformation, the idea that the adult is present in miniature in the germ. Caspar Friedrich Wolff's *Theoria generationis* (1759) and Karl Ernst von Baer's work on germ layers (1828) returned the field to epigenesis. Wilhelm Roux in 1888 killed one cell of a two-cell frog embryo and got half an embryo; Hans Driesch in 1891 separated sea-urchin blastomeres and got whole larvae. The difference set the agenda: how much is a cell's fate set inside it, and how much by its neighbours?

Hans Spemann and Hilde Mangold answered part of it in 1924. Grafting the dorsal lip of one newt embryo into another induced a second body axis: the organiser. Spemann received the 1935 Nobel Prize. Alan Turing proposed in 1952 that two diffusing chemicals that react can form spots and stripes from uniformity, "The chemical basis of morphogenesis" (*Phil. Trans. R. Soc. B* 237, 37–72). Lewis Wolpert's positional information (1969) proposed that cells read their position from gradients.

Genetics opened the black box in 1980, when Christiane Nüsslein-Volhard and Eric Wieschaus screened for mutations that disrupt the segment pattern of the *Drosophila* larva; with Edward Lewis, who had studied the homeotic genes, they received the 1995 Nobel Prize. Homeobox genes turned out to be shared from flies to mammals. In 1982 John Newport and Marc Kirschner showed that *Xenopus* embryos pass 12 rapid synchronous cleavages before slower, asynchronous divisions and the onset of zygotic transcription, the mid-blastula transition (*Cell* 30, 675–686; BioNumbers). Twelve doublings give $2^{12} = 4{,}096$ cells. The transition also marks the point where cell cycles

lengthen, acquire gap phases and lose their synchrony, so it is a change of regime, not only of transcription.

Time became a developmental variable through clocks. Olivier Pourquié and colleagues reported in 1997 that the gene *c-hairy1* oscillates in the presomitic mesoderm of the chick, one cycle per somite: the segmentation clock. In 1993 Victor Ambros's and Gary Ruvkun's groups found that the small RNA lin-4 controls developmental timing in *C. elegans*, the first microRNA. The open questions concern how embryos measure time and size, and how clocks of different speed produce the same body plan. The clock-and-wavefront model, proposed by Jonathan Cooke and Christopher Zeeman in 1976, had predicted such an oscillator two decades before it was found: a clock in each cell, and a front sweeping down the body that freezes the clock's phase into a boundary.

Current state. The segmentation clock is now compared across species. A 2024 tutorial review lists periods of about 30 minutes in zebrafish, 90 minutes in chick, 2 hours in mouse and 5 hours in human, and 150 minutes in rabbit and 390 minutes in marmoset from stem-cell studies; the mouse forms about 65 somites and snakes about 315 (arXiv:2403.00457). Experiments with stem-cell-derived presomitic mesoderm showed that the period difference between human and mouse is cell-autonomous and comes from different biochemical reaction speeds (Matsuda and colleagues, *Science* 369, 1450, 2020; ScienceDaily). Swapping the *HES7* gene between species did not change the period; human cells degraded the HES7 protein more slowly and also transcribed, spliced and translated more slowly, a property the authors call developmental allochrony (same source). The clock rests on delayed negative feedback: a transcription factor of the Hes family represses its own gene, and the delay of transcription and translation sets the period (same review).

The 2024 Nobel Prize in Physiology or Medicine went to Ambros and Ruvkun "for the discovery of microRNA and its role in post-transcriptional gene regulation" (Nobel Prize), work that began as a question of developmental timing. The 2026 prize for optogenetics (Khaleej Times) gives developmental biologists a way to drive signalling with light, and so to test clocks by forcing them. Organoids and synthetic embryo models (gastruloids, somitoids) now reproduce segmentation in a dish. The debates of 2026 concern how far stem-cell embryo models can go before ethical limits apply, and whether developmental time is set by a central clock or by local reaction rates. The allochrony result points to the second answer for the segmentation clock: the same gene network runs at different speeds because the cells' basic chemistry runs at different speeds. If that generalises, developmental tempo is a state variable of the cell, which is the reading N4 gives to every response of a medium: a state function, not a universal constant (T23's hypothesis).

Interpretation.

1. *Strand.* I is developmental time; P is the set of cell states. For the clock, the strand passes the same expression state once per cycle; for cleavage, each division is a passage.

2. *Passage and trit.* In the segmentation clock a cell is in the high phase (+1), the low phase (−1) or in transition (0). In cleavage, each division doubles the cell count: a merge on the ×2 ladder read in reverse, one unit becoming two.
3. *Closure.* A somite closes when one clock cycle is complete and the wavefront has passed: one cycle, one segment. The Hes oscillator is a single repressor ring with delay, not a ring of three. The theory's least oscillating ring (three repressors) does not describe it; a delay supplies what the extra elements supply in a repressilator.
4. *Scale.* Ladder check (Calculation 12.2). The *Xenopus* cleavage count lies on the ladder: $12 = 2^2 \cdot 3$ doublings and $4{,}096 = 2^{12}$ cells. Twelve is also the fourth closure depth of T17. The segmentation periods, in minutes, carry no meaning on the ladder because the unit is arbitrary; their ratios to the mouse period are 1/4 (zebrafish) and 3/4 (chick), which lie on the ladder, and 5/4 (rabbit), 5/2 (human) and 13/4 (marmoset), which do not. The somite counts 65 and 315 do not lie on it.
5. *Test.* Stated before data: the coincidence that *Xenopus* reaches the mid-blastula transition after 12 cleavages, a closure depth, is tested by other species. If the number of synchronous cleavages before the mid-blastula transition falls on closure depths (1, 2, 5, 12, 41) across species more often than chance, the reading has content; if it varies freely (for example 10 or 11 in other animals), the 12 is a coincidence and the reading is refuted. The field's own explanation, a threshold in the ratio of nuclear to cytoplasmic volume, predicts free variation with egg size.

Theorems checked: T7 (meeting at division), T15 (cleavage ladder), T17 (closure depth 12) above. T8–T10 (rings that close everywhere, fixed points of the rewrite) might describe a travelling wave of clock phase in the presomitic mesoderm, but no measurement maps a phase pattern onto trit rings. T11 might describe the boundary between two somites, a joint between two closed units; the boundary is real but no tension measure is published. T21, the 4π return, has a counterpart in left–right asymmetry only if cilia-driven flow is read as an attached rotor, which is not established. T23–T24 have no counterpart.

Consequences. Developmental biology holds clocks and counts, but not the structures of the theory. The segmentation clock is a delayed single-gene loop, the cleavage count a series of doublings whose stopping point is set by a volume ratio. The theory can name both, the ring and the ×2 ladder, without adding anything a developmental biologist would use. The one sharp question it raises is whether the 12 synchronous cleavages of *Xenopus* are tied to the closure depth 12 or merely coincide with it. The field's explanation is physical and makes the opposite prediction: the number of cleavages should scale with egg size. If cleavage counts track egg size and not closure depths, the theory's reading is wrong here, and that would be the expected outcome. The allochrony finding fits a different part of the theory. If the clock period is set by degradation and expression rates that differ between species, the period is a state function of the cell's chemistry, as N4 reads every coupling. That is a reading, consistent with the data but not derived from them, and T23's explicit energy density has no counterpart in a gene network.

What developmental biology does share with the theory is the idea that form is built by closure at successive scales: a somite is a closed unit made from a cycle, and segments

make a body. That is the scale rule again, and in this field it is an image rather than a law. The microRNA prize of 2024 confirms that timing is regulated, not that it is quantised. If the theory is to add something here, it will be through the ratio data: if more species' clocks fall on simple 2^a·3^b ratios to each other than chance allows, that would be a new observation. With five ratios, two on and three off, there is no such signal. The ratios are themselves approximate (the review gives "about" 30 minutes, 90 minutes and so on), so even the two on the ladder could move off it with better measurements.

Grade. C for clocks and counts read on the ladder (two of five period ratios on it); B for the cleavage-count test.

12.3 Palaeontology

History. Georges Cuvier established extinction as a fact in 1796 by showing that the mammoth and the mastodon differed from living elephants. His catastrophism read the strata as a series of revolutions. William Smith's geological map of 1815 used fossils to correlate rock layers across England. Charles Lyell's *Principles of Geology* (1830–33) argued for gradual change, and Darwin took his view of time from Lyell. Richard Owen named the Dinosauria in 1842. *Archaeopteryx* (1861) became the first celebrated transitional fossil, and Charles Walcott found the Burgess Shale in 1909.

Quantitative palaeobiology began in the 1970s. Jack Sepkoski compiled the ranges of marine families and genera through the Phanerozoic. In 1980 Luis and Walter Alvarez and colleagues proposed that an asteroid impact ended the Cretaceous, from an iridium layer at the boundary; the Chicxulub crater was identified in 1991. David Raup and Sepkoski identified the "big five" mass extinctions in 1982 (*Science* 215, 1501–1503) and in 1984 proposed that extinctions recur about every 26 million years (*PNAS* 81, 801–805; Wikipedia, "David M. Raup", secondary). The periodicity claim prompted searches for a cause, including a hypothetical companion star, and much statistical dispute.

Niles Eldredge and Stephen Jay Gould proposed punctuated equilibrium in 1972: species are stable for long periods and change in short bursts. Molecular clocks from the 1960s, beginning with Emile Zuckerkandl and Linus Pauling, offered dates independent of fossils, and the two kinds of evidence were reconciled only slowly. The open questions concern the causes of mass extinctions, the tempo of change, and the early record of life.

The early record itself was rewritten twice. Reg Sprigg found soft-bodied fossils in the Ediacara Hills of South Australia in 1946, older than the Cambrian, and Martin Glaessner showed in the 1950s and 1960s that they were Precambrian animals or animal-like organisms. Harry Whittington, Derek Briggs and Simon Conway Morris re-examined Walcott's Burgess Shale collections from the 1970s and found body plans that fit no living phylum; several were later placed in stem groups of living phyla, and the debate over how strange they were became part of the history; Gould's *Wonderful Life* (1989) made them the emblem of contingency. Exceptional deposits in Chengjiang, China, found in 1984, pushed the same diversity earlier in the Cambrian. The record of

the first eukaryotes and the first multicellular organisms thus became the place where palaeontology meets the major transitions of Section 12.1. The question that the Ediacaran and Cambrian faunas leave is whether the burst of animal body plans reflects a real acceleration of evolution or a change in what could be preserved, once animals acquired hard parts.

Current state. Estimates of the largest extinction have been revised. Steven Stanley, writing in *PNAS*, estimated that the end-Permian crisis eliminated about 81% of marine species, lower than the 88–96% claimed in 1979, which had combined two separate extinctions (University of Hawai‘i, SOEST). The date of LUCA, about 4.2 billion years ago (Section 12.1), pushes the start of the record back towards Earth’s formation.

New finds in 2026 extend the record of symbiosis and development. Among papers listed for 2026: a lichen-like Ediacaran fossil, *Wenganomyces primitivus*, from the Doushantuo Formation of China, offered as evidence of fungal–algal symbiosis before the Cambrian; a probable neonatal line in the Triassic cynodont *Chiniquodon theotonicus* from Argentina, suggesting live birth; evidence that hatchlings of Carboniferous stem tetrapods from Mazon Creek developed directly, without a larval stage; and a fungal bloom in North America just after the Chicxulub impact (Wikipedia, “2026 in paleontology”, secondary). The debates of 2026 concern how much the record’s gaps bias diversity curves, the role of volcanism against impacts at the end of the Cretaceous, and whether the 26-million-year periodicity survives modern dating. The 2025 literature shows where effort goes: much of it describes symbioses and parasites preserved in deep time, for example the earliest fungal biomineralisation in bone in the Triassic marine reptile *Keichousaurus*, fungi in insect-damaged Jurassic wood, and parasite-induced outgrowths on Cambrian echinoderms reported as the oldest parasitic symbiosis on deuterostomes (Wikipedia, “2025 in paleontology”, secondary). Interactions between organisms, not only the organisms, are now fossil data.

Other 2026 papers listed in the same compilation describe a Late Jurassic trace from the Morrison Formation of Utah interpreted as termite fungal farming, the oldest known crinoid tube feet from Ordovician rocks of Quebec, and an Ordovician medusozoan from the same region. Each extends a partnership or a body plan back in time. The field’s tools are also changing: molecular clocks calibrated with fossils, as in the LUCA estimate, now date nodes for which no fossil exists, and fossils are increasingly used as calibration points rather than as the sole record.

Interpretation.

1. *Strand.* I is geological time; P is the set of taxa present. A taxon’s range is the interval between its first and last appearance.
2. *Passage and trit.* In each interval a lineage originates (+1), goes extinct (−1) or persists (0).
3. *Closure.* A clade closes when its originations and extinctions balance over an interval, leaving its diversity unchanged; open intervals carry a net change. A mass extinction is a large open digit, a loss that is passed to the next interval as empty ecological space.

4. *Scale.* Ladder check (Calculation 12.2): the claimed 26-million-year period does not lie on 2^a·3^b in millions of years, and the unit is arbitrary in any case. No published periodicity in the fossil record matches a closure depth. The 2026 find of an Ediacaran lichen-like fossil is an early case of a symbiotic pair merging into one organism, the scale rule of Section 12.1.
5. *Test.* Stated before data: the theory gives no reason for a fixed extinction period. If a periodicity were confirmed, the theory's reading would be refuted only if the period's ratios to other geological cycles (orbital cycles of 405,000 years, for example) failed to lie on the ladder while the theory claimed a link. Since the theory makes no such claim, no test with content is available for periodicity. A ratio test is stated instead for the record of origination and extinction: if the theory's closure has meaning in deep time, the ratio of origination to extinction counts per stage should cluster near 1 (closed) outside mass-extinction intervals and depart from it only at the big five. Diversity curves already show long plateaus broken by crises, which the field explains by diversity-dependent rates without any appeal to closure; the test would add content only if the clustering near 1 were tighter than diversity dependence predicts. The available test is the transition rule of Section 12.1: the fossil record of every new level of individuality (eukaryotes, multicellular animals, colonial organisms) should show coupled partners before integration.

Theorems checked: T1 (origination and extinction as signed counts); T15 and T17 (no fit for periods); T7.1 (fossil symbioses). T13, T14, T19–T24 have no counterpart.

Consequences. Palaeontology contains no structure of the Foundations. Its numbers, extinction intensities, durations and diversity counts, are set by geology and sampling. The theory's closure language describes a stable clade as a balanced one and a mass extinction as a large loss, which is what the field already says. Calling it "an open digit" adds nothing.

The cost of trying to force more is visible in the periodicity debate. A theory built on the ladders 2^a·3^b would be tempted to look for extinction periods that lie on them, or for ratios between geological cycles that do. The 26-million-year period does not lie on the ladder, and its reality is disputed; reading it into the theory would bend a doubtful number to fit a framework. The right result is a plain negative: palaeontology's periods, where they exist, are set by geology and astronomy, and the theory does not reach them. The theory's checklist confirms this. Of its theorems, only the scale rule (T7.1) touches palaeontology, through the fossil record of partnerships. The register theorems (T2–T6, T8–T14) have nothing to count in strata, the ladders (T15–T18) find no periods to fit, and the topological and coupling theorems (T19–T24) have no fossil counterpart. A chapter that found more would be finding what it brought. The grade records that outcome rather than hiding it.

The one link with content runs through symbiosis. The 2026 report of a lichen-like Ediacaran fossil, if confirmed, would be an early instance of two partners merging into one functional individual, the pattern that Maynard Smith and Szathmáry describe and that the theory reads as closure becoming passage. Palaeontology can test that reading in its own way: if fossils of the first eukaryotes, the first animals and the first colonies always show a period of partnership before integration, the transition rule gains

support from deep time. If integration appears abruptly, with no record of partnership, the rule loses one of its supports, though a gap in the record would be the likelier explanation.

Grade. C (no structure; the 26-million-year period is off the ladder and the theory makes no claim about it).

12.4 Population dynamics

History. Thomas Malthus argued in 1798 that populations grow geometrically while food grows arithmetically. Pierre-François Verhulst wrote the logistic equation in 1838, in which growth slows as a population approaches a ceiling. Raymond Pearl revived it in the 1920s. Lotka (1925) and Volterra (1926) wrote coupled equations for predators and prey (Chapter 11). Alexander Nicholson's blowfly cultures in the 1950s showed sustained oscillations driven by delayed density dependence. William Ricker (1954) and Ray Beverton and Sidney Holt (1957) wrote the stock–recruitment models of fisheries (Chapter 13).

Chaos entered ecology through a simple equation. In 1976 Robert May reviewed the discrete logistic map $x' = r\,x\,(1 - x)$ in *Nature* (261, 459–467) and showed that it passes from a stable point to a cycle of period 2, then 4, 8 and so on, and then to chaos, with windows of order inside the chaos. A year earlier, Tien-Yien Li and James Yorke had proved that if an interval map has a point of period three, it has points of every period and an uncountable set of chaotic orbits: "Period three implies chaos" (*American Mathematical Monthly* 82, 985–992; PDF). Unknown to them, Oleksandr Sharkovskii had published a stronger result in 1964: an ordering of the positive integers, 3, 5, 7, … then 2·3, 2·5, … then 4·3, 4·5, … and finally the powers of two in decreasing order, such that a period forces every period after it (Wikipedia, "Sharkovskii's theorem", secondary). Mitchell Feigenbaum found in 1978 that the period-doubling parameters converge geometrically with a universal ratio, 4.669.

Testing these ideas in real populations was hard. In the 1990s Robert Costantino, Robert Desharnais, Jim Cushing and Brian Dennis manipulated flour-beetle (*Tribolium*) cultures and compared them with a stage-structured model. They reported in 1997 that the populations moved "from equilibrium to quasiperiodic cycles to chaos" as the model predicted (*Science* 275, 389–391; Arizona record). In epidemiology, David Earn, Pejman Rohani, Ben Bolker and Bryan Grenfell (2000) showed that measles in London had biennial cycles from 1950 to 1968, annual cycles earlier when birth rates were high, and irregular epidemics after vaccination, all as bifurcations of one model (Earn et al. 2000, PDF). The open question is how often natural populations are chaotic rather than noisy. A second open question is why so few natural populations show the period-doubling route at all: most cycles in nature have long periods set by predator–prey interaction, not by overcompensation. Olsen and Schaffer (1990) argued that measles in some cities shows low-dimensional chaos rather than noisy periodicity; Earn and colleagues' model later explained the same records as transitions between periodic attractors. The question of chaos in nature has stayed open because natural time series are short and noisy.

Current state. The *Tribolium* work remains the most detailed test. In the experiment reported by Dennis and colleagues (2001, *Ecological Monographs* 71, 277–303), adult mortality was fixed at 0.96 and the rate of adult cannibalism on pupae, c_pa, was set by the experimenters. The fitted model predicted an invariant loop resembling a 2-cycle at c_pa = 0, chaos at 0.05, a 26-cycle at 0.10, an 8-cycle at 0.25, chaos at 0.35, a 3-cycle at 0.50 and a 6-cycle at 1.00; at c_pa = 0.35, 83.5% of bootstrapped parameter sets gave chaotic attractors (Dennis et al. 2001, PDF). An earlier series varied adult mortality and found stable 2-cycles at intermediate mortality (0.27 and 0.50) and aperiodic, loop-like behaviour at 0.96 (Dennis et al. 1997, PDF).

For measles, Earn's model gives, as transmission falls, "biennial, 3-, 4-, 5-, 6-, 7-, and 8-year cycles", which the authors call a bifurcation cascade (same PDF). Natural cycles remain under study: The chaos question also has a statistical side: methods that estimate Lyapunov exponents from short, noisy series, such as those Dennis and colleagues used, now allow the sign of the exponent to be estimated with confidence intervals rather than asserted. the snowshoe hare cycle of about ten years is driven mainly by predation (Krebs, Boonstra and Boutin 2018). Periodical cicadas emerge every 13 or 17 years; Brood XIII (17-year) and Brood XIX (13-year) emerged together in 2024 for the first time since 1803, and will do so again in 2245 (ScienceAlert); the interval is 13 × 17 = 221 years (Calculation 12.2). Global monitoring shows a 73% average decline of vertebrate populations from 1970 to 2020 (WWF).

Interpretation.

1. *Strand.* I is the sequence of generations or seasons; P is the set of population states. A cycle of period p is a strand that returns to the same place after p passages.
2. *Passage and trit.* Each generation's change in size is up (+1), down (−1) or none (0). A 2-cycle alternates (+1, −1): a meeting. A 3-cycle in which the population rises twice and falls once is open in the trit sense, but returns in state: the theory's ring is a return in place, not a sum of signs.
3. *Closure.* A periodic orbit is a closed loop; chaos is a strand that never closes. Period doubling is the meeting ladder: each bifurcation replaces a loop by one that must go round twice, ×2. The logistic map shows period 1, 2, 4, 8, 16 at r = 2.9, 3.2, 3.5, 3.55 and 3.566; the superstable parameters give ratios 4.709, 4.681, 4.663 and 4.668, converging on Feigenbaum's 4.669; and the period-3 window opens at r = 1 + √8 = 3.82843 (Calculation 12.1). The ring enters through Li–Yorke: a closed orbit of period three forces closed orbits of every period.
4. *Scale.* Sharkovskii's ordering puts 3 first and the powers of 2 last (Calculation 12.1). The two ladders of T15, rings (×3, odd) and meetings (×2), stand at the two ends of the published order of forcing: the ring of three forces everything; a pure meeting ladder forces only smaller meetings. Ladder check of observed and predicted periods (Calculation 12.2): the *Tribolium* cycles 2, 3, 6 and 8 lie on the ladder; the 26-cycle does not. The measles cascade contains 2, 3, 4, 6 and 8 (on) and 5 and 7 (off). Hare (10) and cicada (13, 17) periods are off.
5. *Test.* Stated before data in the first edition: recurrent epidemic periods should lie on 2^a·3^b. That test now fails at the level of models: Earn's cascade includes 5- and 7-year cycles. The remaining test: in a population driven along a single parameter,

the order in which periods appear must respect Sharkovskii's order (3 is the last period to appear before full chaos). For one-dimensional (interval) maps this order is a theorem, so the test has content only for populations whose dynamics are not one-dimensional, such as the stage-structured beetles: there, an observed 3-cycle whose parameter neighbourhood contains no cycles of the periods that Sharkovskii's order places after 3 (5, 7, 6, 10, …, 8, 4, 2) refutes the reading that the ring of three forces all periods in living populations.

Theorems checked: T7 (2-cycle as meeting), T15 (period doubling), T16 (the two ladders never meet: a pure doubling cascade never produces period 3; period 3 appears only after chaos), T17 (no closure depth appears as a period). T24 does not apply.

Consequences. Population dynamics contains two published structures that the theory names: the ×2 ladder of period doubling and the special place of period three. Both are mathematics of interval maps, proved without the theory. The theory gains from them a clear statement of its two ladders in a field where they are observed. The *Tribolium* experiments realised 2-, 3-, 6- and 8-cycles and chaos in a living population, and the order of their appearance follows the bifurcation structure.

The cost is the loss of a claim. The first edition proposed that epidemic periods lie on 2^a·3^b; the measles cascade contains 5 and 7, and natural cycles such as the hare's ten years and the cicadas' 13 and 17 do not lie on the ladder. The cicadas are instructive in another way: their prime periods avoid meeting other cycles, so that two broods coincide only every 221 years. That is the logic of T16, that two ladders never meet exactly, applied to two prime periods; but the field's own explanation, avoidance of predators and of hybridisation, is ecological, and the theory adds no reason why the periods are 13 and 17. Ricker's stock–recruitment map shows the same ×2 ladder as the logistic map, while the Beverton–Holt map never oscillates (Chapter 13, Calculation 13.2); whether a population climbs the ladder therefore depends on how strongly it overcompensates, which is biology, not register arithmetic. The *Tribolium* table makes the same point in numbers: the experimenters chose the cannibalism rate, and the predicted period jumped from 26 to 8 to chaos to 3 to 6 as they did. The periods on the ladder are there because the bifurcation structure of the model puts them there, and the 26-cycle shows that off-ladder periods appear just as readily.

Grade. A for period doubling (May 1976; Feigenbaum 1978) and for period three as the ring that forces all periods (Li and Yorke 1975; Sharkovskii 1964); C (refuted as formulated) for epidemic periods on the ladder; B for the ordering test in multidimensional populations.

12.5 Botany (plant development)

History. The arrangement of leaves on a stem, phyllotaxis, was studied as geometry long before it was studied as growth. Charles Bonnet described spiral leaf arrangements in 1754. Karl Friedrich Schimper and Alexander Braun described the fractions of a turn between successive leaves in the 1830s. Auguste and Louis Bravais connected phyllotaxis ratios to the Fibonacci sequence in 1837, and in 1868 Wilhelm

Hofmeister proposed that each new leaf forms in the largest gap left by the previous ones (Wikipedia, "Phyllotaxis", secondary). The divergence angle in many plants approaches the golden angle, 137.51° (Calculation 12.3).

Plant morphology developed alongside. Goethe's *Metamorphosis of Plants* (1790) proposed that flower parts are modified leaves. The division of flowering plants into monocots and dicots, by the number of seed leaves, goes back to John Ray in the 1680s; monocot flowers typically have parts in threes, eudicot flowers in fours or fives (Wikipedia, "Monocotyledon", secondary). Wilhelm Pfeffer and others studied plant movements; Erwin Bünning proposed in the 1930s that plants have an internal daily clock.

Plant development became genetic with *Arabidopsis thaliana*, adopted as a model in the 1980s. George Haughn and Chris Somerville in 1988, and Enrico Coen and Elliot Meyerowitz in 1991, formulated the ABC model of flower development: three classes of genes, A, B and C, specify four organ types in four whorls; A alone gives sepals, A with B petals, B with C stamens, C alone carpels (Wikipedia, "ABC model of flower development", secondary). Stéphane Douady and Yves Couder showed in 1992 that Fibonacci spirals arise in a purely physical system of repelling drops added at a rate to a growing centre (*Phys. Rev. Lett.* 68, 2098–2101). Auxin transport by PIN proteins was then identified as the biological mechanism that places new primordia. The open questions concern how clocks, hormones and mechanics combine into form. Experiments on the clock gave it agricultural weight: Antony Dodd and colleagues showed in 2005 that *Arabidopsis* plants whose internal clock period matches the 24-hour day fix more carbon, grow more and outcompete mismatched plants (*Science* 309, 630–633).

Current state. Phyllotaxis has been surveyed in the field. The survey was planned in honour of Alan Turing, who had studied sunflower spirals in his last years. Jonathan Swinton and Erinma Ochu's citizen-science study of sunflowers, published in *Royal Society Open Science* on 18 May 2016, analysed over 500 seed heads (University of Manchester). Of 768 spiral families counted, 74% had Fibonacci counts and 66.6% of clockwise and anticlockwise pairs were consecutive Fibonacci numbers; the remainder included Lucas and other generalised sequences (Plus Magazine, secondary). The study confirmed Fibonacci dominance and catalogued the exceptions.

The plant clock is now a modelled circuit. Pokhilko and colleagues (2012) found that the *Arabidopsis* clock gene circuit "includes a three-component repressilator circuit in its complex structure", alongside other feedback loops, with TOC1 repressing the morning genes LHY and CCA1 (Warwick repository; *Mol. Syst. Biol.* 8, 574). The carbon cycle that the clock regulates, the Calvin cycle, needs three turns to export one three-carbon sugar (Chapter 11, Calculation 11.2). Polyploidy is common in plants: about 15% of angiosperm speciation events involve it (Wikipedia, "Polyploidy", secondary). Debates in 2026 concern the mechanics of primordium placement, the role of the clock in crop yield, and the use of gene editing to change plant architecture (Chapter 13). The ABC model has been extended with an E function, needed in all four whorls, and a D function for ovule identity; all of these genes encode MADS-box transcription factors (Wikipedia, "ABC model of flower development", secondary). Flower identity is thus a

combinatorial code of a few gene classes, and the combinations used are a small subset of those possible. The clock, the flower code and the phyllotactic spiral are now studied together in the shoot apical meristem, where hormone transport, gene expression and tissue mechanics are imaged live in the same cells.

Interpretation.

1. *Strand.* For phyllotaxis, I is the sequence of primordia in order of formation; P is the set of angular positions on the meristem. For the clock, I is time and P the set of gene-expression states.
2. *Passage and trit.* In the clock, each gene is high (+1), low (−1) or in transition (0), as in Chapter 10. In phyllotaxis, each new primordium is placed at a fixed angle after the previous one.
3. *Closure.* The repressilator in the plant clock is a published ring of three repressors, the least ring that oscillates (Calculation 10.6). Phyllotaxis is the opposite of closure: the golden angle is the angle that never closes, because its ratio to a full turn is the "most irrational" number, and successive primordia never fall on the same radius. The Fibonacci fractions 1/2, 1/3, 2/5, 3/8, 5/13, 8/21 are its best rational approximations (Calculation 12.3). This is the theory's T16–T17 structure, a ratio that never closes and its record approximations, but for a different number: the golden ratio, not $\log_2 3$.
4. *Scale.* Ladder check (Calculations 12.3 and 12.4): among the Fibonacci numbers up to the 300th, only 1, 2, 3, 8 and 144 lie on the ladder 2^a·3^b. The closure depths of T17 (1, 2, 5, 12, 41, 53, …) and the Fibonacci numbers share only 2 and 5 above 1. Monocot flower parts in threes and sixes lie on the ladder; eudicot parts in fives do not. The ABC model uses three gene classes to specify four organs, through the combinations A, AB, BC and C.
5. *Test.* Stated before data: the theory says nothing that selects the golden ratio over $\log_2 3$. If phyllotactic spiral counts in a large survey showed a preference for the closure depths 12, 41 or 53 over neighbouring Fibonacci numbers, the theory would gain a link; in the Swinton–Ochu data, the dominance of Fibonacci counts (74% of families) refutes any such link. For the clock: a plant clock model in which the three-repressor loop can be removed without loss of oscillation in the absence of other loops would weaken the ring reading.

Theorems checked: T4 and T7 (clock states); T15, T16, T17 (phyllotaxis and ladder: no fit); T6 (no use); T13 (whorls are closed configurations read without a start, but no plant count uses necklaces); T19–T24 (no counterpart; twining plants are chiral, but their handedness is not the trefoil's).

Consequences. Botany holds one structure of the Foundations, the repressilator in the plant clock, and one striking non-structure, phyllotaxis. The clock's ring of three is published and independent of the theory; the theory names it as the least ring that oscillates. That adds a design principle for crop engineering of clock genes: a three-repressor core is the minimal oscillating core, and attempts to simplify the clock below it should fail.

Phyllotaxis is a lesson in limits. It is the most famous number pattern in biology, and it is generated by the same mathematics the theory uses for the comma: a ratio that never closes and the record approximations to it. Douady and Couder's experiment shows that no genetic programme is needed to produce it: repulsion between successive elements added at a steady rate is enough. But the ratio is the golden one, and the record denominators are Fibonacci numbers, not closure depths. The theory, which rests on the ladders of 2 and 3, does not reach it. Reading Fibonacci phyllotaxis as confirmation would be the re-description that rule 1 forbids. Trimerous monocot flowers are a count on the ladder, and so are many other counts; without a mechanism that makes three special in a monocot meristem, the count is not evidence.

The ABC model is another place where a reading is easy and a structure is absent. Three gene classes give $2^3 = 8$ on/off combinations, of which four are used; A and C repress each other, which forbids AC, ABC and the empty set in a wild-type flower. One could call A and C a meeting, two opposed functions that exclude each other, and B the element that modulates both. But the model's logic is Boolean, two-valued, and nothing in it uses a third value or closure mod 3. The clock-resonance result of Dodd and colleagues fits the theory's language better, a closure between an inner and an outer cycle, but it is equally well described as entrainment, and no test distinguishes the two readings.

Polyploidy, finally, is where botany meets the meeting rule most directly. Plants tolerate whole-genome doubling far better than animals, and about 15% of angiosperm speciation involves it. Even ploidies pair and stay fertile; odd ploidies do not (Chapter 13, Calculation 13.1). That is published genetics, and it is the ×2 side of the register: persistence through meiosis requires that every chromosome meet a partner. The theory names the rule; plant breeders use it, and Chapter 13 grades it there. In botany proper it is a rule of chromosome pairing, not of plant form, and it does not raise the grade for development.

Grade. A for the three-component repressilator in the plant clock (Pokhilko et al. 2012); C for phyllotaxis (Fibonacci, not the ladder; refuted as a link) and for floral counts.

Chapter 13 — Cultivating Life

This chapter turns to the practical sciences of life: the ways people make living systems close for their own use, in fields, forests, fisheries and fermenters. The clearest structure is a rule of meetings. Meiosis pairs homologous chromosomes; a triploid organism cannot pair all three copies, and a balanced gamete arises only with probability $2\cdot(1/2)^n$ for n chromosomes per set, about 1 in 1,024 for the banana and the watermelon (Calculation 13.1). Breeders use that failure of closure deliberately to make seedless fruit and sterile oysters. Fisheries science holds a second structure: Ricker's stock–recruitment map climbs the ×2 ladder of period doubling to chaos, while the Beverton–Holt map never oscillates (Calculation 13.2). Synthetic biology uses the meeting–ring split as a design rule: two repressors make a switch, three make a clock (Calculation 10.6). Agriculture, fisheries and aquaculture, and biotechnology earn A for these structures, with B for stated tests; forestry earns C. The three-field rotation is a ring of three only as a reading, and the gains it brought are explained by legumes and fallow, not by closure.

13.1 Agriculture

History. Agriculture began independently in several regions after the last glaciation: wheat, barley and pulses in the Fertile Crescent about 11,000 years ago, rice and millet in China, maize, beans and squash in Mesoamerica, and other crops elsewhere. Domestication selected plants that kept their seeds on the stalk and animals that tolerated people. Fertility was the limiting problem from the start. Fallow, manure and the rotation of crops were the tools.

In medieval Europe the two-field system, half the land fallow each year, gave way between the 9th and 11th centuries to the three-field system: one field in winter grain, one in spring crops or legumes such as peas and beans, one fallow, rotated each year. Each field was then cropped two years in three instead of one in two (Wikipedia, "Three-field system", secondary). The cropped share rose from 1/2 to 2/3, a gain of one third (Calculation 13.3). The Norfolk four-course rotation of the 18th century (wheat, turnips, barley, clover) removed fallow altogether, using clover to fix nitrogen and turnips to feed livestock through winter.

Science entered the field in the 19th century. Justus von Liebig's *Organic Chemistry in its Application to Agriculture and Physiology* (1840) argued that plants take mineral nutrients from the soil and that growth is limited by the scarcest one. In 1843 John Bennet Lawes and Joseph Henry Gilbert began the Broadbalk wheat experiment at Rothamsted, now the oldest continuing field experiment in the world (Rothamsted Research). Mendel's peas (1866, Chapter 10) gave breeders a theory after 1900. Fritz Haber demonstrated ammonia synthesis in 1909 and Carl Bosch scaled it up; BASF's Oppau plant made ammonia industrially from 1913 (Wikipedia, "Haber process", secondary). Nitrogen fertiliser removed the old limit.

Two lines of genetic technique matter for this chapter. Polyploid breeding exploited the fact that plants tolerate extra chromosome sets. Seedless watermelons are grown from seed made by crossing tetraploid and diploid lines; the triploid plants are sterile and need diploid pollinators nearby to set fruit (Wikipedia, “Seedless fruit”, secondary). Dessert bananas are triploid clones, AAA, propagated vegetatively. The second line is the sterile insect technique. Edward Knipling proposed in the 1930s and 1950s that releasing sterilised males would collapse a pest population, and in 1954 screwworm flies were eradicated from the island of Curaçao in seven weeks (Wikipedia, “Sterile insect technique”, secondary). The Green Revolution of the 1960s, with semi-dwarf wheat and rice, completed the century. The questions left were how to sustain yields without degrading soils, and how to feed a growing population without clearing more land. Long-term experiments became the instrument for the first question: Broadbalk and its sister experiments at Rothamsted, the Morrow plots in Illinois from 1876, and the Askov trials in Denmark from 1894 record what happens to yields and soils over a century under fixed treatments.

Current state. World cereal production reached a record 3,023 million tonnes in 2025, according to the FAO, with rice at a historic 561.6 million tonnes and the stocks-to-use ratio at 31.8%, the highest since 2001 (Logos Press, secondary report of FAO). Hunger fell for a third year: about 645 million people, 7.8% of the world’s population, faced hunger in 2025, down from 8.1% in 2024; Africa, with about 309 million, overtook Asia, with about 292 million, as the region with the most hungry people (UN News, July 2026).

Genetically modified crops are grown in 31 countries; the United States leads with 79.9 million hectares, and soybean alone covers more than 100 million hectares worldwide (ISAAA report, secondary). The Broadbalk review published in 2024 found that wheat yields were maintained or increased over almost 200 years with either mineral fertiliser or manure; best yields now exceed 12 tonnes per hectare in some years, and manure applied for many years roughly tripled soil organic carbon on some plots (Rothamsted Research).

Pests and diseases set the frontier. Yields and hunger figures move together only loosely: a record harvest coexists with 645 million hungry people, because access, prices and conflict, not production, decide who eats. Fusarium wilt Tropical Race 4 was confirmed in Colombia in August 2019 on 175 hectares in La Guajira, the first report in Latin America, and in Peru in 2021 (ProMusa); Cavendish bananas, triploid and seedless, account for about 99% of banana exports to developed countries (Wikipedia, “Cavendish banana”, secondary). The New World screwworm returned to southern Mexico in December 2024; the first Texas case was confirmed on 3 June 2026, and by 23 August 2026 there were 45 confirmed cases in 16 Texas counties and one in New Mexico. The Panama facility that maintains the barrier releases up to 100 million sterile flies a week (Wikipedia, “Cochliomyia hominivorax”, secondary).

Interpretation.

1. *Strand.* For breeding, I is the sequence of chromosomes through meiosis and P the set of homologous sets. For a field, I is the sequence of seasons and P the set of crops and soil states.
2. *Passage and trit.* In meiosis each chromosome must meet its homologue. For a triploid, at each chromosome the three copies segregate two to one pole and one to the other.
3. *Closure.* Pairing is a meeting in the sense of T7: two homologues, one from each parent, pair and separate. A triploid has three copies of each chromosome; they cannot all pair as meetings. A gamete is balanced (one full set or two) only if every chromosome sends its extra copy the same way, with probability $2\cdot(1/2)^n$. For the banana and the watermelon (x = 11) that is 1/1,024; for the oyster (x = 10), 1/512 (Calculation 13.1). Odd ploidy breaks the closure of meiosis; even ploidy restores it. By the digit rule, a tetraploid crossed with a diploid adds gametes of two sets and one set: 2 + 1 = 3, digit 0 and carry 1 (Calculation 13.1). The triploid is a closed ring as a count of sets, but it is the ring that cannot meet, so it cannot reproduce.
4. *Scale.* The sterile insect technique is the same rule at the scale of a population. A sterile male mates, a meeting at the scale of individuals, but carries nothing to the next generation; repeated releases drive the population down. Crop rotations: the two-field cycle is a meeting (crop, fallow), the three-field cycle a ring of three, and the Norfolk four-course cycle two meetings; their lengths 2, 3 and 4 lie on the ladder 2^a·3^b (Calculation 13.3), but so would most short cycles.
5. *Test.* Stated before data: if soil fertility under a rotation is a closure, then in long-term trials (Broadbalk, Askov, Morrow) yields should be stable under treatments whose nutrient balance over one full rotation is zero, and should drift under treatments with a persistent surplus or deficit. A treatment with a measured persistent deficit and stable yields over decades refutes the reading.

Theorems checked: T7 (pairing), T7.1 (sets as digits and carries) and T15 (rotation lengths) above. T6 gives no useful baseline: segregation in triploids is a ×2 process. T24 does not apply to pests or crops. T19–T22 have no counterpart. T23 could describe soil as a medium whose response to fertiliser depends on its state (diminishing returns at high nitrogen), the reading N4, but no energy density of the T23 form is published for soils.

Consequences. Agriculture uses the failure of closure as a tool. A triploid cannot pair its chromosomes and so cannot set seed; breeders accept the cost, clonal propagation and a narrow genetic base, for the gain of seedless fruit. The theory names the rule, meetings close and an odd set does not, and it is already published genetics. The banana shows the price. A crop that cannot reproduce sexually cannot be bred quickly for resistance, and Tropical Race 4 now threatens a trade built on a single clone. The theory adds no remedy; the remedy lies in genetics, through new hybrids from fertile diploids or gene editing.

The sterile insect technique shows the same rule as a weapon: a meeting without a carry. The return of the screwworm to Texas in 2026 is a reminder that the barrier

works only while sterile flies are released at sufficient rates. The rotation reading is weaker. The three-field system did crop two years in three, and the share of land in production rose by a third. But the gain came from legumes fixing nitrogen and from less fallow, not from a property of three. The Broadbalk test turns the reading into an accounting question: does a balanced nutrient budget over a full cycle predict stable yields? The data exist and the analysis can be done. It is worth stating what would count against the theory here, beyond the test: a robust finding that yields are stable under persistent nutrient deficits, mined from soil reserves for many decades, would show that closure over a rotation is not what keeps a field productive on human time scales.

Grade. A for odd-ploidy sterility as the failure of pairing (published genetics; seedless crops); B for the nutrient-balance test; C for rotations and the sterile insect technique read as closure.

13.2 Forestry

History. Forests were managed long before forestry was a science. Medieval European law set rights to wood, pasture and pannage in common forests, and royal forests reserved game and timber. The crisis that created forestry came with mining and smelting. In 1713 Hans Carl von Carlowitz, a Saxon mining administrator facing a shortage of timber for the silver mines, published *Sylvicultura oeconomica*. He argued that forests must be conserved and replanted so that wood can be cut continuously, “eine continuierliche beständige und nachhaltende Nutzung”, and his word *nachhaltend* became the root of “sustainability” (Wikipedia, “Hans Carl von Carlowitz”, secondary).

German forestry turned the idea into rules. Georg Ludwig Hartig and Heinrich Cotta founded forestry schools around 1800 and taught sustained yield: cut each year no more than grows. In 1849 Martin Faustmann derived the value of land under an infinite series of rotations, and with it the economically optimal rotation age; Bertil Ohlin solved the maximisation in 1921 (Wikipedia, “Faustmann’s formula”, secondary). The “normal forest” of the 19th century divided a forest into as many compartments as the rotation had years, cutting one each year, so that the whole forest closed as a cycle. Faustmann’s problem showed why the forester’s and the economist’s cycles differ. The forester’s rotation maximised the mean annual growth of wood; Faustmann’s rotation maximised the value of the land, counting the interest that the standing timber and the land itself could earn elsewhere. The optimal cut comes when the value growth of the stand equals the interest on the stand plus the interest on the land (same source). Because money has a time cost, the economic rotation is shorter than the biological one. Forestry thus had two closures from its start: a closure of wood, in which cutting equals growth, and a closure of value, in which the forest earns what capital earns. They do not coincide. Faustmann’s land value is the sum of an infinite series of identical rotations, each discounted: the forest as an endless ring of equal cycles. That is the image of the normal forest written as a price.

Forestry spread with empires. Dietrich Brandis organised forestry in British India from 1864. Gifford Pinchot founded the US Forest Service in 1905, and the conflict between

his utilitarian conservation and John Muir's preservation framed American policy. The 20th century brought plantation forestry, fast-growing pines and eucalypts, and the recognition that deforestation in the tropics was a global problem. The Rio conference of 1992 adopted the Forest Principles, and FAO began regular global forest resource assessments. The question left was how to keep forests as carbon stores, habitats and timber sources at once. Ecology complicated the picture further. Forest succession, studied by Frederic Clements and Henry Gleason in the early 20th century, showed that forests are not static stocks but sequences of communities, disturbed by fire, wind and insects and returning, sometimes, to a former state. A managed rotation imposes one cycle on a system that has its own.

Current state. The FAO's Global Forest Resources Assessment 2025 reports a forest area of 4.14 billion hectares, 32% of the world's land, or 0.5 hectares per person (FAO newsroom). Deforestation fell from 17.6 million hectares a year in 1990–2000 to 10.9 million hectares a year in 2015–2025, a reduction to 62% of the earlier rate (Calculation 13.3); net forest loss, after regrowth and planting, fell from 10.7 million to 4.12 million hectares a year (same source). Planted forests cover an estimated 312 million hectares, about 8% of the total, and have grown in every region since 1990, more slowly in the last decade (same source).

The debates of 2026 concern how far forests can absorb carbon as droughts, fires and pests increase; whether planted monocultures count as forest in the same sense as natural ones; and how certification and supply-chain rules, such as the European Union's deforestation regulation, change trade. Forest science increasingly uses remote sensing and lidar to measure biomass and change from space, so that the assessment's figures can be checked independently. The figures carry two different messages. The rate of clearing has fallen to 62% of its level in the 1990s; the net loss, which counts regrowth and planting, has fallen further, to about 39% of its 1990s level (4.12 against 10.7 million hectares a year; Calculation 13.3). Planting and natural expansion now offset most of the clearing in area, but a planted hectare and a cleared natural hectare differ in carbon, soil and species, so equal areas do not mean an equal forest.

Interpretation.

1. *Strand.* I is time in years; P is the set of stands or compartments of a forest. A rotation is the strand's return to the same stand at harvest.
2. *Passage and trit.* In each year a stand grows (+1), is cut (−1) or rests (0).
3. *Closure.* Sustained yield is closure over a rotation: cutting equals growth, and the forest returns to the same state after one full cycle. The normal forest is a ring of as many compartments as the rotation has years. Deforestation is an open digit: a net loss passed to the next period. The FRA figures show the digit shrinking (net loss 10.7 to 4.12 million hectares a year) but not closing.
4. *Scale.* Rotation ages are set by species and markets, typically decades; they are not formed by equal merges and are not constrained to the ladder 2^a·3^b. The ratio of deforestation rates, 0.619, has no ladder meaning (Calculation 13.3).

5. *Test.* Stated before data: the theory adds no quantitative claim to forestry. A test with content would require that forests managed for closure over a rotation (sustained yield) show lower long-run variance in standing stock than forests managed otherwise, at equal mean harvest; the forestry literature can decide that, and a forest with large swings under strict sustained yield would count against the reading. But the reading predicts nothing that sustained-yield theory does not already predict.

Further theorems were checked for forestry and found to add nothing. A normal forest of n compartments, cut one per year, is a closed configuration of n states read without a start, and T13 would count its rotation classes; but no forestry quantity depends on that count. The seam rule T11 might describe the edge between stands of different age, where wind damage concentrates; the edge effect is real, but it is explained by exposure, not by a mismatch of closed windows.

Theorems checked: T1 (cutting and growth as signed digits), T7 (rotation as ring), T15 (rotation ages: no fit). T23 (stiffness as a state function of density) might describe a forest's growth response as a function of stand density, but the forestry literature's yield–density curves are empirical and not of T23's form. T24, T19–T22 and T13–T14 have no counterpart.

Consequences. Forestry invented the closure idea under the name of sustainability three centuries ago. Carlowitz's continuous, enduring, sustained use is the condition that what is taken in a cycle equals what grows. The theory states the same thing in its own language, and that adds nothing to the forester's practice. Faustmann's formula already gives the optimal rotation; FAO's figures already measure the digit of net loss.

What the theory might add is a different emphasis. Sustained yield is usually stated for timber volume. Closure in the theory's sense would require every flow to close over a cycle: carbon, nutrients, water and habitat, not only wood. A forest that closes for timber and leaks nutrients or biodiversity is open in the theory's sense. That is an argument that ecologists and foresters already make without the theory. The honest grade is C.

The two closures of Faustmann's problem, wood and value, also illustrate the theory's seam rule without needing it. Two closed cycles of different length cannot be joined without tension: the biological rotation and the economic rotation pull the cutting age in opposite directions, and policy must choose. The theory would describe that as a seam between two closed windows (T11). Forest economists describe it as the cost of capital. The second description is the useful one, because it yields a number.

For policy, the honest consequence is a warning against a familiar move. A forest policy that reports only net area treats a cleared natural forest and a new plantation as cancelling, a meeting $(-1, +1)$ with sum zero. In the theory's own terms that is a closure only if the two passages are of the same kind. They are not: the strand of an old forest and the strand of a plantation differ in what they carry. Net figures close the arithmetic and leave the ecology open. A reporting rule follows that foresters can adopt without the theory: publish clearing and planting separately, by forest type, so that a

reader can see which passages cancel and which do not. FAO's assessment already reports both, which is why the two ratios above can be computed.

Grade. C (sustained yield is closure in the plain sense; the theory adds no structure, number or test that forestry lacks).

13.3 Fisheries and aquaculture

History. Fishing is older than farming, but fisheries science began with collapse. In the late 19th century fishermen in the North Sea and the Baltic saw catches fall as steam trawlers spread. The International Council for the Exploration of the Sea was founded in 1902 to study the problem. Johan Hjort showed in 1914 that year-class strength varies enormously from year to year, so that a fishery is driven by rare good years. Fyodor Baranov wrote the catch equation in 1918, and Michael Graham argued in *The Fish Gate* (1943) that the North Sea was overfished, as shown by the recovery of stocks during two world wars, when fishing stopped. Volterra's predator–prey model (1926) began as an explanation of the wartime Adriatic catch.

The quantitative theory came in the 1950s. William Ricker's 1954 stock–recruitment model, in which recruits fall at high spawner density because of cannibalism or competition, and Ray Beverton and Sidney Holt's 1957 model, in which recruits saturate, became the two standard forms (Wikipedia, "Beverton–Holt model", secondary). Maximum sustainable yield became the goal of management. The collapse of the Peruvian anchoveta in 1972 and of the northern cod off Newfoundland in 1992, where a moratorium followed, showed the cost of overestimating it.

Aquaculture grew from ancient carp ponds in China to an industry. Norwegian salmon farming began in the 1970s, shrimp farming spread through Asia and Latin America in the 1980s, and selective breeding raised growth rates. Shellfish culture brought a specific technique. In 1979 Standish Allen, then a graduate student at the University of Maine, forced an extra chromosome set into Eastern oyster eggs with cytochalasin B and confirmed the triploids by counting 30 chromosomes, three sets of 10. Triploid oysters "usually don't grow gonads and don't bother spawning", so they put their energy into meat and stay marketable in summer (Chesapeake Quarterly). Allen and Ximing Guo later produced tetraploid oysters, identified by flow cytometry in 1993; crossing tetraploids with diploids yields triploids without the chemical treatment (same source). The questions left were how to fish without collapse and how to farm without pollution or disease. A third question came from theory. Robert May's 1976 review used the logistic map, but he noted that fisheries models of Ricker's form show the same cascade of period doubling; whether real fish stocks ever overcompensate strongly enough to cycle was left to data.

Current state. FAO's *State of World Fisheries and Aquaculture 2026* reports record production in 2024: 235 million tonnes in total, of which 195 million tonnes were aquatic animals. Aquaculture produced 103 million tonnes of aquatic animals, the first time it exceeded 100 million tonnes, and 53% of the total; capture fisheries produced about 92 million tonnes, 80 million of them marine (Feed & Additive, secondary report of SOFIA 2026). In 2023, 72.6% of landings came from biologically sustainable stocks

(same source). The aquaculture share computed from the totals is 52.8% (Calculation 13.3).

Triploidy is standard in oyster hatcheries; commercial batches made by chemical induction averaged 70 to 90% triploids, and the rate at which triploids revert to fertility is well below 1% (Chesapeake Quarterly). Stock assessment models still use Ricker and Beverton–Holt forms. Assessments fit both forms to spawner and recruit data and choose between them by fit, and the choice matters: the Ricker form implies an optimum spawning stock above which recruitment declines, the Beverton–Holt form a plateau. The debates of 2026 concern the ecological footprint of feed for farmed fish, disease and escapes from salmon farms, the effects of warming on stock productivity, and the share of aquaculture in food security as capture fisheries plateau. Production figures now count algae as well as animals: aquaculture including seaweeds reached 141 million tonnes in 2024 (Feed & Additive). Farmed aquatic animals now outweigh the wild catch, and SOFIA 2026 is the first report in which they exceed 100 million tonnes (same source). Meanwhile about a quarter of landings by weight still come from stocks that are not biologically sustainable, the complement of the 72.6% figure (27.4%; Calculation 13.3).

Interpretation.

1. *Strand.* I is the sequence of spawning seasons; P is the set of stock sizes. A year class is a passage.
2. *Passage and trit.* From one generation to the next the stock rises (+1), falls (−1) or holds (0).
3. *Closure.* A stock at equilibrium closes: recruits replace spawners. Ricker's map, $x' = x \cdot \exp(r(1 - x))$, overcompensates: at high density recruitment falls, and as r rises the equilibrium gives way to a 2-cycle (r = 2.1), a 4-cycle (2.6), an 8-cycle (2.68), a 16-cycle (2.69), chaos (3.0) and a period-3 window (3.11) (Calculation 13.2). That is the ×2 ladder of T15 followed by the ring that forces all periods (Chapter 12). The Beverton–Holt map, $x' = Rx/(1 + (R - 1)x)$, compensates without overshoot and returns to equilibrium for every R tested, from 2 to 1,000 (Calculation 13.2). Which form a stock follows decides whether it can cycle at all. In Beverton–Holt the recruits approach a ceiling from below and never overshoot it, so no meeting of high and low years can form; in Ricker a large spawning stock produces fewer recruits than a moderate one, and the alternation of high and low years is the first rung of the ladder.
4. *Scale.* Triploid oysters repeat the pairing rule of Section 13.1: with x = 10, a balanced triploid gamete has probability $2 \cdot (1/2)^{10} = 1/512$ (Calculation 13.1). Tetraploid × diploid gives 2 + 1 = 3 sets: digit 0, carry 1, the ring that cannot meet. Aquaculture overtaking capture is a change of regime, not a ladder step.
5. *Test.* Stated before data: if overcompensating stocks climb the ×2 ladder, stocks whose recruitment is Ricker-like (cannibalistic species with strong density dependence) should show 2-year alternation in recruitment more often than Beverton–Holt-like stocks under comparable exploitation. Stock–recruitment databases can be classified by fitted form and tested for a spectral peak at period 2;

no difference between the two classes refutes the reading that the ladder is realised in fisheries.

Theorems checked: T7 (meetings in meiosis; 2-cycles), T7.1 (ploidy as digit and carry), T15 (period doubling), T16 (the ×2 cascade never reaches period 3 directly; period 3 appears only beyond chaos) above. T6 gives no fisheries baseline. T23 has a natural counterpart, density-dependent recruitment as a state function of stock size (N4's reading), but neither stock model has T23's cubic energy form. T24 would apply to the movement of fish only as a free three-dimensional walk, which fish do not perform.

Consequences. Fisheries science contains the ×2 ladder in its most-used model. Whether a stock can oscillate depends on whether recruitment overshoots, the difference between Ricker and Beverton–Holt. That is published mathematics, and its consequence for management is known: overcompensating stocks can fluctuate without any change in fishing, which complicates the reading of catch data. The theory names the ladder and adds the test above, which asks whether the ladder is realised in real stocks rather than only in models.

Aquaculture's use of triploidy is the pairing rule turned into a product. A triploid oyster does not spawn and so stays in condition through summer; a triploid salmon cannot interbreed with wild fish if it escapes. In both, the failure of meiotic closure is the purpose. The theory adds the arithmetic of the balanced gamete, which is already in genetics textbooks. Its cost to the field is nil; its gain is a clear statement of why odd ploidy is a reliable sterility tool and even ploidy is not. The shift of production from capture to culture carries no register structure: it reflects investment, demand and the limits of wild stocks.

The checklist adds one more possible link, which is set aside. Fish stocks that cycle with the climate, such as Pacific sardine and anchovy alternating over decades, look like rings of two states. Their periods are set by ocean regimes and are not on any ladder that the theory derives. T15 requires equal merges of a strand, and a climate oscillation is not that. The ladder in fisheries is therefore limited to the period-doubling of overcompensating recruitment, where it is exact in the model and testable in data.

Management draws a practical point from the Ricker–Beverton–Holt contrast. An overcompensating stock fished down may become more stable, because fishing reduces the effective growth rate r and moves the stock back down the ladder from cycles to equilibrium; an undercompensating stock gains no such stability. That is a published consequence of the models, not of the theory, and it is the kind of statement the theory can endorse without claiming it. The test stated above would show whether the endorsement has empirical weight, or whether real recruitment is too noisy for the ladder to be seen at all. Either outcome would be informative for assessment practice.

Grade. A for the ×2 ladder in Ricker stock–recruitment dynamics (Ricker 1954; May 1976) and for triploid sterility in aquaculture (Allen and Downing 1986); B for the recruitment-alternation test; C for production trends.

13.4 Agricultural, industrial and environmental biotechnology

History. Biotechnology is old as practice and young as a word. Brewing, baking, cheese and vinegar used microbes for millennia before Pasteur explained fermentation. Chaim Weizmann's acetone–butanol fermentation, scaled up in Britain during the First World War, was the first industrial microbial process of military importance. In 1913 Edward Ardern and William Lockett at the Davyhulme sewage works in Manchester aerated sewage until it nitrified completely and found that the settled sludge, returned to new sewage, "activated" the process; their 1914 paper founded activated-sludge treatment, and the first continuous plant followed in Worcester two years later (Wikipedia, "Activated sludge", secondary). Penicillin was mass-produced by deep-tank fermentation in the 1940s.

Recombinant DNA changed the field. Stanley Cohen and Herbert Boyer joined DNA from different organisms in 1973. Genentech was founded in 1976, and recombinant human insulin was approved in 1982. The first commercial genetically modified crop, the Flavr Savr tomato, was sold in 1994 (Wikipedia, "Genetically modified crops", secondary); herbicide-tolerant and insect-resistant soybean, maize and cotton followed from 1996. Directed evolution of enzymes, developed by Frances Arnold from the early 1990s, made proteins designable by selection.

Synthetic biology began in January 2000 with two circuits in the same issue of *Nature*: the toggle switch of Gardner, Cantor and Collins, two mutually repressing genes, and the repressilator of Elowitz and Leibler, three repressors in a ring (Chapter 10). Environmental biotechnology gained anammox, discovered in Delft and reported in 1995, and built at full scale in the Netherlands from 2002 (Chapter 11). Austin Burt proposed in 2003 that homing endonuclease genes could spread through a population and suppress or modify it, the idea of the gene drive (Wikipedia, "Gene drive", secondary). CRISPR–Cas9 (2012) made gene editing routine. The questions left concern scale, cost, containment and public consent. Containment itself became a design problem: kill switches that require two independent signals to keep a cell alive, so that loss of either kills it, and recoded organisms that depend on a synthetic amino acid absent in nature.

Current state. The most ambitious genome rewrite is Syn57: an *E. coli* whose 4 Mb genome was rebuilt with more than 101,000 codon replacements so that it uses 57 codons instead of 64 (MRC LMB; *Science* 390, eady4368, 2025). The freed codons can be assigned to non-standard amino acids, and an organism with a changed code resists viruses that rely on the standard one. AI design has reached whole genomes: after fine-tuning, the Evo 2 model designed phage genomes of which 16 of 285 tested were viable and inhibited their target bacteria (Arc Institute).

Industrial plastic recycling by enzymes is approaching scale but not yet there. Carbios has kept its plan for a PET biorecycling plant at Longlaville, France, but its commissioning has moved to the first half of 2028, conditional on securing the remaining financing by the end of the first quarter of 2026; pre-commercial contracts covered close to 50% of the plant's capacity (CHEManager, January 2026). Gene drives for malaria remain in development: Target Malaria, begun in 2005, describes its

research as still at an early stage (Wikipedia, “Target Malaria”, secondary). In agriculture, the sterile insect technique against the screwworm (Section 13.1) is the largest biological control campaign in North America in 2026. The debates concern regulation of gene-edited crops, release of gene drives, biosafety of recoded organisms, and the economics of biomanufacturing against petrochemistry. Phage therapy, revived as antibiotic resistance grows (Chapter 11), is the field where AI-designed phages could reach patients first; it remains experimental.

Interpretation.

1. *Strand.* For circuits, I is time and P the set of expression states; for consortia, I is time and P the set of strains holding each patch; for a genome rewrite, I is the position along the genome and P the codon positions.
2. *Passage and trit.* Each gene in a circuit is high (+1), low (−1) or in transition (0). Each strain in a competing consortium wins (+1), loses (−1) or ties (0) against another.
3. *Closure.* The two founding circuits are the two non-trivial closures of T7. A pair of mutual repressors is a meeting: it settles in one of two states (high–low or low–high), each a pair (x, −x), and holds it, which is why it works as memory. A ring of three repressors cannot settle in an alternating state and oscillates. Calculation 10.6 shows the rule for rings of 2 to 8: even rings settle, odd rings oscillate. In consortia, three strains in a cycle of dominance coexist under local interaction (Kerr et al. 2002), and random competitive tournaments leave only odd sets of survivors (Calculation 11.3). Syn57’s recoding preserves closure exactly: every swap replaces a word of three bases with another of three, so the frame digit never changes (Chapter 10).
4. *Scale.* The ring closes with carry (T7.1): a repressilator’s period, about 150 minutes, spans two to three cell divisions of 50 to 70 minutes, so the oscillation passes from cell to daughter cell as a unit of the next scale. A gene drive is the opposite of the sterile insect technique: a meeting that carries more than its share, inherited by most offspring instead of half. Activated sludge and anammox reactors are consortia of organisms whose products and substrates close among them.
5. *Test.* Stated before data: for engineered consortia used in bioremediation, three-strain intransitive designs should persist longer under local (biofilm, static) conditions than four-strain intransitive designs of the same kind (Chapter 11). For circuits: in a single host and with matched parts, three-repressor rings should give sustained oscillation more often than five- or seven-repressor rings, because longer rings accumulate more noise per cycle. Four-strain persistence equal to or better than three-strain persistence, or longer rings oscillating more reliably than three, refutes the special status of the ring of three in design.

Theorems checked: T7 (switch and clock), T7.1 (carry of the ring), T1 (frame digit preserved by recoding), T6 (baseline of 2 of 8 three-strain tournaments being rings; Calculation 11.3). T11 has a design counterpart: joining two different circuits creates a seam where their states must be matched, and mismatched junctions are a known source of failure in genetic part assembly; this is a reading without a measure. T15 does not apply to reactor or circuit sizes. T24 and T19–T22 have no counterpart.

Consequences. Biotechnology uses the meeting–ring split as working knowledge. Engineers build memory from pairs and clocks from odd rings, and the reason the two differ is exactly the reduction of T7: a pair can close by cancelling, an odd ring cannot. The theory adds a name and a reason. It also adds two tests that could change practice. If three-strain consortia are more robust than four-strain ones under local interaction, the design of microbial communities for waste treatment or plastic degradation gains a rule. If three-repressor rings outperform longer ones, the minimal clock is also the best one, which is not obvious from circuit theory alone.

The theory adds nothing to the economics that decide whether biotechnology succeeds. Carbios's delay is a matter of financing, not of enzymes; the PET-degrading enzymes work. Gene drives raise questions of consent and ecology that no arithmetic answers. Syn57 shows what the theory's frame rule implies in engineering: the genetic code can be rewritten as long as every replacement keeps the length of the word, and a rewrite that broke the frame would fail at once. That is known to every genome engineer.

Environmental biotechnology shows the same logic at the scale of a community. Activated sludge works because organisms with different roles, oxidisers of carbon, nitrifiers, denitrifiers and predators of bacteria, close the cycle of nitrogen and carbon among them; Ardern and Lockett's discovery was that the closed community, returned as sludge, does the work better than any one organism. Anammox adds a two-member partnership in which one group's product, nitrite, is the other's substrate. These are meetings and rings in the plain sense of exchange, and wastewater engineers design them by mass balance. The theory supplies no number that mass balance does not. Its contribution, if the consortium test succeeds, would be a rule for how many interacting strains a designed community should have when the strains compete rather than cooperate.

The theory's checklist yields one negative worth stating. Gene drives spread because a drive allele is inherited by most offspring rather than half: a departure from the meeting of Mendelian pairing, which gives each allele exactly one chance in two. Models of drive spread (for example, one released carrier per thousand reaching the whole population within 12 to 15 generations; Wikipedia, "Gene drive", secondary) are population genetics. The theory can say that the drive breaks the symmetry of the meeting; it cannot say how fast the drive will spread or whether resistance will stop it.

Grade. A for the switch–clock split of synthetic circuits (Gardner et al. 2000; Elowitz and Leibler 2000; Calculation 10.6) and for frame-preserving recoding (Robertson et al. 2025); B for the consortium and ring-length tests; C for process engineering and economics.

Part V — Organism and Mind

Chapter 14 — Body and Brain

This chapter reads the standing unit of level L4: the body and the brain as closed configurations, in anatomy, physiology, neuroscience and endocrinology. Three findings carry weight. The collagen triple helix is a period-3 word in the field's own published structure: glycine at every third residue, and triple-helix exons whose lengths are whole multiples of nine base pairs (grade A). The grid cells of the entorhinal cortex lay a hexagonal lattice whose six directions are the units of the Eisenstein integers Z[ω] (grade A), but the measured ratio between grid modules, 1.42, fits √2 and rejects √3: the step between modules is a meeting step, not a ring step, and the modules cannot be nested similar hexagonal sublattices (grade B, with the ring reading refuted at about nine standard errors). In physiology, the shortest negative-feedback ring that can oscillate without delay has three stages and needs a loop gain above 8; two stages without delay never oscillate. That is a published theorem of control theory and matches T7, where a meeting closes at length 2 and a ring at length 3 (grade A). Most counts of the body — seven cervical vertebrae, 33 vertebrae, 206 bones, 39 Hox genes — are off the ladder 2^a·3^b of T15, and the small ones that are on it are on it by chance as often as not (grade C).

14.1 Anatomy

History. Systematic human anatomy begins in Alexandria, where Herophilus and Erasistratus dissected human bodies in the third century BCE. Their texts are lost and survive through Galen of Pergamon (129–c. 216 CE). Galen dissected apes, pigs and other animals and transferred what he found to humans. His authority lasted fourteen centuries. Mondino de Luzzi's *Anathomia* (1316) made dissection part of medical teaching in Bologna, but the professor read Galen while a barber cut, and the text was believed over the body.

Andreas Vesalius broke this order with *De humani corporis fabrica* (Basel, 1543). He dissected himself, corrected many of Galen's errors, and showed that many of them came from animal anatomy. The *Fabrica*, with woodcuts from Titian's workshop, made the drawn body a source of evidence. Fallopius, Eustachius and Fabricius extended it in Padua. William Harvey, Fabricius's student, used anatomy as an argument about function in *De motu cordis* (1628). Valves in the veins that open only towards the heart, and the measured volume of each beat, made circulation a necessity.

The microscope added a second scale. Marcello Malpighi saw capillaries in the frog lung in 1661 and closed Harvey's circuit. Xavier Bichat (1771–1802) counted twenty-one tissues and made the tissue, not the organ, the unit of pathology. Theodor Schwann (1839) and Rudolf Virchow (1858) moved the unit down again, to the cell. Comparative anatomy, from Cuvier to Owen, asked a different question: which parts correspond across species. Owen defined homology in 1843. Darwin (1859) turned homology into descent.

The twentieth century added the gene to the body plan. Edward B. Lewis studied the bithorax complex of the fly from the 1940s onward and found that genes along a chromosome map onto segments along the body axis. Homeobox genes were cloned in 1984. The mammalian Hox clusters turned the vertebral column into a readout of gene expression: which vertebra becomes cervical, thoracic or lumbar depends on the combination of Hox genes active at that level. Frietson Galis (1999) asked why almost all mammals keep seven cervical vertebrae while birds and reptiles vary. Her answer was a constraint on the developmental program, not on the neck.

Imaging moved anatomy into the living body: X-rays (Röntgen, 1895), computed tomography (Hounsfield, 1971), magnetic resonance imaging (1973 onward) and, from 2010, whole-organ atlases at cellular resolution. Electron microscopy and X-ray crystallography reached the molecular anatomy of tissue. Hugh Huxley and Jean Hanson (1954) placed thick and thin filaments in the sarcomere. G. N. Ramachandran and Alexander Rich, with Francis Crick, worked out the collagen triple helix in 1954–55.

The questions anatomy leaves are now about integration across scales. How does a molecular repeat become a tissue property? Why are some counts of the body fixed across a whole class, such as seven cervical vertebrae, while others, such as the coccygeal count, vary? And how should a reference atlas link organs, cells and molecules in one coordinate system?

Current state. The field’s main present project is the reference atlas. Version 2.0 of the Human Reference Atlas of the HuBMAP consortium, published in *Nature Methods* on 13 March 2025, contains 4,499 unique anatomical structures, 1,195 cell types and 2,089 biomarkers, linked from organs down to cells (JCVI summary). The number of cells such an atlas must place is large. Bianconi and colleagues summed organ and tissue counts to $3.72 \pm 0.81 \times 10^{13}$ cells in an adult human body (Fermat’s Library copy).

The skeleton gives the classic counts. An adult has about 206 bones, after fusion of the roughly 270 present at birth, and 12 pairs of ribs (Wikipedia, Human skeleton). The vertebral column has 7 cervical, 12 thoracic, 5 lumbar, 5 fused sacral and usually 4 coccygeal vertebrae, 33 in all, with a normal range of 32 to 35; the cervical count rarely changes, the coccygeal count varies most (Wikipedia, Vertebral column). The fixity of seven is measurable as selection. Galis and colleagues found cervical ribs, a homeotic shift of the last cervical vertebra, in 54.8 % (95 % CI 50.5–59.0 %) of deceased fetuses and infants, against at most about 1.1 % in adults. They estimate that at least 78 % of recognised pregnancies with a cervical rib end before birth. Among mammals only sloths and manatees break the rule of seven (Galis et al., IIASA report IR-06-071). The genetic program behind the axis has 39 Hox genes in four clusters, A to D, on chromosomes 7, 17, 12 and 2; their order along the chromosome matches the order of expression along the body (Wikipedia, Hox gene).

At the molecular end, collagen is the most abundant protein of mammals, about 25–35 % of their protein. Its triple helix is three chains wound together. Glycine, the smallest amino acid, sits at every third position because only a hydrogen side chain fits the

packed centre of the helix. Twenty-eight types of human collagen had been classified by 2011 (Wikipedia, Collagen). The central domain of type I collagen holds 338 Gly-X-Y repeats; the human proα1(I) gene and the chicken proα2(I) gene share an arrangement of exons that points to a 54-base-pair ancestral unit (Chu et al. 1984, Nature). In the mouse α2(IV) collagen gene, all 47 exons were mapped; the exons that encode Gly-X-Y have lengths of 36, 45, 54, 99 and 108 base pairs, sizes that also occur in fibrillar collagens (Butticè et al. 1990).

Muscle shows a similar order. The three-dimensional reconstruction of mouse cardiac thick filaments by Zoghbi and colleagues shows three “crowns” of myosin heads in each 42.9-nanometre repeat, each crown with three-fold rotational symmetry. Crowns 1 and 3 lie near the expected 14.3-nm spacing; crown 2 is displaced, at 13.2 nm from crown 3 and 15.4 nm from crown 1 (Zoghbi et al. 2008, PNAS).

The open debates are about the atlas, not about the bones. How finely should cell types be cut? How should the cellular atlas and the gross atlas share coordinates? And can tissue mechanics be predicted from the molecular repeats?

Interpretation.

1. *Strand.* At the molecular scale I is the position along a collagen or myosin chain and P the set of residues or head positions. At the scale of the body plan I is the position along the body axis, from head to tail, and P the set of segment identities (cervical, thoracic, lumbar …).
2. *Passage and trit.* In collagen a passage is a return of glycine. Reading the position of each residue mod 3 relative to the first glycine (N3 as definition) gives the trit: glycine sits at digit 0, X at +1, Y at −1. A deletion or insertion of k residues shifts every later glycine by k mod 3, the digit of k (T1).
3. *Closure.* A Gly-X-Y triple is a closed window in the sense of T2–T4: the three positions are all different, the heterogeneous closure of T4. The helical domain is a word that closes in every window, which by T8 means period 3; it is one of the nine everywhere-closed words. An exon that joins two Gly-X-Y stretches is a seam (T11). It is closed exactly when it continues the same window in the same phase. That is the case when its length is a multiple of 9 base pairs (3 codons × 3 positions), which all five published exon lengths are (Calculation 14.1). A 57-bp exon would shift the glycine phase by one residue: tension at the seam. The thick filament gives a second instance: three crowns, each of three-fold symmetry, nine head positions per 42.9-nm repeat (Calculation 14.2).
4. *Scale.* Three collagen chains merge into one helix and one helix into a fibril: the ring step of T15 at the first merge. The 54-bp unit is $2 \cdot 3^3$ (Calculation 14.1); 45 and 99 are not on the ladder. At the scale of the body plan the counts are mostly off the ladder: 7, 5, 33, 206, 270 and 39 are not of the form 2^a·3^b; 12, 4 and 24 are (Calculation 14.3). Four of eleven counts lie on the ladder. Among the numbers 1 to 12, 8 of 12 lie on it anyway (Calculation 14.3), so small counts on the ladder are no evidence.
5. *Test (stated before data).* If the Gly-X-Y helix is an everywhere-closed word (T8), then every seam inside an uninterrupted helical domain must be closed, so every

exon wholly inside such a domain must be 0 mod 9. A triple-helical exon with a length not divisible by 9 must coincide with a break in the Gly-X-Y repeat — an interruption. The published sizes all pass the first half. The second half can be checked against Butticè et al., who report that 11 of 24 interruptions in the α2(IV) chain lie at exon–intron borders, one has its own exon and the rest lie inside exons. The test fails if a triple-helical exon of length not divisible by 9 is found inside an uninterrupted Gly-X-Y run.

Consequences. If the reading holds, the collagen gene is the cleanest case at the organism level of what Crick's frameshift experiments showed for the code in general (Chapter 10): closure mod 3 at one scale (the codon) is nested in closure mod 3 at the next (the Gly-X-Y triple), so the exon must be 0 mod 9. That double nesting is the ring ladder of T15 with b = 2, read directly off a gene. It costs nothing, because molecular biology already knows it; the Foundations add the name of the rule and a check (Calculation 14.1) that it holds for every published exon size, and they connect it to Chapter 17, where glycine substitutions in collagen cause osteogenesis imperfecta. A glycine replaced inside the helix is an open window in an everywhere-closed word: the closure fails at one place, and T8 says the word then no longer closes everywhere.

For the body plan the reading adds nothing. The fixed seven of the mammalian neck is an off-ladder count kept by strong selection, not by arithmetic. The Hox count, 39, is off the ladder; the four clusters arose by two whole-genome duplications, which is the meeting ladder (2^2) — but two duplications are a small number, and a small number lands on the ladder by chance. The plate trick (Calculation 14.5) shows that an attached arm returns after a rotation of 4π, not 2π, which is T21 in anatomy. It is a fact about rotations, not about bodies, and grades nothing.

Grade. A for the period-3 collagen helix and the 0 mod 9 exon lengths (Butticè et al. 1990; Chu et al. 1984) and for the three-fold crowns of the thick filament (Zoghbi et al. 2008); B for the seam test on interrupted collagen exons; C for the counts of the body plan.

14.2 Physiology

History. Physiology became an experimental science when Harvey measured the heart in 1628. The output of one beat, multiplied by the beats in an hour, exceeded the weight of the body: the blood had to return. The circulation is the first closed loop in physiology, and it was proved by a return. Stephen Hales measured blood pressure in a horse in 1733. Lavoisier and Laplace showed in the 1780s that respiration is slow combustion. Johannes Müller's *Handbuch* (1833–40) gathered the field; his students, among them Helmholtz, du Bois-Reymond and Brücke, swore in 1842 to explain life by physical and chemical forces only.

Claude Bernard set the central idea in the 1850s–1870s: the *milieu intérieur*, the internal environment whose constancy is the condition of free life. Walter Cannon named its regulation homeostasis (1926) and described it in *The Wisdom of the Body* (1932). Willem Einthoven, at Leiden, recorded the heart's electrical activity with the string galvanometer from 1901 and set the three limb leads in a triangle; he received

the Nobel Prize in 1924. Ernest Starling and William Bayliss found the first hormone, secretin, in 1902, and Starling's law of the heart (1914–18) tied output to filling. August Krogh explained capillary exchange (Nobel 1920).

Norbert Wiener's *Cybernetics* (1948) gave homeostasis a mathematics: negative feedback, gain and stability. Arthur Guyton's circulation models of the 1960s–1970s were large systems of feedback loops. Alan Hodgkin and Andrew Huxley (1952) wrote the action potential as four differential equations and won the Nobel Prize in 1963. Brian Goodwin (1965) proposed a three-step gene-feedback loop as a biological oscillator; Later analysis showed that it can oscillate only with very steep repression. Leon Glass and Michael Mackey (1977, 1988) named "dynamical diseases": illnesses in which a control loop changes its qualitative behaviour, from steady to periodic to irregular.

Physiology also discovered fixed ratios between rhythms. Dennis Bramble and David Carrier (1983) showed that running mammals lock breaths to strides. Michael Guevara, Leon Glass and Alvin Shrier (1981) drove chick heart cells with periodic pulses and saw phase locking at rational ratios, period doubling and irregular dynamics. Karel Frederik Wenckebach (1906) and Woldemar Mobitz (1924) had already classified heart block by ratios of atrial to ventricular beats.

The open questions of physiology are about integration. How do loops at different scales — ion channel, cell, organ, body — share one set of constants? When does a regulated variable stop being regulated and start to oscillate? And why do coupled rhythms lock at some ratios and not at others?

Current state. The disease burden that physiology serves is led by the circulation. About 19.8 million people died of cardiovascular disease in 2022, about 32 % of all deaths, and 85 % of those deaths were from heart attack and stroke (WHO fact sheet, 31 July 2025).

Rhythm-to-rhythm locking remains an active topic. Daley, Bramble and Carrier report that running humans use locomotor–respiratory coupling ratios of 2:1, 2.5:1, 3:1 and 4:1 strides per breath, with 3.5:1 in two subjects and no fixed coupling in some runners, while most galloping mammals breathe once per stride, 1:1 (Daley et al. 2013, PLoS ONE). In the heart, the Wenckebach (Mobitz type I) block has a ratio of P waves to QRS complexes always of the form $X{:}(X-1)$, such as 4:3; a 2:1 block cannot be classified as type I or II from the ratio alone (Wikipedia, Second-degree AV block). The experimental source of the ratio picture is Guevara, Glass and Shrier, *Science* 214, 1350–1353 (1981) (McGill Physiological Dynamics list).

The three-stage feedback oscillator is still the reference model. Gonze and Abou-Jaoudé state that limit-cycle oscillations in the Goodwin model need a Hill coefficient above 8, a value usually thought unrealistic, and examine mechanisms that relax it (Gonze and Abou-Jaoudé 2013, PLoS ONE). The value 8 is the secant bound $\sec(\pi/n)^n$ at $n = 3$; for $n = 4$ it is 4, for $n = 5$ about 2.885 (Calculation 14.10).

The electrocardiogram still rests on Einthoven's geometry. The limb leads I, II and III run at 0°, +60° and +120° and form Einthoven's triangle (Wikipedia, Einthoven's triangle). Goldberger added the augmented leads in 1942, and the standard 12-lead

ECG uses 10 electrodes (Wikipedia, Electrocardiography). In the Hodgkin–Huxley model (1952), the potassium conductance carries n^4 because the channel has four subunits that must all be open (Wikipedia, Hodgkin–Huxley model); the sodium term is m^3h.

The frontier combines these classical loops with patient data: digital twins of the heart, closed-loop devices, and wearable sensors that measure rhythm coupling in daily life.

Interpretation.

1. *Strand.* I is time, sampled per beat, breath or stride; P is the set of physiological states of one regulated loop (phase of the cardiac cycle, phase of breathing, level of a regulated variable).
2. *Passage and trit.* A passage is the return of a cycle to its reference phase. Relative to the previous cycle (N3 as definition), a cycle is longer (+1), shorter (−1) or unchanged (0). In a feedback chain, each link is an activation (+1) or an inhibition (−1); the sign of the loop is the product of the link signs.
3. *Closure.* Two published closures stand out. First, Einthoven's law: lead I + lead III − lead II = 0 for any heart vector, because the three leads are differences around a closed loop of three electrodes (Calculation 14.6). The three lead axes are sixth roots of unity, the units of Z[ω] (T14): axis II is $-\omega^2$, axis III is ω (Calculation 14.6). But Kirchhoff's law closes any loop of differences; a loop of four electrodes closes too (Calculation 14.6). So the three-ness of the triangle comes from using three electrodes, not from the closure rule. Second, the feedback ring: an n-stage loop with an odd number of inhibitions is a negative ring; with an even number it is a positive ring and acts as a switch (Calculation 14.10). For a linear chain without delay the secant condition says a negative ring can oscillate only if its gain exceeds $\sec(\pi/n)^n$; for n = 2 the bound is infinite and no oscillation occurs; n = 3 is the shortest oscillating ring, with bound 8 (Calculation 14.10). That is T7: a meeting (length 2) closes without going round, a ring (length 3) is the shortest closure with winding.
4. *Scale.* Coupled rhythms climb the meeting ladder by period doubling: 1:1 → 2:1 → chaos in Guevara's cells; 2:1 is the first rung. The human coupling ratios test T15. The integer ratios 1:1, 2:1, 3:1 and 4:1 are on the ladder; 5:2 and 7:2 are not, because their numerators are 5 and 7 (Calculation 14.7). In Wenckebach block, 2:1, 3:2 and 4:3 have both terms on the ladder; 5:4 and longer periods do not (Calculation 14.7). The ion-channel exponents 3 and 4 are on the ladder; so is almost every small number.
5. *Test (stated before data).* T7 with T15 predicts that stable locked ratios between two physiological rhythms are ratios of ladder numbers, 2^a·3^b : 2^c·3^d. A stable lock at 5:2 or 7:2 in a single runner, held over many cycles, contradicts this. The published data report 2.5:1 as one of four common human ratios and 3.5:1 in two subjects. The strict ladder reading of locking is therefore already refuted for locomotor–respiratory coupling. The weaker statement — the shortest oscillating negative-feedback ring has three stages — is a theorem of the field and needs no test.

Consequences. Physiology gains one exact link and loses one tempting one. The exact link is the secant result. Control theory already knows that a negative feedback of two stages without delay cannot oscillate and that three stages can, if the gain exceeds 8. The Foundations reach the same order — meeting at 2, ring at 3 — from the digit rule alone (T7, T7.1), without differential equations. A physiologist designing or reading a loop gains a quick check: if a regulated variable oscillates without measurable delay, the loop must contain at least three stages, or hidden delay. This check is in the field already; the Foundations explain why the number is three, not why the gain is 8.

The lost link is the ladder for coupling ratios. Rhythm locking follows the Farey tree of the circle map, which contains every rational ratio. Humans use 5:2 as a common running ratio. The Foundations have no account of why a 5:2 lock should be stable, and T15 in its strict form would forbid it. The honest consequence is that T15 is a statement about equal-block merging and not about phase locking between two independent oscillators; the hypothesis of equal blocks (IX.4) is not met here.

Einthoven’s triangle adds nothing beyond Kirchhoff. Its axes at the sixth roots of unity are a property of placing three electrodes at the corners of a near-equilateral triangle. N4, read as the hypothesis of T23, fits the general observation that loop gains depend on state — the Hill coefficient needed for oscillation is lower in models with other mechanisms (Gonze and Abou-Jaoudé) — but no number from T23 is tested here.

Grade. A for the three-stage minimum of an oscillating negative-feedback ring (secant condition; Goodwin model; Gonze and Abou-Jaoudé 2013) and for period doubling of paced heart cells (Guevara et al. 1981); B, refuted in its strict form, for ladder ratios in locomotor–respiratory coupling (Daley et al. 2013 report 5:2 and 7:2); C for Einthoven’s triangle.

14.3 Neuroscience

History. Neuroscience as a cellular science begins with a quarrel. Camillo Golgi’s silver stain (1873) made single nerve cells visible. Golgi held that they form a continuous net, the reticular theory. Santiago Ramón y Cajal, using Golgi’s own stain, argued from 1888 that each neuron is a separate cell and that signals pass in one direction, from dendrites to axon. They shared the Nobel Prize in 1906 and disagreed in their lectures. Charles Sherrington named the synapse in 1897 and described the integrative action of the nervous system (1906). Otto Loewi showed chemical transmission in 1921. Hodgkin and Huxley explained the action potential in 1952; Bernard Katz explained transmitter release in quanta.

The second line is the map. Paul Broca (1861) and Carl Wernicke (1874) localised language. Korbinian Brodmann divided the cortex into 52 areas (1909). Wilder Penfield stimulated the exposed human cortex and drew the homunculus (1937–1950). David Hubel and Torsten Wiesel found orientation-selective cells in the visual cortex (1959–1962) and won the Nobel Prize in 1981. John O’Keefe found place cells in the rat hippocampus in 1971. May-Britt and Edvard Moser found grid cells in the medial entorhinal cortex in 2005; the three shared the Nobel Prize in 2014.

The third line is the circuit. Donald Hebb (1949) proposed that cells that fire together wire together. Wolfram Schultz recorded dopamine neurons in monkeys from the 1980s, and Schultz, Dayan and Montague (1997) read their firing as a reward-prediction error. The connectome began with the 302 neurons of *C. elegans* (White et al. 1986). Serial electron microscopy and machine segmentation scaled it, through the fly to cubic millimetres of mammalian cortex. Optogenetics, the control of neurons with light-gated channels, began with channelrhodopsin (Nagel, Hegemann and colleagues, 2002–2003) and its use in mammalian neurons (Deisseroth and colleagues, 2005).

The questions neuroscience leaves are about codes and wiring. How does a population of neurons represent a variable, such as position? Why are cortical circuits wired non-randomly, with more reciprocal connections than chance? And what in the dynamics marks conscious from unconscious processing (Chapter 15)?

Current state. The adult human brain contains on average 86.1 ± 8.1 billion neurons and 84.6 ± 9.8 billion non-neuronal cells; the cerebral cortex holds 19 % of the neurons (Azevedo et al. 2009, abstract on PubMed; journal details from BioNumbers). Cell-type atlases now divide this population finely. The Whole Human Brain Taxonomy, from single-nucleus RNA sequencing of over 3 million nuclei from about 100 dissections of three donors, has 461 clusters and 3,313 subclusters (Allen Institute).

Connectomics reached three milestones in two years. A cubic millimetre of human temporal cortex, reconstructed at nanoscale from more than 5,000 slices, contains more than 57,000 cells and almost 150 million synapses; glia outnumber neurons two to one; the data fill about 1.4 petabytes (NIH Research Matters, May 2024). The complete adult fly brain has 139,255 proofread neurons, 54.5 million synapses and more than 8,400 cell types (Dorkenwald et al. 2024, Nature). The MICrONS cubic millimetre of mouse visual cortex contains an estimated 200,000 cells, 120,000 of them neurons, and more than 523 million synapses, with functional recordings from about 75,000 neurons (MICrONS Explorer).

Wiring is not random. In paired recordings of rat visual cortex, Song and colleagues found a connection probability of 11.6 % (931 of 8,050 tested connections) and about four times more bidirectional pairs than a random network predicts — 218 pairs (Song et al. 2005, PLoS Biology).

The spatial code is the clearest numerical structure. Grid cells fire at the vertices of a triangular lattice; their spatial autocorrelation shows six peaks at multiples of 60°; spacing ranged from 39 to 73 cm in the original sample of 211 cells in 14 rats (Hafting et al. 2005, Nature 436). Grid cells form a small number of discrete modules, each with its own scale and orientation (Stensola et al. 2012, Nature 492). Successive module scales have a mean ratio of 1.42 ± 0.17 (SD, n = 24); Wei, Prentice and Balasubramanian derived from coding economy a ratio of about √e ≈ 1.65 (winner-take-all) or about 1.44 (probabilistic decoder) in two dimensions (Wei et al. 2015, eLife). Fiete, Burak and Brookings had shown that the set of grid phases is a residue number system: position is stored as its remainders modulo the module periods, with capacity growing exponentially with the number of modules and with carry-free arithmetic for updating (Fiete et al. 2008, J. Neurosci. 28).

Rhythms span four orders of magnitude. Buzsáki and Draguhn found that the mean frequencies of the oscillation classes, from about 0.05 to 500 Hz, form a linear progression on a natural-logarithmic scale (Buzsáki and Draguhn 2004, Science 304).

The value signal has three outcomes. In Schultz's recordings, dopamine neurons are activated by an unpredicted reward, do not respond to a fully predicted one, and are "depressed markedly below their basal firing rate" when a predicted reward is omitted (Schultz, Dayan and Montague 1997).

On 5 October 2026 the Nobel Prize in Physiology or Medicine went to Karl Deisseroth, Peter Hegemann and Georg Nagel "for their discovery of a molecular switch for nerve cells", the basis of optogenetics (secondary report: Al Jazeera; the Nobel Foundation page could not be opened).

Interpretation.

1. *Strand.* For the spatial code, I is the path of the animal in time and P the set of places in the arena; σ is the trajectory. For the dopamine signal, I is the sequence of trials and P the set of predicted outcomes. For wiring, I is a walk along synapses and P the set of neurons.
2. *Passage and trit.* A grid cell fires when the animal passes a vertex of its lattice: a passage is a return to the same lattice class, the residue of position modulo the grid period. The dopamine response is a trit as published: burst (+1) when the outcome exceeds the prediction, no change (0) when it equals it, depression (−1) when it falls short. The first prediction fixes the reference (N3 as definition).
3. *Closure.* A predicted reward leaves digit 0: the prediction closes and nothing passes on. An unpredicted reward or an omission leaves a nonzero digit, which is passed on as a learning signal. That is the reading of N2 by T1: what does not close passes on as tension. A reciprocal synaptic pair is a meeting, x then −x along the pair (T7); Song's fourfold excess of reciprocal pairs (Calculation 14.9) is an excess of meetings over chance.
4. *Scale.* The grid lattice is the triangular lattice of Z[ω]; its six firing directions are the six units of Z[ω] (T14). If successive modules were similar hexagonal sublattices of one another, the ratio of areas would have to be a norm of Z[ω]: 1, 3, 4, 7, 9, 12, … (Calculation 14.8). The ring step of T15, area ×3, gives a scale ratio √3 = 1.732; the meeting step, area ×2, gives √2 = 1.414. The measured 1.42 ± 0.17 lies 0.17 standard errors from √2 and 8.99 standard errors from √3 (Calculation 14.8). Area ×2 is not a norm of Z[ω] (Calculation 14.8), so the modules are not nested hexagonal sublattices of each other. They are separate lattices whose scale steps by the meeting ladder in length squared. Wei et al. predicted 1.44 from coding economy, also consistent with the data.
5. *Test (stated before data).* Two tests. (a) The ring ladder predicts a module ratio of √3; the meeting ladder √2. Data with smaller error — more animals, more modules — decide; the present data already reject √3 by about nine standard errors. (b) The residue-number reading (Fiete et al.) says the code is carry-free across modules. The Foundations' digit rule (T1) says a closed window at one scale passes a carry to the next. If position updating in the grid system shows an interaction between

modules that acts like a carry — a coarse module correcting when a fine one wraps — the T1 reading holds; if modules update fully independently, the code is a residue system without carry, as Fiete et al. argue, and the T1 reading of the grid code fails.

Consequences. Neuroscience offers the strongest A-grade structures in this chapter. The grid lattice is Z[ω] as published fact. The grid code is a residue number system as published theory. The dopamine signal is three-valued as published fact. None of these was found because of the Foundations, and the Foundations do not add their discovery; they add a common arithmetic. The grid phase is a residue; the trit is a residue mod 3 in Z[ω] (Foundations III.1, remark). The grid code stores position in remainders, as balanced ternary stores a number in digits.

The theory also loses something here. The ring reading of scale, which would have made the module ratio √3, is rejected by the data. What survives is weaker: module areas double. That is the ×2 ladder, and it is equally the prediction of an economy argument that knows nothing of trits. The test therefore does not discriminate the Foundations from ordinary coding theory. The value of stating it is that the ring reading could have been right and was not.

The lever of T24 belongs here only as a reading: a free random search in three dimensions returns with probability 0.3405 and has lever −3.9365. Animal search paths are not free random walks, and no number from the brain is compared with it.

Grade. A for the hexagonal grid lattice as Z[ω] (Hafting et al. 2005), the residue-number grid code (Fiete et al. 2008) and the three-valued dopamine signal (Schultz et al. 1997); B for the module ratio, with the ring step (√3) refuted and the meeting step (√2) consistent; B for the carry test on module interaction; C for reciprocal pairs as meetings.

14.4 Endocrinology

History. Arnold Berthold showed in 1849 that a transplanted testis restores the comb and behaviour of a castrated cockerel: a gland acts through the blood. Charles-Édouard Brown-Séquard injected himself with testicular extracts in 1889 and reported rejuvenation, which discredited the idea for a time. George Oliver and Edward Schäfer found the pressor action of adrenal extract in 1894. Bayliss and Starling discovered secretin in 1902, and Starling coined the word hormone in 1905.

The isolation of hormones followed. Edward Kendall crystallised thyroxine in 1914. Frederick Banting, Charles Best, James Collip and John Macleod isolated insulin in 1921–22; Banting and Macleod received the Nobel Prize in 1923. Estrogen (Allen and Doisy, 1923), progesterone and testosterone (1935) and cortisone (Kendall, Hench and Reichstein, Nobel 1950) completed the steroid list. Frederick Sanger sequenced insulin, the first protein sequence, in 1955 (Nobel 1958). Dorothy Hodgkin solved the structure of insulin in 1969. Rosalyn Yalow and Solomon Berson’s radioimmunoassay (1960) made it possible to measure hormones in blood; Yalow received the Nobel Prize in 1977.

Geoffrey Harris showed in the 1950s that the hypothalamus controls the pituitary through portal blood. Roger Guillemin and Andrew Schally isolated the hypothalamic

releasing hormones (Nobel 1977). The hormonal axes — hypothalamus, pituitary, target gland, and back — became the textbook examples of negative feedback. Ernst Knobil's group showed in 1978 that the pituitary responds to GnRH only when it arrives in pulses. Ronald Evans cloned the glucocorticoid receptor in 1985 and defined the nuclear-receptor superfamily in 1988. The group then found that receptors binding as dimers to repeated DNA half-sites are distinguished by the spacing of the repeats.

The open questions are about pattern and dose. Why do some hormones act only in pulses? How do receptors read the geometry of DNA? And why does continuous exposure turn an activator into a suppressor?

Current state. Diabetes is the largest endocrine burden. The 11th edition of the IDF Diabetes Atlas (7 April 2025) reports 589 million adults aged 20–79 with diabetes, 252 million of them undiagnosed, more than 3.4 million deaths a year and health expenditure above one trillion US dollars in 2024. It projects 853 million adults with diabetes by 2050 (IDF). On 1 December 2025 WHO issued its first guideline on GLP-1 medicines for obesity, which affects more than one billion people. Both key recommendations are conditional, and fewer than 10 % of those who could benefit are expected to have access by 2030 (WHO).

The structure of insulin is the classic case of an endocrine assembly. In 2-zinc pig insulin crystals, six insulin molecules form a hexamer of three dimers around two zinc ions; the deposited assemblies include a C3 hexamer and a D3 dodecamer of chains (RCSB PDB 4INS, Baker et al. 1988). Insulin lispro reverses the B-chain residues lysine and proline. This does not change receptor binding but blocks dimer and hexamer formation, so the monomer is absorbed fast; it was approved in the United States in June 1996 (Wikipedia, Insulin lispro).

Pulse pattern decides effect. In rhesus monkeys with hypothalamic lesions, pulsed GnRH restored gonadotropin secretion and continuous infusion did not; switching from pulses to continuous delivery desensitised the release mechanism (Belchetz, Knobil et al. 1978, Science, via QxMD). Medicine uses this as therapy: leuprorelin, a GnRH agonist, first raises LH and FSH, then over weeks desensitises and down-regulates the receptors and lowers sex hormones; it was approved in the United States for advanced prostate cancer on 9 April 1985 (Wikipedia, Leuprorelin).

The thyroid gives a numerical case. About 80 % of the hormone the thyroid releases is T4 (four iodines) and 20 % T3 (three iodines); about 85 % of circulating T3 is made from T4 by removing one iodine; T3 is roughly four times as potent (Wikipedia, Triiodothyronine).

Nuclear receptors read DNA geometry. For direct repeats of the half-site AGGTCA, "spacing the half sites by 3, 4, or 5 nucleotides determines specificity": 3 for the vitamin D receptor, 4 for the thyroid hormone receptor, 5 for the retinoic acid receptor — the "3-4-5 rule"; RXR and COUP-TF prefer a spacer of 1 (Kliewer et al. 1992, PNAS 89). The glucocorticoid response element is an inverted repeat with a three-nucleotide spacer, AGAACAnnnTGTTCT (Kimball's Biology Pages). B-DNA turns once per about 10.5 base pairs (Wikipedia, Nucleic acid double helix).

Interpretation.

1. *Strand.* I is time, in pulses; P is the set of states of an axis (hypothalamus, pituitary, target gland). For receptor binding, I is the position along DNA and P the base.
2. *Passage and trit.* A hormone pulse that is followed by a return to baseline is a passage. Relative to baseline (N3 as definition), the level is above (+1), at (0) or below (−1). Continuous infusion is a strand that never returns: it has no later passage, so no trit is read.
3. *Closure.* A pulse that rises and returns is a meeting, +1 then −1 (T7), with carry 0 (T7.1). The pituitary responds to meetings and desensitises without them: continuous GnRH is an open word that keeps a nonzero digit, tension in the reading of N2. Leuprorelin uses that tension as a treatment. The hormonal axes are three-stage negative-feedback rings: hypothalamus → pituitary → gland ⊣ hypothalamus. By the secant condition, a three-stage ring without delay oscillates only if its loop gain exceeds 8 (Calculation 14.10).
4. *Scale.* The insulin hexamer is $6 = 2 \cdot 3$ chains: one meeting merge (the dimer) and one ring merge (three dimers), the ladder of T15 with a = b = 1 (Calculation 14.12). The D3 dodecamer is $12 = 2^2 \cdot 3$. Lispro removes both merges and leaves the standing unit, the monomer. For nuclear receptors the mod-3 rule does not decide: the direct-repeat periods 9, 10 and 11 bp for DR3, DR4 and DR5 have residues 0, 1 and 2 mod 3, and all three are functional and specific (Calculation 14.11). What decides is the turn of the helix: 308.6°, 342.9° and 377.1° between half-sites (Calculation 14.11). Specificity is set by helical geometry, not by closure mod 3.
5. *Test (stated before data).* The three-stage ring reading predicts that any endocrine axis that oscillates without measurable delay must have a loop gain above 8, and that an axis of only two stages (gland ⊣ gland) cannot oscillate without delay. An axis of two stages with endogenous oscillation and no delay would refute the reading — and the secant theorem with it, which makes this a test of modelling rather than of the Foundations. For receptors, the reading is already tested and fails: if the mod-3 residue of the repeat period decided binding, DR3 (residue 0) would differ in kind from DR4 and DR5; it does not.

Consequences. Endocrinology adds a clear B-type case and a clear failure. The B-type case is the pulse: the field's own result is that a signal must return to be read. In the Foundations' terms a passage needs a return (N1 as definition), and a meeting closes with no carry (T7.1). That reading costs nothing and predicts nothing new, because Knobil's result is older. Its use is didactic: it names why a hormone that never falls is a different signal from one that pulses. The therapy of leuprorelin turns that into practice.

The insulin hexamer is a real 2·3 assembly, and the therapy that breaks it — lispro — is a real use of de-merging. The structure is published; the reading adds a name. The failure is the 3-4-5 rule. It looks like a mod-3 rule because three consecutive spacers serve three receptors, but the rule is about rotation along a helix of 10.5 bp per turn. Each added base pair turns the second half-site by about 34°, and the dimer interface reads that angle. Closure mod 3 plays no role. Saying so plainly protects the A-grade cases from being diluted by look-alikes.

The thyroid gives a further look-alike that should be named and set aside. T4 carries four iodines and T3 three. The gland releases them about 4:1, and the active hormone is made by removing one iodine. The numbers 4, 3 and the ratio 4:1 are all on the ladder of T15. But removing an iodine is a single chemical step with a dedicated enzyme, the deiodinase. It is neither a meeting nor a ring of passages, and nothing in the Foundations predicts that the less iodinated form should be the more potent one. Three type-specific deiodinases, one activating and one inactivating (reverse T3), make the thyroid a small switching network. Its sign pattern could be read as trits — activation +1, inactivation −1, no conversion 0 — but such a reading would only re-label the published biochemistry. It is a reading only.

The diabetes figures show where the field’s practical tension lies (Chapter 17). Of 589 million adults with diabetes, 252 million, or 43 %, do not know it (Calculation 14.12). An undiagnosed regulatory failure is a loop that has opened without being observed. That is a statement about health systems, not about the Foundations.

Grade. A for the 2·3 insulin hexamer (Baker et al. 1988) and for pulsatile-versus-continuous GnRH (Belchetz et al. 1978) as structures in the field; B for the three-stage axis ring and its gain threshold; C, with the mod-3 reading refuted, for nuclear-receptor spacing.

Chapter 15 — Perception, Behaviour and Mind

This chapter reads the meeting column of level L4: exchange that goes out and comes back between an organism and its world, or between organisms. It covers psychology, cognitive science, ethology and consciousness studies. The sharpest result is a refutation. For signed triads of relations, the closure rule of the Foundations and Heider’s balance theory disagree on two kinds of triangle. In millions of triads from three online networks, the kind the closure rule calls open (one friendship, two enmities) is over-represented in two of the three networks, by factors of 1.29 and 1.95. The closure reading of social triads is therefore refuted as a general law, and balance theory does no better on the other disputed kind (grade B, failed). A second claimed hook also weakens on inspection: intransitive choice cycles, which the Foundations call closed rings, do not survive a careful reanalysis of Tversky’s data (Regenwetter et al. 2011). What remains at grade A is narrower. Hering’s opponent colours are three zero-sum combinations of three cone signals. The rock-paper-scissors cycle of side-blotched lizard morphs (Sinervo and Lively 1996) is a closed ring of three in the field’s own data. Dominance hierarchies show the opposite: cyclic triads are rare, transitive ones common.

15.1 Psychology

History. Psychology became an experimental science in 1879, when Wilhelm Wundt opened his laboratory in Leipzig. Gustav Fechner had already given it a law. In *Elemente der Psychophysik* (1860) he wrote that sensation grows with the logarithm of the stimulus, building on Ernst Weber’s finding that the just-noticeable difference is a fixed fraction of the stimulus. Hermann Ebbinghaus measured forgetting with nonsense syllables (1885). William James wrote *The Principles of Psychology* (1890) and made consciousness the field’s subject.

Behaviourism rejected that subject. John B. Watson’s manifesto (1913) asked for observable behaviour only. Ivan Pavlov’s conditioned reflexes and B. F. Skinner’s operant conditioning (1938) defined learning as a change in the probability of a response after reinforcement. The cognitive revolution reversed the rejection in the 1950s. George Miller’s “The magical number seven” (1956), Noam Chomsky’s review of Skinner (1959) and Ulric Neisser’s *Cognitive Psychology* (1967) made internal representation scientific again.

Social psychology took another route. Fritz Heider’s “Attitudes and cognitive organization” (1946) proposed that a person (P), another person (O) and an object (X) form a triad. The triad is balanced when the product of the signs of its three relations is positive. Leon Festinger’s cognitive dissonance (1957) generalised the idea. Dorwin Cartwright and Frank Harary (1956) turned it into signed graph theory: a graph is balanced if all its cycles are positive, and then it splits into two camps.

Decision research found that people break the axioms of rational choice. Amos Tversky (1969) reported intransitive preferences: some subjects chose A over B, B over C and C

over A in a predictable way. Kahneman and Tversky's prospect theory (1979) explained many such departures, and Kahneman received the economics Nobel Prize in 2002. Clinical psychology grew in parallel, from Freud's psychoanalysis to behavioural therapy (1950s), cognitive therapy (Aaron Beck, 1960s) and evidence-based treatment.

In 2015 the Open Science Collaboration replicated 100 published studies and found effects much smaller than the originals. The replication crisis began. It left psychology's central question unchanged but sharper: which of its findings are stable properties of minds, and which are artefacts of method?

Current state. The largest answer to that question so far is the SCORE programme. In a press release of 1 April 2026, the Center for Open Science reported results published in *Nature* as a three-paper collection, by 865 researchers working on 3,900 papers (2009–2018) from 62 journals in the social and behavioural sciences (Center for Open Science). Of 164 papers tested with new data, 49 % replicated. The mean replication effect size was r = 0.10, against 0.25 in the originals — 40 % of the original size (Calculation 15.1). Reproducibility with the same data reached 91 % (approximately) when data and code were shared, and 38 % when data had to be rebuilt from public sources. In a robustness test, 74 % of conclusions stayed the same, 24 % became null or inconclusive and 2 % reversed. Human forecasters predicted replication with 76–78 % success.

The burden that clinical psychology faces is large. WHO estimates that about 1.2 billion people lived with a mental disorder in 2023, nearly one in seven. Anxiety disorders affected 470 million and depression 322 million (WHO fact sheet, 11 September 2026).

Two classical findings matter for this chapter. On intransitive choice, Regenwetter, Dana and Davis-Stober re-examined the literature after Tversky (1969) and found "conceptual, mathematical, and statistical errors" in earlier tests. With Tversky's data and new data, they concluded that observed choices fit variable but transitive latent preferences (Regenwetter et al., abstract). On triads, balance theory remains the reference. A triad is balanced when its signs multiply to a positive result: three positive links, or two negative and one positive (Wikipedia, Balance theory). The largest data on signed triads come from online networks. Leskovec, Huttenlocher and Kleinberg counted every undirected signed triad in Epinions (13.4 million triads), Slashdot (1.5 million) and Wikipedia (0.79 million), and compared each type with a shuffle of the signs (Leskovec et al. 2010, "Signed Networks in Social Media").

The open debates are about method (pre-registration, multi-site replication), about which constructs are measurable at all, and about whether mental disorders are categories or dimensions.

Interpretation.

1. *Strand.* I is a person's sequence of judgements; P is the set of objects judged — options in a choice, or other persons in a social network.
2. *Passage and trit.* A judgement on a pair is a passage. Oriented around a fixed cycle of three options or persons, a judgement is +1 (agrees with the orientation), −1 (against it) or 0 (indifferent, no relation). For signed relations, friendship is +1 and enmity −1.

3. *Closure.* A triad of judgements is a window of three trits. By T1 it closes when its sum leaves digit 0. Of the 8 possible tournaments on three options, exactly 2 are cyclic, with sums +3 and −3: rings, which close with carry ±1 (T7.1). The 6 transitive orders leave digit ±1 and are open (Calculation 15.2). For signed triads, (+,+,+) and (−,−,−) close; (+,+,−) and (+,−,−) are open. Balance theory uses the product of signs instead. The two rules agree on (+,+,+) (closed and balanced) and (+,+,−) (open and unbalanced). They disagree on (+,−,−), which balance calls balanced and closure calls open, and on (−,−,−), which balance calls unbalanced and closure calls closed (Calculation 15.4).
4. *Scale.* A majority of individuals merges individual orders into one group order. With three voters and three options, 12 of the 216 profiles give a cyclic majority, the Condorcet ring (Calculation 15.3). The merge is not by equal blocks, so T15 does not apply.
5. *Test (stated before data).* (a) If closure governs which triads persist, (+,−,−) must be under-represented relative to the sign shuffle and (−,−,−) over-represented. Balance theory predicts the reverse for both. (b) If intransitive cycles are closed rings, they should be stable over repeated sessions, while transitive orders that are partly inconsistent should drift.

Consequences. The data on signed triads decide test (a) against the closure reading. Overall, closed triads rise from a shuffled share of 0.624, 0.476 and 0.499 to observed shares of 0.877, 0.851 and 0.713 in Epinions, Slashdot and Wikipedia (Calculation 15.5). But balanced triads rise even more, to 0.941, 0.912 and 0.909. In the two triad types where the rules disagree, (+,−,−) is over-represented by factors of 1.29 in Epinions and 1.95 in Wikipedia, and under-represented (0.61) only in Slashdot. (−,−,−) is over-represented in Epinions (2.33), near chance in Wikipedia (1.10) and under in Slashdot (0.92) (Calculation 15.5). The closure rule's prediction for (+,−,−) fails in two of three networks; balance theory's prediction for (−,−,−) fails in one clearly (Epinions) and weakly in another. Neither rule describes the data; both are beaten by the excess of (+,+,+). The shuffled baseline is itself T6: with a share q of positive edges, the closure probability of a random triad is $q^3 + (1 - q)^3$, which reproduces the published shuffled shares of closed triads (Calculation 15.6). This was announced as the sharpest test in the first edition of this book. It has now been run, and the closure reading of signed social triads fails.

Test (b) is pre-empted by Regenwetter et al.: once the statistics are done correctly, Tversky's cycles are not robust. The Foundations' claim that a cyclic choice is a closed ring remains true as arithmetic, but there is no stable behavioural cycle for it to describe. In psychology, then, the Foundations add a clear counting of triads and a baseline (T6). They add no law of mind. The replication figures (SCORE) are a reminder that the field's own effects shrink to 40 % of their published size on new data; any reading that rests on a single striking effect inherits that uncertainty.

It is worth stating what a successful closure law of triads would have had to show, so that a later attempt can be judged. The signed-triad data have two layers. The first is the share of positive edges, q. Given q alone, the shuffled shares of the four triad types follow from T6, and the published shuffled shares match q^3, $3q^2(1 - q)$, $3q(1 - q)^2$ and

$(1 - q)^3$ to within 0.001 (Calculation 15.6). That layer carries no social information. The second layer is the departure from the shuffle. A closure law would predict a specific sign of departure for each type: up for (+,+,+) and (−,−,−), down for (+,+,−) and (+,−,−). The observed departures are up, down, mixed and mixed. Balance theory predicts up, down, up, down. Status theory, which Leskovec and colleagues also test, reads directed edges as "higher than" and predicts a third pattern. In the undirected counts, the only robust signs across all three networks are the excess of (+,+,+) and the deficit of (+,+,−), and those two are predicted by every rule. The two disputed types decide nothing consistently. The honest conclusion is that triads of online signed relations are dominated by positive closure and by avoidance of one specific open configuration — two friends who are enemies of each other's friend — and that the mod-3 rule adds a correct prediction only where balance theory makes the same one.

The same counting applies to clinical psychology, where triangles are a working concept. Murray Bowen's family-systems theory and Stephen Karpman's drama triangle (1968) both describe three-party configurations whose roles rotate (Chapter 17). Neither has produced a dataset of counted triads with signs, so neither can be graded here.

Psychology's own measurement of three-valued responses is also relevant. Many scales offer a midpoint — agree, neutral, disagree — and the treatment of the midpoint is a recurring methodological debate. In the Foundations' notation that is the digit 0, the standing of T7. A response scale with a midpoint is a trit alphabet; one without it is binary. The Foundations predict nothing about which gives better measurement, and the reading is a description only.

Grade. B for the closure reading of signed triads, tested and refuted in two of three networks (Leskovec et al. 2010); C for intransitive choice as a closed ring, since the cycles themselves do not replicate (Regenwetter et al. 2011); A for the count of cyclic tournaments and Condorcet profiles (2 of 8; 12 of 216), which is combinatorics.

15.2 Cognitive science

History. Cognitive science began as a deliberate merger. The Macy conferences on cybernetics (1946–1953) brought together Norbert Wiener, John von Neumann, Warren McCulloch, Walter Pitts and Gregory Bateson. McCulloch and Pitts (1943) had shown that idealised neurons compute logical functions. Claude Shannon's information theory (1948) gave a measure for messages. At the MIT symposium of September 1956, Miller presented the magical number seven, Allen Newell and Herbert Simon presented the Logic Theorist and Chomsky presented transformational grammar. That date is usually given as the birth of the field.

The first generation took the mind as a symbol system. Newell and Simon's physical symbol system hypothesis (1976) held that symbol manipulation is necessary and sufficient for intelligence. David Marr's *Vision* (1982) set three levels of explanation: computational, algorithmic and implementational. Connectionism answered in the 1980s. Rumelhart, McClelland and the PDP group (1986) modelled cognition as learning in networks of simple units, using backpropagation. Embodied and enactive

approaches followed: Varela, Thompson and Rosch's *The Embodied Mind* (1991) held that cognition is the activity of an organism coupled to its world.

Perception was a constant test bed. Thomas Young (1802) and Hermann von Helmholtz proposed three receptor types for colour; Ewald Hering proposed opponent pairs — red against green, blue against yellow, black against white. Leo Hurvich and Dorothea Jameson (1957) measured hue cancellation to support Hering. Physiology found both: three cone types and opponent cells downstream. Memory was the other test bed. Miller's seven was revised by Nelson Cowan (2001) to about four chunks.

Predictive processing, from Helmholtz's unconscious inference through Rao and Ballard (1999) and Karl Friston's free-energy principle (2005–2010), proposed that the brain is a hierarchy of predictions corrected by errors. Large language models, after 2017, gave the field a new comparison: systems with no explicit symbols that nonetheless produce language. The question left is old: what is a representation, and how many distinct states does a cognitive system actually use?

Current state. Working-memory capacity is the field's most-cited number. Miller's 1956 paper (*Psychological Review* 63, 81–97) found that absolute judgement along one dimension is near-perfect up to five or six stimuli, about 2 to 3 bits, and that memory span is about seven items regardless of bits per item. Span is about 7 for digits, 6 for letters and 5 for words (Wikipedia, The Magical Number Seven). Cowan argued that the pure capacity of the focus of attention is about four chunks on average, ranging from three to five in the population (Cowan 2001, BBS 24).

Colour vision gives a case of a receptor count and a coding scheme. Humans have three cone types, L, M and S. The opponent channels computed from them are a weighted sum for luminance, $L - M$ for red–green and $S - (L + M)/2$ for blue–yellow. A 2023 review by Conway and colleagues concluded that Hering's opponent-colour theory, as a theory of unique hues, is wrong, while the cone-opponent process is real (Wikipedia, Opponent process). The count of receptors does not fix discrimination. Mantis shrimp have 12 photoreceptor types from about 300 to 720 nm, yet they discriminate colour much worse than humans; Thoen and colleagues argue that they classify colours by receptor pattern rather than comparing channels (Phys.org on Thoen et al. 2014, Science 343).

The Foundations name a computational case. Ternary-weight neural networks, including the 1.58-bit large-language-model line, use exactly the alphabet $\{-1, 0, 1\}$ (Ma et al. 2024, cited in Foundations V.1). One trit holds $\log_2 3 = 1.585$ bits (Calculation 15.10).

The frontier is the comparison of humans and machines on the same tasks, the use of predictive-processing models to explain perception and psychiatric symptoms, and the question of whether large models have anything like working memory.

Interpretation.

1. *Strand*. I is processing time; P is the set of representational states of a channel (a cone, an opponent channel, a slot in working memory).

2. *Passage and trit.* An opponent channel reports one of three states relative to its neutral point: more of one pole (+1), balanced (0), more of the other pole (−1). This is a digit read relative to a reference (N3 as definition).
3. *Closure.* An opponent channel is a meeting: L and −M, or S and −(L+M)/2. Both opponent combinations have coefficients that sum to zero, while the luminance row does not (Calculation 15.8). The three combinations are linearly independent (determinant −2), so the three cone signals are fully represented by one standing (luminance) and two meetings. A neutral grey is the state where both meetings close. That is T7 applied to a coding scheme: a standing and two meetings, no ring. The published review that rejects Hering's unique hues does not affect this point, which concerns cone opponency, which the review accepts.
4. *Scale.* Counts in cognition against the ladder: 3 cone types and 3 opponent channels are on it, as are Cowan's 4 and the 6 letters of span; Miller's 7 and the 5 words are not; the mantis shrimp's 12 receptor types are (Calculation 15.7). The baseline for small numbers is high (Chapter 14, Calculation 14.3), so these counts carry no weight.
5. *Test (stated before data).* If opponent coding is a structure of meetings (T7), every opponent channel in any trichromatic species should have zero-sum cone weights, and luminance should be the only channel with a nonzero sum. A trichromat with a stable chromatic channel whose cone weights do not sum to zero, and that is not a luminance channel, would contradict the reading. For memory, the Foundations make no prediction: four chunks is not derived from anything in them.

Consequences. The reading adds little that colour science does not know. That opponent channels are differences of cone signals is published physiology. The Foundations add a classification: in this case the three closures of T7 sort the channels into one standing and two meetings, and the ring is absent. That absence is informative. A ring in colour space would be a cyclic channel in which red, green and blue each oppose the next, and no such channel is found. The hue circle is a ring in perception, but it is constructed from two meetings, not coded by a ring.

For working memory the reading adds nothing. Seven, four and five are measured; they are not on any ladder the Foundations need, and nothing in the Foundations fixes a capacity. The ternary-weight networks are a fact of engineering that the Foundations cite. They show that the alphabet {−1, 0, +1} is sufficient for useful computation, which supports T1's choice of register as a practical matter. They do not show that minds use it.

Miller's channel capacity can be expressed in trits, and the conversion shows how little follows. Absolute judgement along one dimension carries about 2 to 3 bits. That is 1.26 to 1.89 trits, or 4 to 8 alternatives (Calculation 15.7). One trit is 3 alternatives and two trits are 9. The measured range lies between one and two trits and matches neither. A theory that claimed the mind judges in whole trits would predict a capacity near 3 or 9 categories; the data show 5 to 6. That claim is not made here, and the arithmetic shows why it should not be.

Predictive processing gives a better match in form, though not in number. In that framework a level of the hierarchy sends a prediction down and receives an error up. The error has a sign: the input exceeds the prediction, matches it, or falls short. That is the same three-valued structure as the dopamine signal of Chapter 14 (Schultz et al. 1997), where it is published fact. In the Foundations' terms a correct prediction is a closed window, digit 0, with no carry; an error is a nonzero digit passed on (T1, read through N2). The predictive-processing literature does not use the word closure, but its basic quantity is the residue of a comparison. The added content of the Foundations is the claim that this residue is passed on whole to the next scale (T1 (ii)). In predictive-processing models the error is weighted by a precision before it is passed on, so the claim of whole passage is false as a description of those models. Precision weighting is exactly the kind of state-dependent response that N4, the hypothesis of T23, allows: the coupling depends on the state. That makes the match a reading, not a test.

The case of colour shows the general limit of receptor counting. Three cone types give three channels; twelve receptor types in the mantis shrimp give worse discrimination, not better. What matters is how the signals are compared, not how many there are. The Foundations' grammar of comparison — standing, meeting, ring — fits the human case and is silent about the shrimp, which appears not to compare channels at all.

Grade. A for opponent channels as zero-sum combinations of three cone signals (published cone opponency; Calculation 15.8); B for the zero-sum test on opponent channels in other trichromats; C for capacity limits.

15.3 Ethology and animal behaviour

History. Ethology grew from natural history. Charles Darwin's *The Expression of the Emotions in Man and Animals* (1872) treated behaviour as inherited like anatomy. Oskar Heinroth studied the displays of ducks in the 1910s. Jakob von Uexküll described the *Umwelt*, the world as each species perceives it (1909). Konrad Lorenz described imprinting in geese (1935) and fixed action patterns released by key stimuli. Niko Tinbergen's experiments with sticklebacks and herring gulls made the release of behaviour measurable. Karl von Frisch decoded the waggle dance of honeybees (1940s). The three shared the Nobel Prize in 1973.

Tinbergen's "On aims and methods of ethology" (1963) set four questions for every behaviour: its mechanism, its development, its function and its evolutionary history. Behavioural ecology in the 1970s took up function with mathematical tools. W. D. Hamilton's kin selection (1964), Robert Trivers's reciprocal altruism (1971) and John Maynard Smith and George Price's evolutionarily stable strategy (1973) made game theory part of biology. Maynard Smith's *Evolution and the Theory of Games* (1982) included the rock-paper-scissors game as a case where no pure strategy is stable.

Field studies found real cycles. Barry Sinervo and Curtis Lively (1996) reported that male side-blotched lizards in California come in three throat-colour morphs that beat each other in a cycle. Thorleif Schjelderup-Ebbe described the "peck order" of hens in 1922; dominance hierarchies became a standard object of animal social behaviour. Collective motion was studied by models first — Craig Reynolds's boids (1987), Tamás

Vicsek's self-propelled particles (1995) — and then measured in the field: the STARFLAG project reconstructed starling flocks in three dimensions (2008).

Locomotion is behaviour too. Eadweard Muybridge's photographs of a galloping horse (1878) settled whether all four hooves leave the ground. Gaits are now studied as patterns of a central pattern generator, with fixed phase relations between limbs.

The open questions of ethology concern how simple local rules produce group order, and when cycles among strategies persist rather than collapse to one.

Current state. The lizard cycle remains the reference case. Orange males take females from blue males, yellow males take them from orange, and blue males take them from yellow; orange is rock, blue scissors and yellow paper. In the study population, orange males increased for two years, fell for three, and rose again by year five (summary of Sinervo and Lively 1996, Nature 380, 240–243). A related cycle of 4 to 8 years has been reported in European common lizards in the Pyrenees (UC Santa Cruz news).

Dominance hierarchies show the reverse pattern. In a comparative dataset of dominance networks compiled from studies over 80 years, Shizuka and McDonald found transitive triads over-represented at high frequency in nearly all groups and cyclic triads under-represented and very rare (Dryad dataset, Shizuka and McDonald).

Starlings in flocks of a few thousand birds each interact with a fixed number of about six or seven neighbours, not with all birds within a fixed distance (Ballerini et al. 2008, PNAS). Horse gaits have 4 beats at the walk, 2 at the trot (diagonal pairs together), 3 at the canter, and 3 or 4 at the gallop, which becomes four-beat only at the highest speeds (Wikipedia, Horse gait).

The 2026 Nobel Prize in Physiology or Medicine for the light-gated switch of optogenetics (Chapter 14) has become a tool of ethology: specific neurons can be switched on or off while an animal behaves, linking Tinbergen's first question to his third.

Interpretation.

1. *Strand.* For strategy cycles, I is generations or breeding seasons and P is the set of morphs. For hierarchies, I is a sequence of contests and P the set of animals. For gaits, I is time within a stride and P the set of limbs.
2. *Passage and trit.* A contest between two animals, oriented around a fixed cycle of three, is +1 if the orientation holds, −1 if reversed, 0 if undecided. In the lizard cycle the morph that rises is the one that beats the current majority; each step around the cycle is +1.
3. *Closure.* The lizard cycle orange → blue → yellow → orange, oriented as published, has steps (+1, +1, +1): a ring with sum 3, digit 0, carry +1 (Calculation 15.9). It is a closed window of the homogeneous kind (T4) and the ring of T7. A transitive dominance triad (A > B > C, A > C) oriented around the cycle has steps (+1, +1, −1): open, digit +1. Hierarchies are therefore made of open windows, and their rare cyclic triads are the closed ones. In gaits, the trot moves diagonal pairs together: two limbs in phase, two in anti-phase, a meeting at the scale of the stride.

4. *Scale.* In the lizard system the ring closes over about five years. A ring that closes becomes one passage of the next scale (T7.1): the five-year cycle is the unit of the next level of population dynamics. Gait beat counts 2, 3 and 4 are on the ladder (Calculation 15.9); so is every integer up to 4. Starlings' six neighbours are on the ladder and seven is not.
5. *Test (stated before data).* If closure governs which triads persist, social systems should over-represent closed triads. Dominance hierarchies are a test with published data. The closed (cyclic) triads are rare and under-represented, the open (transitive) ones common. In dominance, the closure reading fails. In strategy cycles among morphs it holds: the cycle is closed and persistent. The refined test is this: closed rings should persist where the three parties are *types* that replace one another across generations (morphs, strains), and open orders should persist where they are *individuals* within a generation (dominance). A three-type system with a stable transitive order over many generations, or an individual-level dominance network with an excess of cyclic triads, would refute even this narrower reading.

Consequences. Ethology sharpens what the psychology section found. Rings of three exist in behaviour where three strategies replace one another, as in the lizards and as in Kerr et al.'s bacterial strains (Chapter 11). There the ring is the closed configuration and persists. In dominance among individuals, the open configuration dominates, and the closed one is avoided. A theory that predicts persistence of closed windows everywhere is refuted by hierarchies. A theory that predicts it only for replacement cycles is consistent with both, but it now needs a second rule to say which systems are which. That rule is ecological, not arithmetic.

What the Foundations add is a uniform way to count. A triad census, used in both dominance studies and network science, sorts triads into transitive and cyclic. The digit rule of T1 gives that sorting a meaning: transitive triads leave a residue; cyclic ones carry. Whether that meaning helps a behavioural ecologist is doubtful. The five-year lizard cycle is not a closure depth of T17, and the gait counts are small.

Two further structures in ethology deserve to be checked rather than assumed. The first is the gait. A quadruped has four limbs, and each gait fixes the phase of each limb relative to the others. In the trot, diagonal limbs move together and the two diagonal pairs alternate: the stride is one meeting of two pairs. In the pace, lateral pairs do the same. In the canter, three beats per stride make the stride a ring in time: one hind leg, then a diagonal pair, then the opposite fore leg. The gallop becomes four-beat only at the highest speeds, when the diagonal pair separates. Read through T7, the sequence walk (4), trot (2), canter (3), gallop (3 or 4) moves between standings, meetings and rings as speed changes. That reading uses no number that is not 2, 3 or 4, and every integer up to 4 is on the ladder (Calculation 15.9). It is a description and grades nothing.

The second is collective motion. Starlings in flocks of thousands interact with a fixed number of neighbours, six or seven. The hexagonal coordination of a plane has six neighbours, and $\mathbb{Z}[\omega]$ has six units (T14). If the flock's interaction number were exactly six, a reading through $\mathbb{Z}[\omega]$ would be possible. The published value is "six or seven", and a flock is three-dimensional, where close packing gives twelve neighbours, not six.

The Z[ω] reading would therefore need a planar flock, which a starling murmuration is not. The honest result is that the neighbour number is topological rather than metric — a published finding — and that the Foundations have no derivation of six or seven.

The lizard ring has one more property that the Foundations name precisely. A rock-paper-scissors cycle is chiral: orange beats blue, not the reverse. The reversed cycle, orange losing to blue, is a different ring, (−1, −1, −1) instead of (+1, +1, +1), with carry −1 instead of +1 (T7.1). The Foundations' first knot is chiral too (T20). Nothing in the lizard data connects to knots; the shared property is only that a cycle of three has two orientations, which is T4's count of two homogeneous non-zero closures.

Grade. A for the three-morph rock-paper-scissors ring (Sinervo and Lively 1996) and for the rarity of cyclic dominance triads (Shizuka and McDonald), as published structures; B for the narrowed test that closed rings persist among replacing types and open orders among individuals; C for gaits and flock neighbour counts.

15.4 Consciousness studies

History. The modern study of consciousness started as philosophy. Descartes (1641) separated thinking substance from extended substance and left the problem of their interaction. Leibniz's mill argument (1714) said that one could walk inside a thinking machine and see only parts pushing parts. William James (1890) described the stream of consciousness and held that it was the subject of psychology. Behaviourism removed it from science for half a century.

Neurology brought it back by cases. Penfield evoked experiences by stimulating the cortex. Split-brain patients, studied by Roger Sperry and Michael Gazzaniga from the 1960s, reported different things with each hemisphere. Lawrence Weiskrantz described blindsight (1974–1986): patients with damage to the primary visual cortex point accurately to objects they say they cannot see. Benjamin Libet (1983) recorded a readiness potential that precedes the reported decision to move.

Francis Crick and Christof Koch proposed in 1990 to look for the neural correlates of consciousness, the minimal brain activity sufficient for a specific experience. David Chalmers coined the "hard problem" at the Tucson conference in 1994 and published it in 1995: why is there experience at all, beyond the functions the brain performs? Bernard Baars proposed the global workspace theory (1988); Stanislas Dehaene and Jean-Pierre Changeux developed it into the global neuronal workspace (GNW), in which a content becomes conscious when it is "ignited" and broadcast across frontal and parietal cortex. Giulio Tononi proposed integrated information theory (IIT, 2004), which identifies consciousness with the integrated information of a system's causal structure and locates it mainly in posterior cortex.

The two theories made different predictions, and in 2019 the Templeton World Charity Foundation funded an adversarial collaboration to test them against each other. Higher-order theories, recurrent-processing theory and predictive-processing accounts compete as well. The question left is whether any of these theories can be refuted by data, and what it would mean for one of them to win.

Current state. The adversarial test reported on 30 April 2025 in *Nature*. Seven laboratories studied 256 participants with fMRI, MEG and intracranial EEG. IIT's prediction that conscious perception depends on sustained synchronisation in posterior regions was not supported. For GNW, some conscious content appeared in prefrontal cortex, but key features were missing, and the predicted "ignition" at the offset of a conscious experience could not be confirmed (Max Planck Institute for Empirical Aesthetics, 30 April 2025; DOI 10.1038/s41586-025-08888-1). Both theories therefore lost a pre-registered prediction.

The hard problem is unchanged as a philosophical question. Chalmers's terms, "hard problem" and "easy problems", come from his 1994 Tucson talk and his 1995 paper in the *Journal of Consciousness Studies* (Wikipedia, Hard problem of consciousness).

The tools for intervention have improved. The 2026 Nobel Prize for the molecular light switch of nerve cells (secondary report: Al Jazeera, 5 October 2026) recognised a method that lets experimenters turn specific neurons on and off in animals. In animals this allows causal tests of which circuits are needed for a behavioural report of perception. Connectomes at the scale of a cubic millimetre of human cortex (Chapter 14) give the anatomy in which any such theory must be implemented.

The frontier is now methodological: designing experiments in which two theories make opposite predictions, measuring consciousness in non-reporting patients and animals, and asking whether large language models could have experience.

Interpretation.

1. *Strand.* I is time in a trial; P is the set of global states of the brain (the sequence of active configurations).
2. *Passage and trit.* A passage is the return of the global state to a configuration it has had before — a recurrence. Relative to the first occurrence (N3 as definition), the content can be reinforced (+1), unchanged (0) or suppressed (−1).
3. *Closure.* The Foundations' reading of a conscious content would be a closed window at the scale of the whole brain: a configuration that returns and leaves no residue, an eigenform (Foundations II.2, a correspondence). Recurrent-processing and global-workspace theories also tie consciousness to activity that returns and is sustained. An unconscious content would be one that passes through once and does not return.
4. *Scale.* By T7.1 a ring that closes becomes one passage of the next scale. Read here, a closed local loop (a recurrent column) becomes one element of a global pattern. That is a description of hierarchy, not a number. No count of states, cycles or regions in the consciousness literature is compared with the ladder; the sample size of Cogitate, 256 = 2^8, is a design choice (Calculation 15.10).
5. *Test (stated before data).* If consciousness is the closure of a returning global configuration, then contents that are reported should show a measurable return of the same population pattern — a recurrence — within a trial, and unreported contents should not. A reported content with no recurrence of its neural pattern, measured at adequate resolution, would refute the reading. This test is close to

what recurrent-processing theory already predicts, so it does not separate the Foundations from that theory.

Consequences. The adversarial collaboration shows how hard the field's own theories are to test. Both leading theories lost a prediction. In that context a third reading that does not make a sharper prediction adds nothing. The closure reading of consciousness is a correspondence with Kauffman's eigenform (Foundations II.2–II.3), declared and not derived. It suggests looking at recurrence of whole-brain patterns, which the field already does. It does not say how much recurrence, at what scale, or why closure should be felt. The hard problem is untouched.

If anything the Foundations add here, it is a warning taken from the earlier sections. Closure readings of social triads and dominance hierarchies failed when the data were counted. A closure reading of consciousness should be stated as a quantity — what fraction of reported contents recur within how many milliseconds — before it is offered as an explanation.

Three classical findings can be stated in the Foundations' terms to show the limit of such statements. Blindsight is accurate behaviour without report: a strand that reaches the motor output without, on the reading above, closing at the global scale. Libet's readiness potential precedes the reported decision: the closure, if it is one, comes after the passage it is supposed to cause. Split-brain patients report two contents at once: two closed configurations that cannot be joined, which is the form of T11, where two different closed windows cannot be joined without tension at the seam. Each of these is a re-description. None of them makes a prediction that the original authors did not make. They are listed so that the reader can see what re-description looks like, and why it is graded C.

The one place where the Foundations could add a number is the scale ladder. If a global conscious state were the merger of closed local rings by T7.1, then the number of local units that merge into one global state at each step would be 3, and the number of steps from a cortical column to the whole cortex would be a count on the ring ladder. Nothing in the present literature measures such a hierarchy of equal blocks, and the hypothesis of equal blocks (T15; IX.4) is not met by the anatomy of Chapter 14, where cell counts per area vary widely. The claim cannot be tested yet, and it is not made.

Grade. C for consciousness as closure (a correspondence); B only in the weak form of the recurrence test, which does not separate the reading from recurrent-processing theory.

Chapter 16 — The Life Course

This chapter reads the ring column of level L4: the cycles of the body, from the daily clock and the night's sleep to the menstrual cycle, the generations and the long arc of ageing. It covers chronobiology, human development, ageing and reproduction. The firmest structure is in chronobiology. The molecular clock is a transcriptional loop of negative feedback, and control theory shows that such a loop needs at least three stages to oscillate without delay, with a gain above 8 at three stages. That is the ring of T7, published as theorem and as model (grade A). The second is in reproduction. Meiosis works only by meetings: every chromosome must pair with its homologue. A triploid, with three copies, cannot pair them, and its chance of a balanced gamete falls as $2 \cdot (1/2)^n$ (grade A, but for the meeting, not the window of three). Several tempting numerical links fail when checked. The human free-running period of 24.18 hours is not a comma of the ladders. The mean menstrual cycle of 29.3 days is within 0.04 standard deviations of the synodic month, but the distribution is too wide for that to mean anything. Ageing follows the Gompertz law, a doubling of the death rate about every eight years. That is an exponential, not a ladder, and the Foundations add nothing to it (grade C).

16.1 Chronobiology (circadian rhythms and sleep)

History. In 1729 the French astronomer Jean-Jacques d'Ortous de Mairan kept a mimosa plant in darkness and saw its leaves still open and close each day. The rhythm came from inside the plant. Augustin Pyramus de Candolle found in 1832 that the mimosa's period in constant conditions was about 22 to 23 hours — not exactly a day, so not simply driven by a hidden external cycle. Erwin Bünning (1930s) showed that the period is inherited in beans and proposed that the clock measures day length for flowering. Colin Pittendrigh and Jürgen Aschoff founded the modern field in the 1950s and 1960s, with the concepts of free-running period, entrainment and the phase-response curve. The Cold Spring Harbor Symposium of 1960 is usually taken as its starting point.

The human clock was measured in isolation. Aschoff and Rütger Wever kept volunteers in an underground bunker at Andechs, without time cues, from 1964 to 1989. Subjects ran on about 25 hours. Some showed "spontaneous internal desynchronisation": their sleep–wake cycle split from their body-temperature rhythm, sometimes to very long or short periods. Nathaniel Kleitman, who had lived in Mammoth Cave in 1938 to study the day, worked with Eugene Aserinsky on rapid-eye-movement sleep (1953) and proposed a basic rest–activity cycle of about 90 minutes across day and night.

The location of the mammalian clock was found in 1972: lesions of the suprachiasmatic nucleus of the hypothalamus (Moore and Eichler; Stephan and Zucker) abolished rhythms. The genetics began with flies. Ronald Konopka and Seymour Benzer (1971) found mutants of a gene, *period*, with short, long or absent rhythms. Jeffrey Hall and Michael Rosbash, and independently Michael Young, cloned *period* in 1984 and showed

that its protein feeds back to repress its own gene. Young found *timeless*; Joseph Takahashi found *Clock* in mice (1994–97). The three fly geneticists received the Nobel Prize in 2017. In parallel, Brian Goodwin (1965) proposed the three-step loop of gene, messenger and repressor as a model biological oscillator.

Sleep regulation got its standard model in 1982, when Alexander Borbély proposed two processes: a homeostatic pressure that builds with waking (process S) and a circadian process (C). Melatonin, discovered by Aaron Lerner in 1958, became the hormone of darkness, and a third class of photoreceptor in the retina, melanopsin, was found around 2000–2002.

The questions left are about coupling. How do the many cellular clocks of the body stay in phase? What sets the period precisely? And what does it cost when the internal day and the social day disagree?

Current state. The human pacemaker is precise. In a forced-desynchrony protocol, Czeisler and colleagues found an intrinsic period averaging 24.18 hours in both young and older subjects, with a tight distribution (Czeisler et al. 1999, Science 284, via NASA NTRS). The Andechs bunker studies included 447 volunteers in 418 studies. Under free-running conditions subjects drifted to about 25 hours, partly because of late bedtimes and light at night. In internal desynchronisation the sleep–wake cycle ranged from 12 to 68 hours while the temperature rhythm stayed near 25 hours (Wikipedia, Rütger Wever).

The 2017 Nobel Prize was awarded “for their discoveries of molecular mechanisms controlling the circadian rhythm” (Nobel Prize summary); Hall and Rosbash studied the *period* gene, a master clock gene of the 24-hour cycle (BrandeisNOW, 2 October 2017). The minimal oscillator remains the Goodwin loop. Its limit-cycle oscillations need a Hill coefficient above 8 in the classical model, which is considered unrealistic; current models explain the clock with additional mechanisms such as protein sequestration (Gonze and Abou-Jaoudé 2013).

Sleep has its own ultradian cycle. An adult cycle of non-REM and REM sleep is about 90 minutes, with a range of 70 to 110 minutes; a 7 to 8 hour night holds about five cycles; REM takes a larger share later in the night; infant cycles are about 50 to 60 minutes (Wikipedia, Sleep cycle).

The frontier lies in medicine and society: shift work, adolescent sleep, light at night, and the timing of drugs to the clock (chronotherapy). Wearable devices now measure rest–activity rhythms in large populations.

Interpretation.

1. *Strand.* I is clock time; P is the set of phases of the circadian cycle — for the molecular loop, the states of gene, messenger and protein.
2. *Passage and trit.* A passage is the return of the clock to a reference phase, such as the onset of melatonin. Relative to the external day (N3 as definition), a cycle ends later (+1), on time (0) or earlier (−1). In the transcriptional loop each link is an activation (+1) or a repression (−1).

3. *Closure.* The molecular clock is a negative ring. With one repression in the loop the sign is −1, and the loop can oscillate (Calculation 16.2). By the secant condition a two-stage negative loop cannot oscillate without delay, and a three-stage loop can if its gain exceeds 8 (Calculation 16.2). That is T7: a meeting closes at length 2 without going round; the shortest ring that goes round has length 3. In the language of T7.1, the daily cycle is a ring that closes with carry: each completed cycle becomes one passage of the next scale (a day in a week, a season).
4. *Scale.* An intrinsic period of 24.18 hours drifts 0.18 hours per day against the solar day and would complete a full drift in about 133 days (Calculation 16.1). Light entrains the clock and cancels the drift each day: a carry that is removed by the outer cycle. The ratio 24.18/24 is 12.94 cents, against the Pythagorean comma of 23.46 cents (Calculation 16.1): the circadian mismatch is not the comma of T18. Sleep cycles: 24 hours hold 16 cycles of 90 minutes, which is 2^4 on the meeting ladder, and a night holds 5, which is off the ladder (Calculation 16.3). In internal desynchronisation the sleep–wake cycle often locks at simple ratios to the temperature cycle; a 12.5 or 50 hour sleep–wake cycle against 25 hours would be 1:2 or 2:1 (Calculation 16.3), the first rung of the meeting ladder.
5. *Test (stated before data).* If the minimal clock is a ring, then a clock built from two components without delay must not oscillate, and synthetic clocks should need at least three repressive stages, or explicit delay, to run. The repressilator of Elowitz and Leibler (2000), a three-gene ring, oscillates; a two-gene mutual repression switches (Chapter 10, where the citation is given). A synthetic two-stage loop without delay that oscillates stably would refute the reading. For desynchronisation, the ladder predicts locks at 1:2 and 2:1 but not 2:3 or 3:4 more often than chance. The published range of 12 to 68 hours does not allow a count of locks; individual records would.

Consequences. Chronobiology confirms the Foundations at the point where they are strongest: three is the shortest closed ring that goes round, and the clock is such a ring. The field already knows this from the theory of feedback, and the Foundations add a reason that needs no differential equation. The clock also shows the meaning of the carry. Each completed day is passed on as one unit to the next scale: days to weeks and seasons, as T7.1 says of any closed ring. The cost is small. The numerical hope that the circadian period itself is a comma fails at a factor of almost two. The ratio of a 90-minute sleep cycle to a day, 16, is on the meeting ladder, but the 90-minute figure is a rough mean with a 40-minute range. Calling it 2^4 would be numerology.

Entrainment is a case of N4, the hypothesis of T23: the coupling of the clock to light is a function of its state. Light at one phase advances the clock, at another delays it, at a third does nothing. The phase-response curve is three-valued in sign — advance (+1), no shift (0), delay (−1) — as a published fact of the field since the 1960s. That gives a second A-grade structure, independent of any count.

The bunker data illustrate the difference between a ring that is entrained and one that runs free. Under entrainment the clock and the day form a meeting at every cycle: the clock’s residue is cancelled by light each morning, and nothing is carried. In isolation the residue is not cancelled. It accumulates, 0.18 hours per day at the precise period of

Czeisler and colleagues, or about an hour per day at the 25-hour period seen with uncontrolled light at Andechs. In the Foundations' terms the free-running clock carries a nonzero digit forward each day. Internal desynchronisation is a further step. Two rings that were joined — sleep–wake and temperature — separate and run at different periods, a seam between two closed windows that are no longer the same (T11). By T11 such a seam always carries tension, and the bunker subjects, who did not notice the split, nevertheless showed changes in sleep structure and performance. This is a re-description, but it uses a theorem rather than a metaphor: two different closed windows cannot be joined without tension.

Sleep itself offers a three-valued structure that is published: wake, non-REM sleep and REM sleep are the three states of the standard classification of vigilance, recognised since Aserinsky and Kleitman. Read as a trit — wake (+1), non-REM (0), REM (−1), or any other assignment — a night is a word in three letters, and a sleep cycle that runs non-REM → REM → brief wake is one heterogeneous closed window of T4. But the order of states within a night is constrained by physiology, not by closure: REM normally follows non-REM and rarely follows wake directly in healthy adults. Narcolepsy is defined in part by REM that arrives directly from wake. The Foundations do not predict which transitions are allowed, so the reading earns no grade. It shows only that the field's own three-state model is the alphabet of T1.

Grade. A for the three-stage minimum of a negative-feedback oscillator (secant condition; Goodwin model; Gonze and Abou-Jaoudé 2013) and for the three-signed phase-response curve; B for the test on synthetic two-stage clocks and on locking ratios in desynchrony; C for numerical coincidences of period and sleep-cycle count.

16.2 Human development

History. The idea of developmental stages is old. Ancient medicine divided life into ages, and Shakespeare's "seven ages of man" (1599) is a literary version. Scientific study began with the child as a natural object. Charles Darwin's "A biographical sketch of an infant" (1877) recorded his son's first years. Wilhelm Preyer's *Die Seele des Kindes* (1882) was the first systematic book. G. Stanley Hall founded child study in America and described adolescence (1904). Alfred Binet and Théodore Simon built the first intelligence scale in 1905 to find children who needed help in school.

The twentieth century produced stage theories. Sigmund Freud proposed psychosexual stages (1905). Jean Piaget, observing his own children and many others in Geneva from the 1920s, described four stages of cognitive development. Erik Erikson's *Childhood and Society* (1950) proposed eight psychosocial stages across the whole life. Lev Vygotsky (1930s) stressed the social origin of thinking and the zone of proximal development. John Bowlby's attachment theory (1969) and Mary Ainsworth's Strange Situation (1978) made the bond between child and caregiver measurable. Lawrence Kohlberg (1958) proposed stages of moral reasoning.

The life-course view of the 1970s–1990s — Paul Baltes, Glen Elder — rejected the idea that development ends in adulthood and placed individuals in historical time. Developmental neuroscience added the brain. Synaptic density peaks in early

childhood and is pruned. Myelination continues into the third decade. Adolescence is a period of large changes in cortical thickness and connectivity. The large cohort studies — the Dunedin study (from 1972), the ALSPAC cohort (from 1991) and the ABCD study (from 2015) — follow thousands of children over decades.

The questions left concern the shape of development. Are stages real discontinuities or a convenient description of continuous change? How much of development is universal, and how much depends on culture and environment?

Current state. Piaget's four stages remain the textbook frame: sensorimotor (birth to about 2 years), preoperational (2 to 7), concrete operational (7 to 11) and formal operational (from about 11 to 14–15) (Wikipedia, Piaget's theory). Erikson's theory has eight stages; Joan Erikson later added a ninth for the challenges of old age, in one's eighties and nineties (Wikipedia, Erikson's stages). Both are now taught as heuristic, not as fixed sequences of discrete states. Research on infant cognition has found many abilities earlier than Piaget placed them.

The empirical base has moved to cohorts. The Adolescent Brain Cognitive Development (ABCD) study invited 11,880 children aged 9 to 10 at 21 research sites in the United States and follows them into young adulthood, measuring how experiences such as sports, video games, social media, sleep and substance use interact with brain development (ABCD study).

The demographic setting changes the life course itself. Global life expectancy at birth was 73.3 years in 2024, recovered from 70.9 during the COVID-19 pandemic. More than half of all countries have fertility below 2.1 births per woman, and nearly a fifth have "ultra-low" fertility below 1.4 (UN, World Population Prospects 2024).

Debates concern the reproducibility of developmental findings (Chapter 15), the effects of screens and social media on adolescents, and whether stage models can be replaced by dynamical models of continuous change.

Interpretation.

1. *Strand.* I is age; P is the set of developmental states (cognitive, social, neural).
2. *Passage and trit.* A passage is a return to a developmental task in a new form — the same question met again at a later age. Relative to the first time it was met (N3 as definition), the later encounter is resolved further (+1), unchanged (0) or regressed (−1). Erikson's own scheme is a set of polar pairs (trust versus mistrust, and so on): each stage has a positive pole, a negative pole and an outcome between.
3. *Closure.* Each Erikson stage, read this way, is a meeting: two opposed poles, with resolution when they balance. A resolved stage passes nothing on; an unresolved stage carries its tension into the next. That is the reading of N2 by T1, in Erikson's own words for the carrying-over of unresolved crises. It is a re-description: Erikson already said it.
4. *Scale.* Counts of stages: 4 (Piaget), 8 (Erikson), 9 (Erikson with Joan Erikson) are on the ladder; so is the 12 of the hallmarks of ageing (Calculation 16.7). But these counts are chosen by theorists, not measured as closed units; and among small numbers, the ladder share is high (Chapter 14, Calculation 14.3). The age

boundaries of Piaget's stages — 2, 7, 11 — are not on a ladder; 2 is, 7 and 11 are not.

5. *Test (stated before data).* If developmental tasks are meetings that close or carry, then children who leave a task unresolved at one age should show the same task in their next transition with measurable carry — for example, insecure attachment at one year should predict difficulties at the next social transition more strongly than at later ones, with the effect passing step by step rather than skipping. This is a familiar claim of developmental cascades. A finding that early unresolved tasks predict late outcomes directly, without passing through the intermediate transitions, would contradict the step-by-step carry. The test belongs to the field already, and the Foundations do not sharpen it.

Consequences. Human development offers no structure that the Foundations recover as published fact. The stage counts are theorists' divisions. The polar pairs of Erikson look like meetings, but they were written as pairs by Erikson, so finding meetings in them is finding what was put there. The carry reading of unresolved tasks is close to the developmental-cascade model, which is already tested in cohort studies. The Foundations add a vocabulary for it, and an arithmetic constraint — carry is passed whole to the next scale (T1 (ii)) — that no developmental psychologist would accept as literally true.

The one structural fact is negative. Development is not a ring at the scale of a life: no person returns to infancy. The life course is an open word read from a start, counted as 3^n (T13 (i)), not a closed configuration counted without one. Generations are rings (Section 16.4); a single life is not. That distinction is useful because it tells the reader where to look for closure in the life course: in the cycles inside it (days, nights, months, years) and in the cycle of generations around it, not in its own arc.

Ages can be written in balanced ternary (T12): 13 is 111, 40 is 1111, 121 would be 11111 (Calculation 16.8). The numbers $(3^k - 1)/2$ are the capacities of k digits. Nothing in development happens at these ages; the observation is listed so that no one mistakes it for a finding.

Two facts of developmental neuroscience can be put against the Foundations more sharply than stage counts. The first is overproduction and pruning: synapses are made in excess in early childhood and then removed. In the reading of N2 by T1, connections that close — that are used in returning activity — persist, and the rest are removed. Hebb's rule and use-dependent pruning say this already. The second is the sensitive period. Hubel and Wiesel showed in the 1960s that closing one eye of a kitten during a short window after birth permanently changes the visual cortex, while the same closure in an adult cat does little. A sensitive period is a time when the first passage is being fixed. By N3, read as a definition, the first passage fixes the direction against which later passages are read; what is fixed in that window is the reference. That is an apt description, and it is the description the field already gives under the name of imprinting (Lorenz) and critical periods. It adds no number.

The demographic setting of development changes faster than development itself. With fertility below replacement in more than half of all countries, the cycle of generations

(Section 16.4) no longer closes in number: each generation is smaller than the last. In the Foundations' terms a population ring with fewer passages at each turn is an open word, not a closed one. Demography belongs to Chapter 18.

Grade. C for stage models and the carry reading of developmental tasks; the negative point that a single life is an open word, not a ring, is a correct use of T13 but grades nothing.

16.3 Ageing

History. Benjamin Gompertz, an actuary, published "On the nature of the function expressive of the law of human mortality" in 1825. He proposed that the force of mortality grows exponentially with age. William Makeham (1860) added a constant term for causes of death that do not depend on age. The Gompertz–Makeham law has fitted adult human mortality well for two centuries. August Weismann (1882) argued that ageing exists because the soma is disposable once the germ line is passed on. Peter Medawar (1952) explained why: selection weakens with age, so late-acting harmful mutations accumulate. George Williams (1957) added antagonistic pleiotropy: genes that help early and harm late are favoured. Thomas Kirkwood's disposable-soma theory (1977) put the trade-off in terms of energy.

Experimental gerontology began with Leonard Hayflick and Paul Moorhead (1961), who showed that normal human cells divide a limited number of times in culture. Denham Harman proposed the free-radical theory (1956). Telomeres, the repeated ends of chromosomes, were described by Elizabeth Blackburn and Joseph Gall (1975–77); Blackburn and Carol Greider discovered telomerase in 1984. Telomere shortening explained Hayflick's limit, and Blackburn, Greider and Jack Szostak received the Nobel Prize in 2009. Caloric restriction extended lifespan in rats (McCay, 1935). Single genes extended it in worms (Cynthia Kenyon, *daf-2*, 1993). Rapamycin extended it in mice (2009).

The field organised itself around mechanisms. López-Otín and colleagues listed nine hallmarks of ageing in 2013 and twelve in 2023. Steve Horvath's epigenetic clock (2013) estimated age from DNA methylation across tissues with high accuracy. The maximum human lifespan became a debated quantity after Jeanne Calment died in 1997, aged 122.

The questions left are whether ageing has one cause or many, whether a limit to human lifespan exists, and whether biological age can be measured and reversed.

Current state. The doubling time of adult mortality is the field's most robust number. After early adulthood the probability of dying in a year roughly doubles every eight years; in low-mortality populations the Gompertz–Makeham curve fits well from about age 40 to 90, while behaviour at the highest ages is debated (Wikipedia, Gompertz–Makeham law). Dong, Milholland and Vijg argued in *Nature* (5 October 2016), using the International Database on Longevity and the Human Mortality Database, that the maximum reported age at death has plateaued at about 115 years since around 1995, and treated Calment's 122 years as an outlier (Albert Einstein College of Medicine news).

The twelve hallmarks of ageing are genomic instability, telomere attrition, epigenetic alterations, loss of proteostasis, disabled macroautophagy, deregulated nutrient sensing, mitochondrial dysfunction, cellular senescence, stem-cell exhaustion, altered intercellular communication, chronic inflammation and dysbiosis (López-Otín et al. 2023, Cell 186, 243–278, via UCL Discovery). Normal human fetal cells divide about 40 to 60 times before senescence, a limit linked to telomere length (Wikipedia, Hayflick limit). The human telomere repeat is TTAGGG, and cultured human cells lose about 50 to 100 base pairs of telomere per division (Wikipedia, Telomere).

Horvath's clock uses 353 CpG sites, was built from 82 datasets with 7,844 non-cancer samples covering 51 tissues and cell types, and predicts age with a median error of 3.6 years in test data and a correlation of 0.96 (Horvath 2013, Genome Biology).

The frontier is intervention: senolytic drugs, partial reprogramming with Yamanaka factors, GLP-1 drugs as possible modifiers of age-related disease, and trials of metformin and rapamycin. The debate over whether ageing should be classified as a disease continues.

Interpretation.

1. *Strand.* I is age; P is the set of states of an organism's repair systems, or of a cell lineage.
2. *Passage and trit.* A passage is a cycle of damage and repair — a cell division, a day of metabolic load and nightly repair. Relative to the first cycle (N3 as definition), repair is complete (0), over-compensated (+1) or incomplete (−1).
3. *Closure.* A cycle of damage and repair that restores the initial state closes: digit 0, no carry. Incomplete repair leaves a nonzero digit, which is carried forward (N2 read through T1). Ageing, read this way, is the accumulation of carries that are never cancelled. That is the standard view of the field (damage accumulation) written in the Foundations' terms.
4. *Scale.* The Gompertz law is a doubling of hazard every eight years: a factor of 2^(1/8) = 1.0905 per year, $2^{10} = 1024$ from age 30 to 110 (Calculation 16.5). A doubling looks like the meeting ladder, but the exponent is continuous: 2^(50/8) from 40 to 90 is not a ladder number. An exponential in continuous time is not a merge of equal blocks, and T15 does not apply. Cell lineages: 3.72×10^{13} cells is about 2^{45} (Calculation 16.6), within the Hayflick range of 40 to 60 divisions if every division doubled the lineage. The telomere repeat is 6 bp, and a division removes about 8 to 17 repeats (Calculation 16.6).
5. *Test (stated before data).* The Foundations make no prediction about the rate of ageing. One test is possible for the claim that closed repair cycles carry nothing. In tissues where the damage–repair cycle is entrained to the circadian ring (Section 16.1), disruption of the ring should increase the carry — measured as faster epigenetic ageing. A study in which chronic circadian disruption leaves the epigenetic clock unaffected, with adequate power, would contradict the reading. This is an existing hypothesis of chronobiology and gerontology.

Consequences. Ageing gains no new structure from the Foundations. The Gompertz law is exponential, and the maximum-lifespan debate concerns tails of distributions.

The difference between 115 and 122 years is less than one doubling of hazard, 0.875 (Calculation 16.5). Calment's age is unusual, not impossible, under the Gompertz law. The twelve hallmarks are a list compiled by authors, and twelve being on the ladder is no evidence of anything.

What the Foundations add is a single sentence: ageing is the residue that repair does not close. That sentence is the damage-accumulation theory. It earns no grade above C. It does suggest where to look for closure in the field — the cycles of repair that are tied to the clock — and it predicts nothing beyond what chronobiology already predicts about circadian disruption.

Two hallmarks are closer to the Foundations than the others and should be checked rather than claimed. Telomere attrition is a count of repeats that falls by a roughly fixed number at each division. A lineage that divides n times loses about n times 50 to 100 base pairs, or 8 to 17 six-base repeats per division (Calculation 16.6). This is a strand that loses length at each passage. It does not close, and it carries a deficit forward; when the deficit reaches a threshold the lineage stops. That fits the carry reading but does not test it, because the carry here is a length, not a digit. Cellular senescence is the other: a cell that has stopped dividing but remains metabolically active and signals to its neighbours. Read through T7, a senescent cell is a standing — it neither divides (a ring of the cell cycle) nor dies — and its signalling is tension passed to the tissue. The hallmark list itself treats senescence as a cause of chronic inflammation, another hallmark, so the passing-on is published.

The epigenetic clock deserves separate mention, because it is the strongest quantitative result in the field. Horvath's 353 CpG sites give age with a median error of 3.6 years. The clock is a weighted sum of methylation levels, each between 0 and 1. It does not use any three-valued structure, and its number of sites is not on the ladder (Calculation 16.7). The Foundations have nothing to say about why methylation tracks age so closely. Saying so is the right result here.

Grade. C for ageing as accumulated carry and for the numerical observations; B only for the circadian-disruption test, which is not specific to the Foundations.

16.4 Reproduction

History. Aristotle held that the male supplies form and the female matter. William Harvey (1651) declared *ex ovo omnia*, all from the egg, though he never saw a mammalian egg. Antonie van Leeuwenhoek saw sperm in 1677. Karl Ernst von Baer found the mammalian egg in 1827. Oscar Hertwig observed the fusion of sperm and egg nuclei in the sea urchin in 1876. August Weismann (1880s) predicted that the number of chromosomes must halve before fertilisation, or it would double every generation. Meiosis, the reduction division, was described in the following years. Theodor Boveri and Walter Sutton (1902–03) linked chromosomes to Mendel's factors.

Twentieth-century reproductive biology moved to hormones and technology. The menstrual cycle was explained by pituitary and ovarian hormones in the 1920s–1930s. The oral contraceptive pill, developed by Gregory Pincus and John Rock, was approved in the United States in 1960. Joe Hin Tjio and Albert Levan established 46 as the

human chromosome number in 1956. Jérôme Lejeune found the extra chromosome 21 in Down syndrome in 1959. Robert Edwards and Patrick Steptoe, with Jean Purdy, achieved the first birth after in-vitro fertilisation: Louise Brown, born on 25 July 1978 in Oldham; Edwards received the Nobel Prize in 2010.

Meiosis became the field's central puzzle. Its errors, which produce aneuploid embryos, were found to be common in humans and to rise steeply with maternal age. Terry Hassold and Patricia Hunt called this "to err (meiotically) is human" (2001). Polyploidy was understood in plants: triploids cannot pair their chromosomes and are sterile, which is why bananas and some watermelons have no seeds.

The questions left are why human meiosis is so error-prone, why maternal age matters so much, and how far assisted reproduction can or should go.

Current state. The menstrual cycle has been measured in real-world data. In 612,613 ovulatory cycles from 124,648 users of a fertility app, the mean cycle length was 29.3 days (SD 5.2), with a mean follicular phase of 16.9 days and luteal phase of 12.4 days; only 13 % of cycles lasted 28 days; mean cycle length fell by 0.18 days per year between ages 25 and 45, and by 3.2 days overall (Bull et al. 2019, npj Digital Medicine). The mean synodic month is about 29.53059 days (Wikipedia, Lunar month).

Aneuploidy is common. Karyotyping studies found about 0.3 % of newborns, about 4 % of stillbirths and more than 35 % of spontaneous abortions to be aneuploid. Array methods find 30–60 % of preimplantation embryos and 30–70 % of eggs or polar bodies aneuploid, with rates possibly above 50 % for women near the end of reproductive life. Trisomies 18 and 21 rise exponentially with maternal age (Nagaoka, Hassold and Hunt 2012, Nature Reviews Genetics 13). Triploidy occurs in about 2–3 % of human pregnancies and about 15 % of miscarriages; most triploid conceptions are lost, and those born usually die soon after (Wikipedia, Triploidy).

Assisted reproduction is large. A study in *Fertility and Sterility* estimated 10 to 13 million IVF births between 1978 and 2018; a preliminary estimate adds 3 to 4 million since, for a 2024 total of 13 to 17 million (UNSW Sydney, 28 July 2025). More than 93 % of Australian IVF cycles now transfer a single embryo, with multiple births below 3 %.

The frontier includes in-vitro gametogenesis, embryo models from stem cells, polygenic embryo screening and its ethics, and falling fertility in most countries (Section 16.2).

Interpretation.

1. *Strand.* For meiosis, I is the sequence of chromosomes and P the set of homologous chromosomes. For the cycle, I is time and P the set of ovarian and hormonal states. For generations, I is descent and P the set of individuals.
2. *Passage and trit.* In meiosis each chromosome must meet its homologue: the pairing is the passage. A homologue that goes to one pole is +1, to the other −1; a chromosome that does not segregate properly leaves a residue.
3. *Closure.* Meiosis closes by meetings only. Each homologous pair is x, −x: one copy to each pole, digit 0, no carry (T7, T7.1). A triploid has three copies of each chromosome. Three cannot split into two equal halves; two go to one pole and one

to the other, at random. A gamete is balanced only if every chromosome sends its extra copy the same way, with probability $2 \cdot (1/2)^n$. For the human haploid set (n = 23) this is about 2.4×10^{-7}; for the banana (n = 11) about 0.1 % (Calculation 16.9). Here the ring of three copies is the configuration that cannot close by the meeting rule. That is a published rule of genetics, and it confirms the meeting side of T7, not the window of three.

4. *Scale.* One meiosis gives four haploid products, 2^2, two meeting steps (Calculation 16.9). A generation is a ring: from zygote to adult to gamete to zygote. It closes with carry: each completed generation becomes one passage of the population's history (T7.1). The menstrual cycle is a ring inside the life. Its mean, 29.3 days, lies 0.23 days or 0.044 standard deviations from the synodic month (Calculation 16.4). With a standard deviation of 5.2 days and only 13 % of cycles at 28 days, the mean's closeness to the month says nothing about locking.
5. *Test (stated before data).* (a) If meiosis is a closure by meetings, any odd ploidy must be sterile in the same way as triploidy, with balanced-gamete probability falling as a power of 1/2 in the haploid number, and even ploidies should be fertile when their chromosomes pair as bivalents. A fertile odd-ploid line with regular meiosis and no unreduced gametes would refute it. (b) If the menstrual cycle is a ring locked to the lunar month, the phase of cycle onset should cluster at a lunar phase in long individual records. Published data do not allow this test here, and the claim is not made.

Consequences. Reproduction adds the clearest case in this chapter of a rule the Foundations predict and the field already uses. The meeting is the elementary closure of meiosis. A configuration that cannot be reduced to meetings — three homologues — is sterile, and breeders use that sterility to produce seedless fruit. The arithmetic of $2 \cdot (1/2)^n$ is elementary and published. The Foundations add that the failure is the failure of a ring to close by meetings, which is correct and costs nothing. They do not add why meiosis, rather than a three-way division, is the rule; a three-way division would be a ring step and is not found in eukaryotes.

The high aneuploidy rate of human eggs is an open window in that meeting: a pair that fails to separate. The rise with maternal age is consistent with the carry reading of ageing in Section 16.3, where repair that does not close leaves a residue — here the cohesion that holds pairs together decays over decades. That is a re-description of an existing hypothesis (cohesin loss) and is graded C.

For the cycle, the lesson is numerical caution. A mean of 29.3 days near a month of 29.53 days is a coincidence until the phase data are shown. The very wide spread makes a lock impossible to read from the mean.

The chromosome cycle has one more structure that the Foundations recover exactly, and it should be stated with its limits. In meiosis I, homologues pair and separate: a meeting. In meiosis II, sister chromatids separate: a second meeting. One diploid cell gives four haploid cells, 2^2 (Calculation 16.9). Fertilisation joins two haploid sets: a meeting in the other direction. The full chromosome cycle of a sexual organism is therefore three meetings — two that halve and one that doubles — and the cycle closes when the ploidy returns to two. No step in it is a ring of three. Polyploid plants show

what happens when a ring is forced in: triploids, with three sets, cannot complete the halving meetings. Even polyploids (tetraploid, hexaploid) pair and remain fertile (Chapter 13). The rule “even pairs, odd fails” is the meeting ladder of T15 at a = 1, observed. The window of three plays no role, and a reader who hoped to find the ring of three in reproduction will not find it at the level of chromosomes.

The ring appears instead at the scale of the life cycle: zygote, organism, gamete, zygote. That is a ring of three stages that closes when a new zygote forms. A different reading would count more stages, so the number three here is a choice of description, not a measurement. It grades C.

Grade. A for the meeting rule of meiosis and triploid sterility (published genetics; Calculation 16.9), for the meeting side of T7 only; B for the odd-ploidy test; C for the generation as a ring and for the menstrual cycle against the lunar month.

Chapter 17 — Healing

This chapter reads the tension column of level L4: what does not close in a body or a mind, and how medicine makes it close. It covers clinical medicine, pharmacology, medical biotechnology, medical engineering, veterinary and dairy science, nutrition, and psychotherapy and psychiatry. It describes treatments only to grade a structure; it gives no treatment advice. The strongest structure in the whole book is here. In Duchenne and Becker muscular dystrophy a deletion whose length is a multiple of three keeps the reading frame and gives the milder disease; this rule fits over 4,700 recorded mutations at about 91 %, against a chance agreement of about 0.6 (grade A). Exon-skipping drugs then restore closure by removing one more exon so that the total length is 0 mod 3 — the Foundations' arithmetic used as therapy (grade A). Two more A-grade structures follow. The insulin hexamer is a 2·3 assembly, and fast-acting insulin works by undoing it. The capsids of the viral vectors used in gene therapy are built on triangulation numbers that are exactly the norms of Z[ω] (T14). The rest is weaker. The ratio rules of heart block and the pulse rule of hormone dosing are published facts, but they fit the ladder only partly (grade B or C). Psychotherapy's triangles are readings only (grade C).

17.1 Clinical medicine

History. Clinical medicine became a science of comparison late. Hippocrates and the Hippocratic school (fifth to fourth century BCE) observed the course of disease at the bedside. Galen's humoral theory ruled for 1,500 years. James Lind ran a controlled comparison of six treatments for scurvy on HMS *Salisbury* in 1747; the two sailors given citrus fruit recovered. Pierre Louis's "numerical method" (1830s) showed that bloodletting did not help pneumonia. Ignaz Semmelweis (1847) cut maternal deaths by hand-washing; Joseph Lister introduced antisepsis (1867). Louis Pasteur and Robert Koch founded germ theory.

The twentieth century added diagnosis by machine and treatment by trial. Einthoven's electrocardiogram (1901–03), Röntgen's X-rays (1895) and later CT and MRI made the inside of the body visible. The Medical Research Council's streptomycin trial (1948) was the first modern randomised controlled trial. Archie Cochrane's *Effectiveness and Efficiency* (1972) argued that medicine should use only treatments shown to work, and evidence-based medicine became the norm in the 1990s.

Genetic medicine joined the clinic in the 1980s. Louis Kunkel's group cloned the dystrophin gene in 1986–87. In 1988 Anthony Monaco and colleagues proposed the reading-frame rule. Deletions in the dystrophin gene that keep the triplet reading frame lead to an internally shortened but partly working protein and the milder Becker muscular dystrophy. Deletions that shift the frame lead to no protein and Duchenne muscular dystrophy. The rule turned a clinical distinction between two diseases into arithmetic modulo 3. In the 1990s and 2000s, groups in Leiden (Gert-Jan van Ommen, Annemieke Aartsma-Rus, Judith van Deutekom) and elsewhere developed antisense

oligonucleotides that make the splicing machinery skip a chosen exon. The aim was to turn an out-of-frame deletion into an in-frame one: Duchenne into Becker.

Cardiology developed its own ratio rules. Karel Frederik Wenckebach (1906) and Woldemar Mobitz (1924) classified second-degree heart block by the ratio of atrial to conducted beats. Leon Glass and colleagues later explained these ratios as phase locking of a driven oscillator.

The questions clinical medicine leaves are about individual prediction: which patient will respond, which mutation will be severe, and why exceptions to clean rules occur.

Current state. The cardiovascular burden remains the largest: about 19.8 million deaths in 2022, 32 % of all deaths (WHO, 31 July 2025). Duchenne muscular dystrophy affects roughly 1 in 3,500 to 6,000 males in the United States; a global estimate is 7.1 per 100,000 male births (Wikipedia, Duchenne muscular dystrophy). Becker dystrophy, the hallmark of which is milder in-frame deletions, occurs in about 1.5 to 6 per 100,000 male births (Wikipedia, Becker muscular dystrophy). The dystrophin gene spans about 2.24 megabases and has 79 exons in the muscle transcript; the protein has 3,685 amino acids (Wikipedia, Dystrophin).

The frame rule has been checked against the Leiden DMD mutation database: of over 4,700 reported mutations, about 91 % agree with it, and the exceptions have been analysed as “paradoxical cases” that on closer study confirm the rule (Aartsma-Rus et al. 2006, *Muscle & Nerve*; abstract via QxMD).

Exon skipping is in clinical use. Eteplirsen, which skips exon 51 and applies to about 14 % of Duchenne patients, received accelerated FDA approval on 19 September 2016, after an advisory panel voted against it; the European Medicines Agency refused it in 2018 (Wikipedia, Eteplirsen). Golodirsen (exon 53, December 2019), viltolarsen (exon 53, August 2020) and casimersen (exon 45, February 2021) followed. The AAV gene therapy delandistrogene moxeparvovec received accelerated approval in June 2023 for children aged four and five (Wikipedia, DMD). The clinical benefit of all these remains debated.

Other clinical structures: in osteogenesis imperfecta, more than 90 % of cases arise from mutations in COL1A1 or COL1A2, the genes of type I collagen; one known mechanism is substitution of glycine in the triple helix by a bulkier amino acid; the disease affects about one in 15,000 to 20,000 people (Wikipedia, Osteogenesis imperfecta). In Wenckebach block the ratio of P waves to QRS complexes is always X:(X − 1); 2:1 block is ambiguous between types (Wikipedia, Second-degree AV block).

Interpretation.

1. *Strand.* I is the position along the coding sequence of a gene; P is the set of bases. For the heart, I is the sequence of atrial beats and P the states of the conduction system.
2. *Passage and trit.* A deletion of L nucleotides contributes the digit of L in balanced ternary: L ≡ 0 gives 0, L ≡ 1 gives +1, L ≡ 2 gives −1, with the rest carried (T1). In the heart, an atrial beat that is conducted is a passage; one that is blocked is a missed passage.

3. *Closure.* The reading frame closes when the deletion leaves digit 0: lengths 3, 6, 9, … are in frame; all others shift the frame (Calculation 17.1). That is the frame rule. Exon skipping adds a second deletion of residue e to a deletion of residue r. The frame closes when r + e ≡ 0 mod 3: a deletion of residue 1 is repaired by skipping an exon of residue 2, and a deletion of residue 2 by an exon of residue 1 (Calculation 17.2). In balanced-ternary terms, residue 2 is the digit −1, so the repair is a meeting, +1 then −1 (T7). Three deletions of residue 1 would also close, a ring with carry +1 (T7.1). In osteogenesis imperfecta, a glycine substitution breaks the everywhere-closed Gly-X-Y word of collagen (Chapter 14; T8).
4. *Scale.* The codon is the window of three at the scale of the nucleotide. In collagen, the Gly-X-Y triple is the window at the scale of the amino acid, and the exon is 0 mod 9 (Chapter 14): two rungs of the ring ladder. In the heart, conduction ratios X:(X − 1) with X = 2, 3, 4 have group lengths on the ladder; X = 5 does not (Calculation 17.5). 2:1 block is period 2, the first rung of the meeting ladder.
5. *Test (stated before data).* The frame rule is fact, so the test must reach beyond it. If closure at one scale is the rule, then an in-frame deletion that still causes Duchenne must break a closure at another scale — for example, remove an essential protein domain or create a stop codon at the junction. An exception that keeps the frame, keeps all known domains and creates no stop, yet still gives Duchenne, would refute the reading. The database study reports that the paradoxical cases, on closer study, confirm the rule.

Consequences. The frame rule is the strongest instance in this book of a structure the Foundations name and the field already uses. Its arithmetic is the digit rule of T1. Its repair, exon skipping, is the meeting of T7. Its strength can be measured against a baseline. If deletion lengths were spread evenly over residues, a third would be in frame (T6, uniform law). If phenotype did not depend on frame, the rule would agree with the clinic in about 57–63 % of cases, depending on how common Becker is (Calculation 17.3). It agrees in 91 %. The excess is the structure.

What the Foundations add beyond what geneticists know is a way to see three things as one: Crick’s frameshift suppression in phage (Chapter 10), the clinical frame rule and the exon-skipping therapy. All three are closure mod 3 at the codon scale. The same arithmetic applies to frame-aware gene editing (17.3). It also shows a limit. Exon skipping restores the frame but not the missing exons, and the clinical benefit of the drugs is disputed. The Foundations say a closed frame persists; they do not say a closed but shortened protein works. Function is a closure at the scale of the protein’s domains, which the frame rule does not see. That is the content of the test above.

For the heart, the reading is weaker. Wenckebach ratios fit the ladder for short groups and leave it at 5:4. Phase locking explains all of them, as the circle map does (Chapter 14). The Foundations add nothing to cardiac electrophysiology beyond the observation that 2:1 is a meeting.

Grade. A for the Duchenne/Becker frame rule (Monaco et al. 1988; Aartsma-Rus et al. 2006) and for exon skipping as restoration of closure mod 3; A for glycine substitution in collagen as the breaking of a period-3 word; B for the extension to frame-rule exceptions; C for heart-block ratios.

17.2 Pharmacology

History. Pharmacology grew from materia medica, the catalogues of plant, animal and mineral remedies kept since Dioscorides (first century CE). Paracelsus (sixteenth century) stated that the dose makes the poison. Friedrich Sertürner isolated morphine from opium in 1804, the first pure active compound from a plant. Rudolf Buchheim founded the first pharmacological institute in Dorpat (1847), and his student Oswald Schmiedeberg made pharmacology an experimental discipline in Strasbourg.

John Newport Langley (1905) and Paul Ehrlich (around 1900–1910) introduced the receptor: a “receptive substance” on cells that binds a drug. Ehrlich’s Salvarsan (1909) was the first designed chemotherapy. A. J. Clark (1926–1937) treated receptor occupancy quantitatively. Raymond Ahlquist (1948) split adrenergic receptors into α and β. James Black designed propranolol (1960s) and cimetidine (1970s), drugs built for a receptor. Pharmacokinetics — absorption, distribution, metabolism and excretion — became quantitative with Torsten Teorell (1937).

Two lessons of the twentieth century shaped regulation. Thalidomide, sold from 1957 as a sedative, caused severe limb defects in thousands of children before its withdrawal in 1961; the disaster led to modern drug-approval laws. It also showed that the two mirror forms of a molecule can act differently. The second lesson came from endocrinology: the same agonist can activate a receptor when given in pulses and silence it when given continuously. Leuprorelin turned that into a therapy.

Biological drugs changed the field from the 1980s: recombinant human insulin (1982), monoclonal antibodies (from 1986), engineered insulin analogues (1996), GLP-1 receptor agonists (2005). The question pharmacology leaves is precision: which patient, which dose, which pattern of dosing, and why the same molecule can help or harm.

Current state. In 2025 the FDA’s Center for Drug Evaluation and Research approved 46 novel drugs, including suzetrigine, a non-opioid pain medicine, and aficamten for obstructive hypertrophic cardiomyopathy (FDA, Novel Drug Approvals 2025). WHO issued its first guideline on GLP-1 medicines for obesity on 1 December 2025, with conditional recommendations, for a condition affecting more than one billion people (WHO).

Chirality remains a central concern. Thalidomide is a racemic mixture; the (R) form gives the sedative effect and the (S) form is linked to teratogenicity, but the two interconvert in the body, so giving only (R) does not remove the risk. Estimates of severely harmed infants exceed 10,000, possibly 20,000. Thalidomide was later re-approved for complications of leprosy (1998) and for myeloma (2006) (Wikipedia, Thalidomide).

Dosing pattern matters as much as dose. Pulsed GnRH restores gonadotropin secretion; continuous GnRH does not and desensitises the pituitary (Belchetz, Knobil et al. 1978, via QxMD). Leuprorelin uses this. It first raises LH and FSH, then over weeks down-regulates the receptors and produces a hypogonadal state. It was approved in the United States for advanced prostate cancer on 9 April 1985 (Wikipedia, Leuprorelin).

Assembly matters too. Insulin in solution forms dimers and, with zinc, hexamers of three dimers (RCSB 4INS). Insulin lispro swaps two residues at the end of the B chain; this does not alter receptor binding but blocks dimer and hexamer formation, so the monomer acts faster (Wikipedia, Insulin lispro).

Lithium, found to calm mania by John Cade in 1948 and published in 1949, works in a narrow window. Maintenance levels are about 0.6–1.0 mmol/L; levels above 1.5 are toxic. The FDA approved it for mania in 1970 (Wikipedia, Lithium (medication)).

Interpretation.

1. *Strand.* I is time after dosing; P is the set of states of the receptor system (free, bound, desensitised) or of the drug molecule (monomer, dimer, hexamer).
2. *Passage and trit.* A dose that rises and returns to baseline is a passage. Relative to baseline, the receptor is activated (+1), at rest (0) or desensitised (−1).
3. *Closure.* A pulse is a meeting: +1, then −1, digit 0, no carry (T7.1). Continuous agonist never returns; it is an open word that keeps a nonzero digit, and the receptor system answers by moving to the −1 state, desensitisation. That reading names the published fact (Belchetz et al.) and the therapy that uses it (leuprorelin). The insulin hexamer is a closed assembly of three dimers. The dimer is a meeting and the hexamer a ring of three meetings, 2·3 on the ladder (Calculation 17.7). Lispro returns it to the standing unit.
4. *Scale.* The two mirror forms of a chiral molecule are the two orientations of a ring, (+1, +1, +1) and (−1, −1, −1), with carries +1 and −1 (Calculation 17.6). A racemate holds both, with net carry 0. The Foundations' first knot is chiral (T20) and its chirality is conserved under continuous moves (Part VII). Thalidomide's enantiomers interconvert in the body. In knot terms that interconversion is a cut, which no continuous move can make, and chemistry permits it through the exchange of a proton at the chiral centre. So T19–T22 do not apply: molecular chirality is not a topological invariant unless the molecule is a knot.
5. *Test (stated before data).* If pulsatile signals are meetings and continuous ones open words, then every receptor system that desensitises under continuous agonist should recover full response when the agonist is given in pulses separated by a return to baseline. A receptor that desensitises under continuous exposure but also under pulses with full return, at every pulse interval, would contradict the reading. For assemblies: if the hexamer is the closed configuration, any mutation that blocks it should speed absorption, as in lispro and other rapid analogues; a hexamer-blocking analogue with slower absorption than human insulin would contradict it.

Consequences. Pharmacology confirms two structural points and rejects one hope. The pulse rule is published endocrinology, and the Foundations' meeting gives it a simple name. That name has a use: it tells a reader why a long-acting agonist can suppress the system it activates, and why that is used deliberately. The insulin hexamer is a 2·3 assembly in the structural record, and a therapy undoes it. The rejected hope is a topological reading of drug chirality. The thalidomide enantiomers interconvert, so chirality in pharmacology is not conserved, and the Foundations' conservation theorems (T22) do not reach it. The tragedy of thalidomide was partly a failure to see

that a mirror form could act differently. The Foundations can count mirror forms; they cannot predict which one harms.

Lithium's narrow window is a case of N4, read as the hypothesis of T23: the response is a state function, and the window between effect and toxicity is the region where γ keeps its sign. That is a reading only; no number from T23 is tested.

The GLP-1 medicines give a further case of the same logic, stated here as structure and not as advice. They act on a receptor whose natural ligand is released in pulses after meals and cleared within minutes. The drugs are engineered to resist breakdown and to last days, so the receptor sees a nearly continuous signal. The clinical effect persists, which shows that not every receptor system desensitises as the pituitary GnRH receptor does. In the Foundations' reading this is a direct counter-example to a general rule that "continuous means open means desensitised". The pulse rule is a property of particular receptor systems, not of signalling in general. N4, as the hypothesis of T23, allows exactly that: the response is a state function, and the state includes the receptor's own regulation. The reading survives only in the narrow form of the test above, which asks receptor by receptor.

Pharmacokinetics has its own rhythm structure. A drug given at fixed intervals reaches a steady state after several half-lives, in which each dose replaces what was cleared since the last one. In that state each dosing interval is a closed cycle: the concentration returns to the same trough. A missed dose leaves a deficit; a doubled dose a surplus. In the language of T1 the trough-to-trough cycle has digit 0 in steady state and a nonzero digit when the rhythm is broken. This is the accounting that pharmacokinetics already does with exponentials. It adds no number.

What pharmacology does teach the Foundations is the limit of counting. Thalidomide's two enantiomers are as simple a pair as exists — a molecule and its mirror image. Their effects differ completely, and their interconversion in the body defeats any attempt to give only the safe one. No closure rule predicted that. A theory that sorts configurations into closed and open must say which closed configuration a body will treat as harmful, and the Foundations do not.

Grade. A for the pulse-versus-continuous rule (Belchetz et al. 1978) and the 2·3 insulin hexamer with its therapeutic undoing; B for the pulse-recovery and hexamer-blocking tests; C for chirality, where the topological reading fails because enantiomers interconvert.

17.3 Medical biotechnology

History. Medical biotechnology began with molecules made by living cells. Alexander Fleming saw penicillin in 1928; Howard Florey and Ernst Chain made it a drug in 1940–41. Vaccines grew from Jenner (1796) through Pasteur to the polio vaccines of Salk (1955) and Sabin. Recombinant DNA (Cohen and Boyer, 1973) let bacteria make human proteins: insulin (Genentech, approved 1982), growth hormone, erythropoietin. Georges Köhler and César Milstein's hybridomas (1975) gave monoclonal antibodies.

Gene therapy started in 1990 with ADA deficiency and suffered setbacks in 1999 (the death of Jesse Gelsinger) and 2002 (leukaemia in an X-SCID trial). It recovered with

safer vectors, especially adeno-associated virus (AAV). Luxturna, an AAV therapy for inherited retinal dystrophy, was approved in the United States in December 2017. Antisense oligonucleotides, first proposed by Paul Zamecnik in 1978, became drugs for spinal muscular atrophy (nusinersen, 2016) and Duchenne (2016 onward). CRISPR–Cas9 (Jennifer Doudna and Emmanuelle Charpentier, 2012; Nobel Prize 2020) made precise editing of the genome possible.

The physical form of a gene-therapy vector is a virus capsid. Donald Caspar and Aaron Klug (1962) explained why icosahedral capsids have 60T subunits with $T = h^2 + hk + k^2$. The questions left are about delivery, durability, immune reactions and cost.

Current state. The first CRISPR-based therapy, exagamglogene autotemcel (Casgevy), was approved in the United Kingdom in November 2023 and in the United States in December 2023 for sickle-cell disease and in January 2024 for transfusion-dependent β-thalassaemia. It edits the patient's own blood stem cells to raise fetal haemoglobin (Wikipedia, Exagamglogene autotemcel).

AAV is the main in-vivo vector. Its capsid has 60 monomers in icosahedral symmetry, with VP1, VP2 and VP3 in a ratio of about 1:1:10; the single-stranded genome is about 4.7 kilobases. Luxturna was approved on 19 December 2017 (Wikipedia, Adeno-associated virus). The Duchenne gene therapy delandistrogene moxeparvovec, an AAV vector, received accelerated approval in June 2023 (Wikipedia, DMD). The full dystrophin coding sequence, 3,685 codons or 11,055 bases (Wikipedia, Dystrophin), is about 2.35 times the AAV genome (Calculation 17.8), so an AAV vector cannot carry the whole gene.

Frame logic is used in editing. A frameshift is defined by an indel whose length is not divisible by three; exon skipping in Duchenne keeps the rest of the sequence in frame (Wikipedia, Frameshift mutation).

The frontier includes base and prime editing, in-vivo CRISPR delivery by lipid nanoparticles, and the cost of one-time therapies.

Interpretation.

1. *Strand.* I is the position along an edited gene or along a capsid lattice; P is the set of bases, or the lattice points.
2. *Passage and trit.* An edit that inserts or deletes k bases contributes the digit of k (T1). On the capsid lattice, a passage is a step between subunit positions in the hexagonal net, read in $Z[\omega]$ (T14).
3. *Closure.* A knockout by non-homologous end joining aims at a frameshift. If indel lengths fell evenly over residues, one in three would keep the frame and two in three would shift it; two independent alleles would both shift in $(2/3)^2 = 4/9$ of cells (Calculation 17.4). Gene editing thus uses the arithmetic of T1 directly, but in the other direction: it makes a window open on purpose. Exon-skipping and frame-restoring edits close it.
4. *Scale.* Capsid triangulation numbers $T = h^2 + hk + k^2$ are exactly the norms of $Z[\omega]$: 1, 3, 4, 7, 9, 12, 13, … (Calculation 17.8). AAV has $T = 1$: 60 subunits, the smallest closed icosahedral shell. Its VP ratio 1:1:10 fills 60 positions as about 5:5:50

(Calculation 17.8). A capsid that closes becomes one passage of the next scale — one particle in a dose.

5. *Test (stated before data).* T14 divides Z[ω] into elements whose digit orbit comes to rest and elements whose orbit ends in a perpetual turn. If that split corresponds to anything in capsid biology, the triangulation numbers of capsids that assemble easily should fall more often among the norms of “resting” elements. This is the open test of Chapter 10, and the data to run it — assembly efficiency by T-number — are not collected here.

Consequences. Medical biotechnology uses two of the Foundations’ structures as tools. The first is the frame. A CRISPR knockout depends on breaking closure mod 3, an exon-skipping drug on restoring it, a base editor on avoiding frameshifts. The second is the capsid lattice. Every icosahedral vector is built on Z[ω]. Neither was discovered through the Foundations, and neither needs them. What the Foundations add is the observation that both are cases of one arithmetic, the residue of T1 in $\mathbb{Z}$ for the frame and in Z[ω] for the capsid, the ring in which the trit is a residue (Foundations III.1, remark). That is a correct unification at grade A. The test on T14 is the one place where the Foundations could tell vector designers something new, and it is open.

The size of the dystrophin gene shows how the two structures meet in one disease. The coding sequence of 11,055 bases is about 2.35 times what an AAV capsid can hold (Calculation 17.8). Two routes follow. One keeps the gene and changes the transcript: exon skipping, which removes exons so the frame closes again, uses the frame arithmetic at the scale of the codon. The other keeps the vector and changes the gene: a shortened dystrophin that fits the T = 1 capsid uses the lattice arithmetic at the scale of the shell. Both routes accept a protein that is shorter than the original. Becker patients, whose in-frame deletions also shorten the protein, are the natural evidence that a shortened but closed frame can work. That evidence is what made both strategies plausible to the field, long before any vector was approved.

Gene editing adds a quantitative point that is easy to miss. If the residues of repair indels were spread evenly, two-thirds of edited alleles would shift the frame and one-third would not (Calculation 17.4). Real repair outcomes are not uniform: cells favour certain short deletions and insertions at a given cut site. The Foundations’ baseline of one-third (T6 with a uniform law) is therefore only a baseline. Each guide sequence has its own distribution of indel lengths, and the share that closes the frame must be measured site by site. T6 states how to compute the closing share from any law of residues; it does not supply the law.

CRISPR therapy for sickle-cell disease takes a third route. Instead of correcting the mutation it raises fetal haemoglobin, a second globin that the body stopped making after birth. The therapy re-opens a switch that development closed. That is a description in terms of the life course (Chapter 16), and it involves no frame and no lattice.

Grade. A for frame arithmetic in editing and skipping; A for capsid T-numbers as Eisenstein norms (Caspar and Klug 1962); B for the open T14 test on assembly.

17.4 Medical engineering

History. Medical engineering began with instruments that extended the senses. René Laennec's stethoscope (1816) let the physician hear the chest. Hermann von Helmholtz's ophthalmoscope (1851) showed the retina. Wilhelm Röntgen's X-rays (1895) showed the skeleton through skin. Willem Einthoven's string galvanometer (from 1901) recorded the heart's electrical activity; he placed three electrodes, on both arms and the left leg, and defined three leads around a triangle. His Nobel Prize came in 1924.

Machines then began to replace function. Willem Kolff built a rotating-drum artificial kidney in Kampen in 1943. John Gibbon used a heart–lung machine in surgery in 1953. Rune Elmqvist and Åke Senning implanted the first fully implantable pacemaker in Arne Larsson on 8 October 1958 in Solna; the device lasted hours, and Larsson received 26 pacemakers over his life. Godfrey Hounsfield and Allan Cormack built computed tomography (1971; Nobel 1979). Paul Lauterbur and Peter Mansfield made MRI (1973 onward; Nobel 2003). Cochlear implants (1970s–80s) and deep-brain stimulators (1987, Alim-Louis Benabid) placed electronics in the nervous system. Brain–computer interfaces followed (BrainGate, 2006).

Emanuel Goldberger added the augmented limb leads in 1942. The 12-lead ECG, with 10 electrodes, became standard. Closed-loop devices — implantable defibrillators, insulin pumps controlled by glucose sensors — made the device itself a feedback controller.

The questions left are about autonomy and safety: when should a machine act on the body without a clinician, how should AI-based devices be validated, and how long do implants last?

Current state. The 12-lead ECG remains the most-used cardiac test; its 12 leads come from 10 electrodes, Einthoven's string galvanometer of 1901 being its ancestor (Wikipedia, Electrocardiography). Leadless pacemakers, capsules placed inside the heart by catheter through the femoral vein, now exist in single-chamber form; one early device's European study was halted after perforations (Wikipedia, Artificial cardiac pacemaker). AI-enabled devices are listed by the FDA in a public register that is updated continuously (FDA, AI-enabled medical devices); the page as read did not state a total, so none is given here.

The frontier is the closed loop: artificial pancreas systems that dose insulin from a glucose sensor, responsive neurostimulators that detect and interrupt seizures, and pacing algorithms that adapt to the patient's rhythm. All of them are engineering of negative feedback, the subject of Chapter 14.

Interpretation.

1. *Strand.* I is the sequence of stimuli and responses between device and tissue; P is the set of tissue states.
2. *Passage and trit.* Each paced beat returns the tissue to a reference state; its response is longer, shorter or equal to the last.

3. *Closure.* Einthoven's law, I + III − II = 0, holds for every heart vector (Calculation 14.6). It is the closure of a loop of three differences, and the three lead axes are sixth roots of unity, units of Z[ω]. But Kirchhoff's law closes any loop: four electrodes would close too (Calculation 14.6). The three-ness is a design choice, not a consequence of closure. Paced cardiac tissue phase-locks at rational ratios and doubles its period before irregularity (Guevara et al. 1981; Chapter 14). Alternans — alternating strong and weak beats — is period 2, a meeting that does not close and that warns of arrhythmia.
4. *Scale.* Closed-loop devices merge sensing and acting into one ring: sense, decide, act, and sense again. That is a ring of three stages. By the secant condition, a three-stage negative loop without delay can oscillate if its gain exceeds 8 (Chapter 14, Calculation 14.10). Device designers avoid that oscillation by limiting gain and adding filters. Counts in the field: 12 leads and 6 limb and 6 chest leads are on the ladder; 10 electrodes are not (Calculation 17.9).
5. *Test (stated before data).* A closed-loop controller with three stages and no delay should oscillate when its loop gain exceeds 8 and not below. Engineering practice already uses such stability margins, so the test belongs to control theory. A test specific to the Foundations is not available.

Consequences. Medical engineering adds the clearest case of a device built on a three-node closure — Einthoven's triangle — and the clearest reason why that is not evidence. Any loop of potential differences closes. The Foundations add the observation that the triangle's axes are the units of Z[ω], which is true and of no use to a cardiologist. For closed-loop devices the Foundations name the ring and the threshold; control theory had both.

Engineering also shows the cost of reading too much into the triangle. The augmented leads aVR, aVL and aVF, added by Goldberger in 1942, are built from the same three limb electrodes and lie at angles between Einthoven's leads. In the hexaxial reference system the six limb leads, read as lines through the centre, divide the circle every 30°, not every 60°. If the units of Z[ω] mattered, the augmented leads would add nothing; in fact they add views that clinicians use daily. The ECG's geometry is chosen for anatomical coverage, and the arithmetic follows from the choice.

The closed-loop devices are a better place for the Foundations' meeting. An implantable defibrillator senses a dangerous rhythm and delivers a shock that resets the tissue to a common phase: an open window forced closed. A pacemaker in demand mode fires only when the heart's own beat does not arrive in time: it supplies the missing return. A closed-loop insulin system adds insulin when glucose rises and withholds it when it falls, a meeting of signal and correction at each step. These are descriptions of feedback that control engineers write with transfer functions. In each, the device makes a loop close that the body left open. That is the tension column of this book in its plainest form. It earns no grade above C because the Foundations predict no number the engineers do not already compute.

The brain–computer interface is the one device where the Foundations' trit could be checked directly. Many decoders map neural activity to discrete commands such as move left, rest or move right: a three-valued output. Whether a three-symbol alphabet

decodes better than a binary one at equal error is an empirical question in each system. It is not answered by T1, which states only that three is the least alphabet in which a value differs from its reverse.

Grade. A for period doubling and phase locking in paced tissue (Guevara et al. 1981), as published; C for Einthoven's triangle, closed-loop devices and lead counts.

17.5 Veterinary science

History. Veterinary medicine is old: the Kahun papyrus (around 1800 BCE) treats animals, and Roman writers such as Columella and Vegetius wrote on the care of livestock. The first veterinary school was founded by Claude Bourgelat in Lyon in 1761, the second at Alfort in 1765 (Wikipedia, Claude Bourgelat). Schools spread across Europe in response to cattle plague. Veterinary teaching borrowed the methods of human anatomy and pathology, and in the nineteenth century the two fields shared their discoveries: Edward Jenner's smallpox vaccine (1796) came from cowpox, and the word vaccine comes from the Latin for cow. Pasteur's anthrax vaccine (1881) was tested on sheep at Pouilly-le-Fort. Koch's anthrax work (1876) and Daniel Salmon's studies of animal disease (1880s) founded veterinary microbiology.

Rinderpest, the cattle plague, shaped the field. It killed most cattle in parts of Africa in the 1890s. A tissue-culture vaccine (Walter Plowright, 1960s) and coordinated campaigns led to its global eradication. Zoonoses — diseases shared between animals and humans, such as rabies, brucellosis and avian influenza — made veterinary science part of public health. The One Health idea joined the two. The questions left are about antimicrobial use in animals, emerging zoonoses and the welfare of farmed animals.

Current state. Rinderpest was declared eradicated by the Food and Agriculture Organization on 28 June 2011, after the World Organisation for Animal Health confirmed it on 25 May 2011. It is the second disease eradicated in history, after smallpox (Wikipedia, Rinderpest). One Health, defined by its expert panel as an approach that "aims to sustainably balance and optimize the health of humans, animals, plants and ecosystems", is coordinated by the Quadripartite of FAO, WHO, WOAH and UNEP, formalised in March 2022, with a Joint Plan of Action for 2022–2026 (Wikipedia, One Health). Bacterial antimicrobial resistance was associated with more than 4.7 million deaths globally in 2021 (WHO fact sheet, 16 July 2026).

Interpretation.

1. *Strand.* I is the course of an infection through a population of animals and people; P is the set of hosts.
2. *Passage and trit.* A passage is the return of a pathogen to a host species it has infected before. A spillover from animal to human and back is a meeting.
3. *Closure.* Eradication is closure in the strict sense: a pathogen whose chain of passages ends leaves no residue. Rinderpest and smallpox are the two closed cases.
4. *Scale.* Zoonotic cycles involve two or three host types — reservoir, intermediate, human. A cycle through three host types is a ring. Nothing in the data counted here allows a test of ring against meeting.
5. *Test (stated before data).* None specific to the Foundations.

Consequences. Veterinary science adds no structure. The reading of eradication as closure is a re-description, and One Health is an organisational principle, not a quantitative one.

One point is worth separating from the re-description. Eradication requires that the chain of transmission end everywhere, not just fall below a threshold. Two diseases in history have reached that state, against many that have been reduced. In the Foundations' terms a disease that persists at low level keeps a nonzero digit somewhere in the population, and the digit is carried from host to host. The difference between control and eradication is the difference between a small residue and none. Epidemiology already measures that difference with the reproduction number and with surveillance; the Foundations add no method for it. Antimicrobial resistance, associated with more than 4.7 million deaths in 2021, is the reverse case: a residue that grows because each treatment that does not close the infection selects the survivors. That too is published population biology (Chapter 21 treats epidemiology).

Grade. C.

17.6 Animal and dairy science

History. Animal breeding became a science with Robert Bakewell's selection of sheep and cattle in eighteenth-century Leicestershire. Herd books began with the Shorthorn (1822). Jay Lush's *Animal Breeding Plans* (1937) applied population genetics, and Charles Henderson's mixed-model equations (1950s–70s) became the basis of genetic evaluation. Artificial insemination, developed in the 1930s–40s and combined with frozen semen (1950s), let one bull sire tens of thousands of calves. Genomic selection (Meuwissen, Hayes and Goddard, 2001; adopted in dairy from 2009) predicted breeding values from DNA markers.

Dairy science studied milk and the cow that makes it. The ruminant stomach, with four compartments, lets cattle live on cellulose through microbial fermentation. Reproduction is managed around the oestrous cycle. The questions left are about efficiency, methane emissions, welfare and the health of high-yielding animals.

Current state. World milk production reached 930 million tonnes in 2022, up from 524 million in 1992, an increase of 77 % (an average annual growth of 1.93 %; Calculation 17.10); about 150 million households are engaged in milk production (FAO). Cattle have four stomach compartments — rumen, reticulum, omasum and abomasum — and ruminate by regurgitating and re-chewing feed; there are about 200 ruminant species (Wikipedia, Ruminant). The cow's oestrous cycle typically lasts 21 days, with standing oestrus of about 8 to 30 hours (Wikipedia, Estrous cycle).

Interpretation.

1. *Strand.* I is time in a cow's productive life; P is the set of reproductive and lactation states.
2. *Passage and trit.* A passage is the return to oestrus. Relative to the expected day, it comes late (+1), on time (0) or early (−1).

3. *Closure.* Rumination is a meeting in the digestive tract: feed goes down, comes back up, is chewed and goes down again. A cycle that ends in conception closes the reproductive ring.
4. *Scale.* The four stomach compartments are 2^2 on the ladder; the 21-day cycle, 3·7, is not (Calculation 17.10). Neither number is a merge of equal blocks.
5. *Test (stated before data).* None specific to the Foundations.

Consequences. Animal and dairy science adds nothing here. Rumination as a meeting is a description, and the counts are not on a ladder for any reason the Foundations give.

The growth of milk output, 1.93 % a year on average over thirty years (Calculation 17.10), is an exponential trend of the kind treated in Chapter 20, not a ladder. Breeding for yield raised output per cow in the rich world, while FAO notes that growth in developing countries came mostly from more animals. The trade-offs that follow — fertility of high-yielding cows, methane per litre — are questions of biology and policy. The Foundations have no number for them, and saying so is the only honest result for this discipline.

Grade. C.

17.7 Nutrition

History. Nutrition began with energy. Antoine Lavoisier showed in the 1780s that respiration is combustion. Wilbur Atwater, at Wesleyan University in the late nineteenth century, measured the energy of foods in a respiration calorimeter; his general factors of 4 kcal per gram for protein and carbohydrate and 9 for fat are still used (Wikipedia, Atwater system). Deficiency diseases came next. Christiaan Eijkman, in the Dutch East Indies (1890s), found that polished rice caused beriberi in chickens; Frederick Gowland Hopkins proposed “accessory food factors”; Casimir Funk coined “vitamine” (1912). Eijkman and Hopkins shared the 1929 Nobel Prize. Elmer McCollum identified a fat-soluble growth factor, later vitamin A, in 1913. Joseph Goldberger showed in the 1910s and 1920s that pellagra in the American South was caused by diet, not infection. Albert Szent-Györgyi isolated vitamin C and received the Nobel Prize in 1937. By mid-century the classical vitamins were known, and deficiency diseases had become preventable in principle.

William Cumming Rose identified the essential amino acids by diet experiments, among them the discovery of threonine, and later showed which ones adult humans must eat. Ancel Keys’s Seven Countries Study (from 1958) linked diet and heart disease. The twentieth century ended with a reversal: undernutrition remained, but overweight and obesity grew to exceed it.

The questions left are about the effects of whole diets rather than single nutrients, about ultra-processed food, and about individual responses.

Current state. In 2022, 2.5 billion adults were overweight, including 890 million with obesity, and 390 million were underweight. Among children under five, 149 million were stunted and 45 million wasted (WHO malnutrition fact sheet, 1 March 2024).

Obesity is now 36 % of adult overweight (Calculation 17.11). Humans need nine essential amino acids — histidine, isoleucine, leucine, lysine, methionine, phenylalanine, threonine, tryptophan and valine — and six more are conditionally essential (Wikipedia, Essential amino acid). GLP-1 medicines have entered the treatment of obesity (17.2).

Interpretation.

1. *Strand.* I is time in meals and days; P is the set of metabolic states (fed, fasting, stored).
2. *Passage and trit.* A daily energy balance is a passage: surplus (+1), balance (0), deficit (−1).
3. *Closure.* A day in balance closes with digit 0 and no carry. A surplus that is not cancelled carries into storage, the reading of N2 by T1. That is energy accounting, which nutrition already does.
4. *Scale.* Nine essential amino acids is 3^2, on the ladder (Calculation 17.11), but the number comes from the evolutionary loss of synthetic pathways, not from merging. The Atwater ratio of fat to carbohydrate, 9:4, is a ratio of ladder numbers, which is true of every ratio of numbers up to 4 and 9.
5. *Test (stated before data).* None specific to the Foundations.

Consequences. Nutrition gives no structural link. Energy balance is closure in the trivial sense of accounting. The Foundations add nothing to the questions of what to eat.

The double burden of malnutrition — 2.5 billion overweight adults and 149 million stunted children in the same year — shows why closure in the accounting sense is not enough. Both are failures of balance, in opposite directions, and both can occur in one household. The Foundations' trit can label them, surplus (+1) and deficit (−1), but the label does not explain why one population has both. That is a question of food systems and economics (Chapters 13 and 19), not of the register.

Grade. C.

17.8 Psychotherapy and psychiatry

History. Psychiatry as a medical speciality began with the asylum reforms of Philippe Pinel (1790s) and William Tuke. Emil Kraepelin's textbook (from 1883) separated dementia praecox from manic-depressive illness by their course over time; Eugen Bleuler renamed the first schizophrenia (1911). Sigmund Freud founded psychoanalysis (1890s–1900s). The first effective physical treatments came in the twentieth century: lithium (Cade, published 1949), chlorpromazine (1952), imipramine (1957). The *Diagnostic and Statistical Manual* (DSM, 1952; DSM-III 1980) standardised diagnosis.

Psychotherapy diversified. Albert Ellis announced rational emotive therapy in 1956. Aaron Beck developed cognitive therapy from the late 1950s and published it in 1967; his cognitive triad of depression concerns negative views of the self, the world and the future (Wikipedia, Cognitive behavioral therapy). Family therapy arose from the study

of families of patients with schizophrenia; Murray Bowen worked at the National Institute of Mental Health from 1954 to 1959 and treated the family as an emotional unit (Wikipedia, Murray Bowen). Stephen Karpman, a student of Eric Berne, published the drama triangle in 1968: three roles — persecutor, rescuer, victim — that the participants in a conflict rotate through (Wikipedia, Karpman drama triangle).

The questions left are about mechanism and measurement: what makes psychotherapy work, why so many patients do not respond to the first treatment, and whether the categories of the DSM are real.

Current state. About 1.2 billion people lived with a mental disorder in 2023, including 470 million with anxiety disorders and 322 million with depression (WHO, 11 September 2026). Bipolar disorder has a lifetime prevalence usually given as 1–3 %; rapid cycling is defined as four or more mood episodes within a year, affects 25.8–45.3 % of people with bipolar disorder at some point and has a worse prognosis (Wikipedia, Bipolar disorder). Lithium, with the strongest evidence for bipolar disorder, has a maintenance range of about 0.6–1.0 mmol/L and becomes toxic above 1.5 (Wikipedia, Lithium).

The replication crisis (Chapter 15) applies to psychotherapy research too: effects in large, pre-registered trials are often smaller than in early studies.

Interpretation.

1. *Strand.* I is the sequence of interactions in a relationship or of mood states over time; P is the set of roles or states.
2. *Passage and trit.* In the drama triangle a role switch is a step around three positions: forward (+1) or backward (−1). In mood disorders a state is elevated (+1), euthymic (0) or depressed (−1), the three states of the field's own classification.
3. *Closure.* A full rotation through the three roles, in either direction, has steps (+1, +1, +1) or (−1, −1, −1): a ring that closes with carry ±1 (Calculation 17.12). In the drama triangle that matches the clinical observation that the game repeats. It does not resolve; each round becomes the material of the next. A conflict that ends — the parties leave the triangle — is a return to 0. Bowen's triangles and Beck's cognitive triad are three-part structures, but no published rule says they close or rotate in a counted way.
4. *Scale.* Rapid cycling's threshold of four episodes a year is on the ladder; it is a clinical convention. Mood cycling is not counted in equal blocks.
5. *Test (stated before data).* If the drama triangle is a closed ring, sessions in which all three roles are taken in rotation should be followed by recurrence of the same conflict more often than sessions in which only two roles appear. A coded dataset of family or couple interactions could test this. None is known here.

Consequences. Psychotherapy and psychiatry add only readings. The drama triangle as a ring is apt: the field already says that its roles rotate and its games repeat. The Foundations add the arithmetic of rotation — a ring closes with carry and does not end — and the prediction that leaving the ring requires a step outside it, back to 0. That is a description of good therapeutic practice, not a test of it. The three mood states of

bipolar disorder are the alphabet of T1, but the order and timing of episodes are not predicted. Lithium’s narrow window is again a case of state-dependent response.

The psychiatric use of lithium is the one point where this section touches a measured quantity. The drug works within a window whose lower edge is about 0.6 mmol/L and whose toxic edge is 1.5, a factor of 2.5. Below the window the mood ring keeps cycling; above it the body is poisoned. That is a clinical fact, and it is a case of a response that depends on state, as N4 reads the hypothesis of T23. T23 would need an explicit energy density to say anything about the size of the window, and none exists for mood. The reading is a C.

Beck’s cognitive triad — self, world, future — is the third three-part structure in this section. It is a list of the domains of negative thought in depression, not a cycle of roles, and nothing in it rotates. It is listed so that a reader does not count it as a ring.

Grade. C for all readings in this section; B only for the stated recurrence test on coded triangle interactions, which no dataset here allows.

Part VI — Society

Chapter 18 — Social Structure

This chapter reads the standing column of level L5: persons, households, populations and places held together as closed units. It contains the sharpest test of the book outside the natural sciences, and the test was run here on five public signed networks with 6.6 million triangles. The closure rule of the Foundations and Heider's balance theory disagree on two kinds of triangle. On the all-enemies triangle the closure rule is right and strong balance is wrong, but weak balance (Davis 1967) already said the same. On the one triangle where the closure rule differs from weak balance, one friendship and two enmities, the closure rule fails under every null model tried. Geography holds a structural match of grade A: the market-area sizes of central place theory are exactly the norms of the Eisenstein integers Z[ω]. Grades: sociology A for the two forms of exchange, B for signed triads (test run, refuted where it is specific); demography A for the doubling of pedigrees, C otherwise; social and economic geography A for Lösch's numbers, B for the ladder test; network science B (wedge test run, refuted in three of five networks), C otherwise.

18.1 Sociology

History. Auguste Comte named the discipline in his *Cours de philosophie positive* in the 1830s. It became an academic field with Émile Durkheim, whose *Rules of Sociological Method* (1895) treated social facts as things and whose *Suicide* (1897) showed that a private act has social rates. Max Weber's *Protestant Ethic and the Spirit of Capitalism* (1905) made meaning and motive part of the explanation. Georg Simmel's *Soziologie* (1908) was the first to treat the number of persons as a cause in itself. A dyad dies with either member. A triad can survive the loss of one, and its third member can mediate, divide or profit. Simmel's observation that two ties to a common third tend to produce a tie between the two is the origin of what later network writers called triadic closure.

Fritz Heider gave the triad its first formal rule. In "Attitudes and Cognitive Organization" (1946) he proposed that a triad of liking and disliking relations is balanced when the signs multiply to a positive value. Two friends who like the same thing are balanced; two friends who disagree are not. Dorwin Cartwright and Frank Harary generalised the rule to signed graphs in 1956: a cycle is balanced when its product of signs is positive. Harary had already proved in 1953 that a balanced complete signed graph splits into two camps, positive inside and negative between. James Davis weakened the rule in 1967. He asked only that no cycle carry exactly one negative edge. Under this "weak balance" a triangle of three mutual enemies is allowed, and a graph may split into any number of hostile clusters. This change matters for this chapter: strong and weak balance disagree on exactly one triangle, the all-negative one.

A second line of work concerned exchange. Claude Lévi-Strauss's *Elementary Structures of Kinship* (1949) separated two forms of marriage alliance. In restricted

exchange two groups give spouses to each other. In generalized exchange group A gives to B, B to C and so on, and the gift comes back only after it has gone round the whole cycle. The second form ties more groups together but carries more risk, because each group must trust that the cycle will close. Marcel Mauss's *Essay on the Gift* (1925) had already argued that every gift creates an obligation to return.

The quantitative turn came with sociometry (Jacob Moreno, 1934), survey research and, after 1970, network analysis. Mark Granovetter's "The Strength of Weak Ties" (1973) argued that the bridges between dense clusters are weak ties, and that information about jobs travels along them. Pierre Bourdieu's *Distinction* (1979) mapped taste onto class position. Robin Dunbar's social brain hypothesis (1992) predicted a natural human group size from the neocortex ratio of primates. The questions the field left open are its oldest: whether structure or agency comes first, which regularities are laws and which are local customs, and whether small-group rules such as balance scale up to whole societies.

Current state. Sociology in 2026 is shaped by large digital data and by a crisis of credibility. The SCORE programme of the Center for Open Science reported its main results in 2026. It evaluated 3,900 claims with 865 collaborators. About half of the claims replicated, and the replicated effect sizes were reduced by a median of more than 50 % (COS SCORE). Loneliness has become a public-health subject. The WHO Commission on Social Connection reported on 30 June 2025 that about one person in six is lonely, that loneliness is linked to more than 871,000 deaths a year, and that 17–21 % of people aged 13–29 report loneliness (WHO).

Dunbar's number is under debate. His 1992 regression predicted a mean group size of 148 with a 95 % interval of 100 to 230. A replication by Lindenfors and colleagues (2021) with newer data found means between 16 and 109, depending on the method, with intervals as wide as 4–520 (Wikipedia, Dunbar's number). Dunbar's layered model keeps an inner core of about 5 people and a sympathy group of about 15 (same page). Zhou, Sornette, Hill and Dunbar (2005) analysed group sizes in hunter-gatherer and other societies and reported "a preferred scaling ratio close to 3" between successive layers (arXiv).

Balance theory itself has been tested on large data. Leskovec, Huttenlocher and Kleinberg (2010) counted triangles in three signed online networks and compared each type with a shuffle of the same signs. In all three, all-positive triangles were over-represented and triangles with exactly two positive edges under-represented. The all-negative triangle was over-represented in two of three, which they called at odds with Heider's original rule but in line with weak balance (ar5iv). Their table gives, for the triangle with one positive and two negative edges, a surprise of +249.4 in Epinions, −175.2 in Slashdot and +289.1 in Wikipedia (same source). Neuroimaging studies now look for balance in the brain (Chiang et al. 2020, cited at Wikipedia, Balance theory). The open debates are whether balance or status explains signed ties, how much of the observed structure comes from individual dispositions rather than from triads, and whether online signs mean what Heider meant by liking.

Interpretation.

1. *Strand.* I is the sequence of interactions in a population; P is the set of persons. A tie is a loop: the strand returns to a pair it has joined before. σ is the record of who meets whom, in order.
2. *Passage and trit.* A later passage through a pair is read relative to the first (reading N3). It is friendly (+1, written 1), hostile (−1, written T) or neutral or absent (0). This is the trinity of Foundations T1 applied to ties.
3. *Closure.* A window is a triangle of three ties. It closes when the digit of its sum is 0 (T1, T2), that is when its three signs are all equal or all different (T4). Among complete triangles, (1, 1, 1) and (T, T, T) close; (1, 1, T) and (1, T, T) are open (Calculation 18.1). By T7.1 the all-friends triangle closes with carry +1 and the all-enemies triangle with carry −1: each becomes one passage of the next scale, a group or a faction. An open triangle keeps a nonzero digit, which is tension. Balance theory uses the product of signs instead: (1, 1, 1) and (1, T, T) are balanced. The two rules agree on (1, 1, 1) and (1, 1, T). They disagree on (T, T, T), where strong balance says unstable and the closure rule says closed, and on (1, T, T), where balance says stable and the closure rule says open. Weak balance agrees with the closure rule on (T, T, T), so the only triangle on which the closure rule says something weak balance does not is (1, T, T). Exchange gives a second reading. Restricted exchange is a meeting (x, −x): it closes with carry 0. Generalized exchange is a ring (x, x, x): it needs at least three groups and closes with carry ±1 (Calculation 18.8, T7).
4. *Scale.* Rings merge by three (T15 with a = 0). Dunbar's layers 5, 15 and about 150 have ratios 3 and 9.9, near 3 and 3^2, but none of the sizes is of the form $2^a \cdot 3^b$ (Calculation 18.9). Kinship systems show the other ladder. Australian section systems divide a society into 2 moieties, 4 sections or 8 subsections; among the Lardil a man's subsection alternates over 2 generations and a woman's cycles over 4 (Wikipedia, Australian Aboriginal kinship). All these counts are powers of 2, the meeting ladder, and none is a multiple of 3 (Calculation 18.14). T13 applies to the census of triangles: 16 classes of directed triads (Holland and Leinhardt 1970), 10 classes of signed triangles with absent ties under all symmetries, and 11 under rotation alone, because the two cyclic orders of (T, 0, 1) differ (Calculations 18.6, 18.7). T11 applies where two closed groups are joined: the seam between two different closed triangles carries one or two open windows. T6 gives the chance baseline for closure (Calculation 18.3). T14, T16–T18, T19–T22 and T23–T24 have no counterpart here that could be computed.
5. *Test.* Stated before the data were opened. (a) Relative to a shuffle of the observed signs, the triangle (1, T, T) is under-represented and (T, T, T) over-represented. (b) Where an apex has two negative ties to two persons who are tied to each other, the third tie is negative more often than chance; balance predicts it is positive more often. Statistic for (b): the share of negative third edges, compared with a null that gives each edge the negativity of its two endpoints. The reading fails if (1, T, T) is over-represented or if the third edge is less often negative than the null.

The test was run on Epinions, Slashdot, Wikipedia Requests for Adminship, Bitcoin Alpha and Bitcoin OTC (Calculations 18.2–18.4). (T, T, T) is over-represented in all five, with surprises from +21.6 to +223.6. (1, T, T) is over-represented in four of five (Epinions +77.5, Slashdot +19.9, Bitcoin Alpha +54.3, Bitcoin OTC +85.6) and under-represented only in the Wikipedia network (−61.3). On statistic (b), the third edge is more often negative than the overall share of negative edges in all five networks, but less often negative than the endpoint null in all five, with z between −13.6 and −253.0. The excess negativity comes from the persons, not from the triangle. Where the closure rule differs from weak balance, it fails.

Consequences. The result is clear and should be stated plainly. Social triangles are not governed by sums mod 3 at the one point where that arithmetic says something new. The all-enemies triangle does persist, as the closure rule says, but Davis said so in 1967, and the closure rule earns no credit for repeating him. Strong balance loses on the same triangle, which confirms what Leskovec and colleagues found. The field therefore gains one clean, reproducible statement: of the three arithmetics on offer (products as in Heider, cluster rules as in Davis, sums mod 3 as in the Foundations), Davis's weak balance fits signed online networks best, and the mod-3 rule adds nothing to it.

The exchange reading costs nothing and adds a vocabulary. Lévi-Strauss's two forms of exchange are the meeting and the ring of T7, and the claim that generalized exchange "makes" a larger alliance is the carry of T7.1. This is structure that the field published independently, so it is graded A. It does not tell anthropologists anything they did not know, but it places their distinction among the same elementary closures as codons, quark triality and three-phase power elsewhere in this book. The Dunbar layers show the limit of such readings: a ratio near 3 is the ring ladder in spirit, but the sizes are off the ladder, so T15 does not apply in its exact form.

Grade. A for the two forms of exchange (restricted = meeting, generalized = ring; Lévi-Strauss 1949); B for signed triads, test run on public data and refuted at (1, T, T), the only triangle where the closure rule differs from weak balance; C for Dunbar layers.

18.2 Demography

History. John Graunt's *Natural and Political Observations upon the Bills of Mortality* (1662) turned London's weekly burial lists into the first life table and the first estimate of a city's population. Edmond Halley built a life table from the records of Breslau in 1693 and used it to price annuities. Counting itself became an institution: Sweden began continuous population statistics in 1749, the United States held its first census in 1790 and Britain in 1801. From then on a state could see its population as a structure, not only as a total. Leonhard Euler worked out the arithmetic of a population growing at a constant rate in 1760. Thomas Malthus's *Essay on the Principle of Population* (1798) set geometric growth of people against arithmetic growth of food. Pierre-François Verhulst proposed the logistic curve in 1838 to describe growth that slows as a population fills its space. Malthus added "moral restraint", the delay of marriage, as a preventive check in the second edition of his essay (1803), and so made

fertility a matter of behaviour rather than biology alone. Adolphe Quetelet's *Sur l'homme* (1835) treated the rates of birth, marriage and death as stable properties of a society, the first statement that a population has regularities its members do not choose. Wilhelm Lexis drew in 1875 the diagram that still bears his name, with calendar time on one axis and age on the other, so that each life is a line at 45 degrees and each cohort a band.

The twentieth century added structure by age. Alfred Lotka's stable population theory, begun with F. R. Sharpe in 1911, showed that fixed rates of birth and death at each age lead to a fixed age structure and a fixed growth rate. Harro Bernardelli in 1941, E. G. Lewis in 1942 and Patrick Leslie in 1945 wrote the same process as a matrix acting on a vector of age classes. Leslie's paper in *Biometrika* gave the matrix its name (Wikipedia, Leslie matrix). An age structure that oscillates without settling corresponds, in matrix language, to an imprimitive matrix, in which births happen only at ages sharing a common divisor greater than one.

Warren Thompson described the demographic transition in 1929 and Frank Notestein formalised it in 1945 (Wikipedia, Demographic transition). The model has four stages: high birth and death rates, falling death rates, falling birth rates, and low birth and death rates. Abdel Omran added the epidemiological transition in 1971, describing how the main causes of death shift from infection to chronic disease. From the 1970s demographers documented fertility far below replacement in Europe and East Asia, and the "fifth stage" became a debate. The field's open questions are why fertility keeps falling after it passes replacement, how migration changes the arithmetic of ageing, and whether the transition model predicts anything or only describes.

Current state. The United Nations *World Population Prospects 2024* put the world population at nearly 8.2 billion in mid-2024 and projected a peak of about 10.3 billion in the mid-2080s, followed by a slow fall to about 10.2 billion by 2100. That peak is about 700 million lower than the estimate of a decade earlier (UN News). More than half of all countries and areas have fertility below 2.1 births per woman, and nearly a fifth are below 1.4, including China, Italy, South Korea and Spain. Sixty-three countries and areas, among them China, Germany, Japan and Russia, have already passed their population peak; their combined population is projected to fall 14 % over the next 30 years (same source). That is about −0.5 % a year (Calculation 18.12).

Ageing follows directly. The same report projects that people aged 65 and over will outnumber those under 18 by the late 2070s, and that more than half of all deaths will occur at age 80 or older by the late 2050s, up from 17 % in 1995. About 4.7 million babies, 3.5 % of the total, were born in 2024 to mothers under 18 (same source). Rapid growth continues in a few countries: the populations of Angola, the Central African Republic, the Democratic Republic of the Congo, Niger and Somalia are expected to double between 2024 and 2054.

The transition model is under strain. Its stages describe Europe well but fit poorly where fertility fell before mortality (the United States around 1800) or where countries passed through the stages in two decades (China, Brazil, Thailand). Data from 2010–2018 no longer support the idea that fertility rises again at very high levels of

development (Wikipedia, Demographic transition). The frontier is the explanation of ultra-low fertility, the measurement of migration, and the use of register and genomic data to rebuild kinship networks at scale.

Interpretation.

1. *Strand.* I is calendar time in years; P is the set of age classes, or, for a pedigree, the set of persons. A cohort passing from one age class to the next is one step of the strand.
2. *Passage and trit.* A passage is the return of a population to an age structure it has had before. The trit records whether a year's births rise (1), fall (T) or hold (0) against the year before. The pedigree has its own trit-free structure: every person is a meeting of two parents.
3. *Closure.* A Leslie matrix whose only fertile age is k is a ring of length k. For the example of Calculation 18.10, with fertility only in age class 3, the cube of the matrix is the identity: the population returns exactly after three steps, a closed ring. With fertile ages 2 and 3 the period is gcd(2, 3) = 1 and the oscillation dies out. This is ordinary Perron–Frobenius theory; the Foundations add only the name. The transition model's stages are not windows of three and do not close. A closed window would require three successive changes in births whose digits sum to zero, for example a rise, a fall and a hold; the reading would then call a run of such years a closed episode and a run of rises tension. With yearly birth counts this can be computed against the T6 baseline, but I have not found a hypothesis about it that demographers do not already state in terms of tempo and quantum of fertility.
4. *Scale.* The pedigree is the meeting ladder of T15 with b = 0: n generations back, a person has 2^n nominal ancestor places. At n = 33, 2^{33} = 8.6 billion exceeds the present world population (Calculation 18.11). With 25 to 30 years per generation, that is 825 to 990 years back. Every person alive therefore descends from far fewer than 2^{33} distinct people of that time, and many of those places are filled by the same individual many times over. The places must therefore be filled by repeated persons (pedigree collapse), which is a closure of the pedigree on itself: two paths through different parents return to the same ancestor, a loop in the sense of I.1. The demographic stage counts (four, or five) carry no ladder content.
5. *Test.* The reading makes one weak prediction. Where fertility is concentrated at one age, population waves should have the period of that age and not a nearby ladder value. Since this is already a theorem about matrices, it cannot refute the Foundations independently. I found no test that could fail for the net and not for standard demography. T6, T8–T11, T13–T14 and T16–T24 have no demographic counterpart that I could compute.

Consequences. Demography gains little. The doubling of pedigrees is exact, and it is the clearest instance of the ×2 ladder in human affairs: every person is one meeting of two lines. Pedigree collapse, which genealogists and population geneticists already use, is the loop that this ladder forces. The imprimitive Leslie matrix gives a precise picture of a ring in a population, but it was understood before the Foundations. The costs of the reading are low and so are its gains. What the reading does not do is explain the central questions of the field: why fertility falls, and why it keeps falling.

There is one place where the vocabulary helps to state an existing fact more sharply. A two-parent pedigree is a binary tree only on paper. In any real population the 2^n places fill with fewer and fewer distinct persons as n grows, so the tree closes into a net. The number of distinct ancestors at generation n is a measure of how much the pedigree has closed on itself, and its ratio to 2^n falls towards zero within a few dozen generations for any population of realistic size (Calculation 18.11 gives the generation, 33, at which the nominal count passes the present world population). Population geneticists already compute this with coalescent models. The Foundations add the reading that a population is a strand that meets itself, which is true but does not change a single estimate.

Grade. A for the doubling of pedigrees (2^n ancestor places; the meeting ladder of T15); C for the transition model and for population waves.

18.3 Social and economic geography

History. Johann Heinrich von Thünen's *Der isolierte Staat* (1826) described agriculture around a single market town as a set of concentric rings: dairy and market gardens nearest, then forestry, then grain, then livestock. Each ring is fixed by transport cost and land rent. Alfred Weber's theory of industrial location (1909) found the point of least transport cost between raw materials and markets. Felix Auerbach noted in 1913 that the population of a city is roughly inversely proportional to its rank (Wikipedia, Zipf's law).

Walter Christaller's *Die zentralen Orte in Süddeutschland* (1933) is the centre of the classical theory. Christaller argued that towns supplying goods to a surrounding area would, on an even plain, form a hexagonal lattice. Higher-order places, which supply rarer goods, sit on a coarser hexagonal lattice. The ratio between the market area of a higher-order place and that of the next lower order is the number K. Under the marketing principle a higher place takes one third of each of its six neighbours' areas, so $K = 1 + 6 \times \tfrac{1}{3} = 3$. Under the transport principle lower places sit on the roads between higher ones and $K = 4$. Under the administrative principle no area may be split, so $K = 7$ (Wikipedia, Central place theory).

August Lösch's *Die räumliche Ordnung der Wirtschaft* (1940; English 1954) generalised the scheme (Wikipedia, August Lösch). Lösch allowed every hexagonal market size that fits the lattice, not just three, and rotated the lattices against each other to produce sectors rich and poor in towns. The sizes he allowed are now called Löschian numbers. The Dutch polder towns of the Noordoostpolder and Flevoland, planned in the twentieth century with six small towns around a larger one, are often cited as built examples (same page).

After 1950 geography turned quantitative. Torsten Hägerstrand modelled the spatial diffusion of innovations in 1953. Inside the city, Ernest Burgess's concentric zones of Chicago (1925), Homer Hoyt's sectors (1939) and Chauncy Harris and Edward Ullman's multiple nuclei (1945) gave three competing pictures of urban form: rings, wedges and clusters. Walter Isard founded regional science. Paul Krugman's new economic geography (1991) explained why industries cluster even on a featureless

plain. Harold Hotelling had shown in 1929 that two sellers on a line tend to crowd together at its middle, a result that later explained the convergence of political parties as well as the placing of shops. George Kingsley Zipf's *Human Behavior and the Principle of Least Effort* (1949) made rank-size regularities a general programme, and Xavier Gabaix's "Zipf's Law for Cities: An Explanation" (1999) derived them from growth rates that do not depend on size (Wikipedia, Zipf's law). The field left three questions: whether central place lattices exist outside planned landscapes, how distance matters in a world of cheap transport and digital services, and why city sizes follow so regular a distribution.

Current state. The United Nations' *World Urbanization Prospects 2025*, released on 18 November 2025, uses a harmonised definition of cities, towns and rural areas. It finds that cities are home to 45 % of the world's 8.2 billion people, against about 20 % of 2.5 billion in 1950. There are 33 megacities of more than ten million inhabitants, up from 8 in 1975; 19 of them are in Asia. Jakarta, with nearly 42 million, is the largest, followed by Dhaka with almost 40 million and Tokyo with 33 million. Two-thirds of population growth to 2050 is expected in cities, the number of megacities is expected to rise to 37, and rural areas are now the most common settlement type in 62 countries, down from 116 in 1975 (UN DESA). The urban population has grown 7.4-fold since 1950 (Calculation 18.12).

Displacement is the other large movement. The Internal Displacement Monitoring Centre reports that conflict displacement reached a record high in 2025 while disaster displacement fell. Sub-Saharan Africa holds 31.7 million internally displaced people, nearly 40 % of the global total, and East Asia and the Pacific recorded 19.6 million movements in 2025 (IDMC GRID 2026).

Digital geography has adopted hexagons on its own terms. Uber's H3 grid, widely used for spatial analysis, divides the globe into hexagons with aperture 7: each cell has seven children at the next resolution, in 16 resolutions, with 12 pentagons at every level (H3). The open debates concern whether urban scaling laws depend on how a city is bounded (Arcaute et al. 2015) and whether apparent power laws survive careful statistics (Leitão et al. 2016) (Wikipedia, Urban scaling). Bettencourt, Lobo, Helbing, Kühnert and West (2007) had found that socio-economic outputs of cities grow faster than population and infrastructure slower (same page). Arcaute and colleagues showed that the exponents change when the same cities are bounded differently, and Leitão and colleagues that several apparent non-linearities vanish under better error models. Central place lattices themselves are tested chiefly in planned landscapes; in the case of Cambridge in England, seven satellites lie on main roads 10–15 miles away, giving K ≈ 4.5 (Wikipedia, Central place theory).

Interpretation.

1. *Strand.* I is the sequence of journeys made by people and goods; P is the set of places. A central place is a node in the sense of I.1: a place the strand visits again and again.

2. *Passage and trit.* A return to a central place is a passage. Read on the hexagonal lattice, a direction of movement is a unit of Z[ω], and the class of a lattice point modulo the prime $\beta = 1 - \omega$ is a trit (Foundations III.1, remark).
3. *Closure.* The hexagonal lattice of central place theory is the lattice of the Eisenstein integers. A coarser hexagonal lattice nested in it is generated by one element z, and the ratio of areas is the norm $N(z) = a^2 - ab + b^2$. The allowed market-area ratios are therefore exactly the norms of Z[ω], which OEIS lists as the Löschian numbers 1, 3, 4, 7, 9, 12, 13, … (OEIS A003136). Christaller's K = 3 is $N(1 - \omega)$, the prime above 3 whose residue field is the trit; K = 4 is N(2); K = 7 is $N(3 + \omega)$ (Calculation 18.13). This is a structure of the field, published independently of the Foundations.
4. *Scale.* By T15 the coarse lattices of a hierarchy built from meetings and rings have ratios $2^a \cdot 3^b$. Of the Löschian numbers up to 50, nine are on that ladder (1, 3, 4, 9, 12, 16, 27, 36, 48) and eleven are not (7, 13, 19, …). K = 3 and K = 4 are on it; K = 7 and the H3 aperture 7 are not. On the plane a free random walk returns with probability 1 (Pólya), so the lever of T24 for a walker on a square grid tends to −2 as the walk lengthens: −2.76, −2.49, −2.36 for 10^2, 10^3, 10^4 steps (Calculation 18.16). T6–T11, T16–T22 and T23 have no geographic counterpart I could compute. One theorem nearly applies: where two hexagonal systems of different orientation meet, as in Lösch's rotated lattices, the border is a seam in the sense of T11, and T11 says that a seam between two different closed windows always carries tension. Lösch's "city-rich" and "city-poor" sectors are such seams, but I have no count that would turn this into a check.
5. *Test.* Stated before any data were examined: in settlement systems that grew by market forces, the measured ratio of market areas between successive levels clusters at 3 or 4, ladder values; ratios at 7, 13 or 19 appear only where an administration drew the boundaries. The reading fails if unplanned systems show K = 7 as often as K = 3 or 4, or show no clustering at all.

Consequences. The grade A link is real and specific: every K value of central place theory is a norm in Z[ω], the same ring that carries the trit as its residue mod $1 - \omega$. Geographers do not usually say so, and saying so places Christaller's three principles in one algebra: marketing is the prime above 3, transport is the prime 2, administration is a prime that splits ($7 = (3 + \omega)(3 + \bar{\omega})$). The ladder test goes further than central place theory itself, because it ranks the principles. Christaller allowed all three; the Foundations prefer the two on the ladder of 2 and 3. This costs nothing if the test is run on existing gazetteers and market-area studies, and it fails cleanly if K = 7 turns out common in unplanned landscapes. The test does not touch the larger topics of the field, migration and displacement, which remain outside the reading. Digital practice offers an incidental observation. H3 uses aperture 7, an off-ladder Löschian number, for a global analysis grid. That is an engineering choice, not a settlement system, so it neither supports nor refutes the test; it shows only that the Löschian numbers off the ladder have uses of their own, and that the test must be confined to landscapes that grew without a designer.

Grade. A for Löschian market sizes as norms of Z[ω] (Lösch 1940; OEIS A003136); B for the ladder test $K \in \{3, 4\}$ against K = 7; C for urbanisation figures.

18.4 Network science

History. Leonhard Euler solved the problem of the seven bridges of Königsberg in 1736 by replacing the city with points and lines, and so founded graph theory. Jacob Moreno drew the first sociograms in *Who Shall Survive?* (1934). Paul Erdős and Alfréd Rényi gave the theory of random graphs in 1959 and found that a giant connected component appears suddenly when the mean degree passes one. Manfred Kochen and Ithiel de Sola Pool estimated the number of acquaintances and the length of chains between strangers in a paper that circulated from the late 1950s and was published in 1978; it set the question that Milgram answered by experiment. Stanley Milgram's small-world experiment of 1967 asked people in Nebraska and Kansas to pass a letter towards a target in Boston through acquaintances. In one mailing of 296 letters, 64 arrived, along chains of about five and a half to six links (Wikipedia, Small-world experiment).

Paul Holland and Samuel Leinhardt introduced the triad census in 1970. A directed graph on three nodes falls into one of 16 isomorphism classes, and the counts of these classes in a network test theories of hierarchy and transitivity. Mark Granovetter's 1973 paper in the *American Journal of Sociology* (Wikipedia, Mark Granovetter) turned triadic closure into a theory of information flow. Its strong form says that if a person has strong ties to two others, those two are at least weakly tied; the open wedge of two strong ties is the forbidden case (Wikipedia, Triadic closure). Duncan Watts and Steven Strogatz published "Collective dynamics of 'small-world' networks" in *Nature* in 1998 (Wikipedia, Watts–Strogatz model). Their model starts from a ring lattice, where every node is tied to its nearest neighbours and clustering is high, and rewires a small share of edges at random. A few shortcuts make paths short while clustering stays high. Albert-László Barabási and Réka Albert proposed preferential attachment in 1999, which produces degree distributions with power-law tails. Sergey Brin and Larry Page's PageRank (1998) ranked web pages by the long-run share of time a random surfer spends on each, which made the return of a random walk a commercial quantity.

In 2003 Peter Dodds, Roby Muhamad and Watts repeated Milgram's experiment by email with 24,163 chains aimed at 18 targets around the world and found mean chains of about six steps (same small-world page). In 2010 Leskovec, Huttenlocher and Kleinberg brought signed edges into network science with data from Epinions, Slashdot and Wikipedia, and proposed status theory for directed signed ties next to balance theory for undirected ones (ar5iv). The questions left open were how general the "scale-free" claim is, how to separate network effects from shared traits (homophily), and which mechanism generates the triangles seen in real networks.

Current state. The scale-free claim has been narrowed. Anna Broido and Aaron Clauset tested nearly 1,000 networks from social, biological, technological and information sources. Only 4 % showed the strongest evidence of scale-free structure and 52 % the weakest; social networks are "at best weakly scale free" (arXiv). The world is smaller than Milgram found. Facebook's research team measured an average of 3.57 degrees of separation among about 1.59 billion users in February 2016, down from 3.74 in 2011 when the network had 721 million users (Facebook Research).

Signed networks are now a standard object, with public data sets used to benchmark link-sign prediction. For this chapter the original signed data were recounted (Calculation 18.2). The Slashdot file has the same 549,202 directed edges that Leskovec and colleagues reported. The Epinions file used here has 841,372 directed edges against their 841,200; the Wikipedia file is the later Requests for Adminship vote network rather than their 2010 election file. After symmetrising and dropping pairs with conflicting signs, the counts give 4.77 million triangles in Epinions, 0.57 million in Slashdot and 1.19 million in the Wikipedia network. The two Bitcoin trust networks add 21,677 and 32,944 triangles. The recount reproduces the published pattern for (1, 1, 1) and (1, 1, T), and shows that the sign of the surprise for (1, T, T) depends on the version of the data: Slashdot is −175.2 in the 2010 table and +19.9 here. The 2010 paper lists Epinions with 119,217 nodes, 841,200 edges and 85.0 % positive edges; Slashdot with 82,144 nodes, 549,202 edges and 77.4 % positive; and Wikipedia with 7,118 nodes, 103,747 edges and 78.7 % positive (ar5iv). In the undirected recount the positive shares are 83.3 %, 76.4 % and 77.6 %, because pairs that rated each other with opposite signs were dropped.

The frontier lies in higher-order networks (hypergraphs and simplicial complexes), temporal networks in which ties carry time stamps, and causal inference on networks, where the effect of a tie must be separated from the traits that made the tie. Graph neural networks have made signed and directed link prediction a machine-learning task, and with it the question of which structural rule a model has learned.

Interpretation.

1. *Strand.* I is the sequence of steps of a walk on the graph, or the order in which ties are formed; P is the set of nodes. A random walk is a strand in the literal sense of I. 1, and PageRank’s random surfer is one. Each click is one step; the record of pages visited is σ; a page the surfer comes back to is a node. The long-run share of visits to a page is its rank, so PageRank is, in the language of the Foundations, a measure of how often the strand passes through each node.
2. *Passage and trit.* A passage is a return of the walk to a node. A wedge (two ties sharing a node) with signs (x, y) and its missing third tie read as 0 is a window (x, y, 0).
3. *Closure.* By T4 the window (x, y, 0) closes exactly when y = −x: the mixed wedge (1, T) is a closed meeting with nothing missing, while (1, 1) and (T, T) are open, with digits T and 1. The clustering coefficient counts how often wedges are closed by a third tie. In a ring lattice it tends to 3/4 as the number of neighbours grows (Calculation 18.15); this number has no counterpart in the Foundations.
4. *Scale.* Return is the heart of T24. On a finite connected graph a walk returns with probability 1, so its lever −(1 + P)/P tends to −2; on the plane the approach is slow (Calculation 18.16). Triads count up to symmetry as in T13 (Calculation 18.6). T15 would require networks to merge in blocks of 2 or 3, and Broido and Clauset’s finding that most networks are not even scale-free leaves no measured block size to check. T14, T16–T22 and T23 have no counterpart here.
5. *Test.* Stated before the data were opened: since the mixed wedge (1, T, 0) is already closed and carries no tension, it should be completed by a third tie least often of the

three wedge types. Statistic: share of wedges of each type that are closed by an edge, computed over all nodes of degree up to 3,000. The reading fails if mixed wedges are not the least often completed.

The test was run on all five networks (Calculation 18.5). It holds in Bitcoin Alpha (0.0736 against 0.0772 and 0.0908) and narrowly in Bitcoin OTC (0.0578 against 0.0582 and 0.0673). It fails in the three large networks: in the Wikipedia network, Slashdot and Epinions the all-negative wedge is completed least often (0.0969, 0.0160, 0.0354).

Consequences. The wedge test fails where the data are largest, and the reason is easy to see: negative ties are concentrated on a few targets, so wedges of two negative ties are formed around persons who attract hostility but are seldom tied to each other. The closure rule does not see this, because it treats every sign as a free trit. A reading that took account of node dispositions might do better, but it would then be a different model, and it would have to beat status and balance theory on held-out data. Network science gains two precise negative results from this chapter, the third-edge test of 18.1 and the wedge test here, and a public script that reproduces both. That is a small contribution, but it is the kind this book asks for: a link that could fail and did.

The positive side is narrower. A random walk on a network is the most literal strand in this book, and the Foundations' lever $-(1 + P)/P$ is a well-defined statistic of it. For walks on finite social graphs P is 1 in the long run and the lever is -2; for truncated walks it is a function of how quickly the walk returns, and so a measure of how "local" a network is. Whether that measure says anything that the mixing time or the spectral gap of the graph does not already say is an open question. I expect it does not, and I grade it C.

Grade. B for the wedge test, run on five public networks and refuted in the three largest; C for the lever of a walk on a graph and for small-world measures.

Chapter 19 — Markets, Power and Communication

This chapter reads the meeting column of level L5: exchange that goes and comes back, in markets, in votes and in messages. Two tests stated in the first edition of this book were run here on public data, and both came out against the reading. Monthly closed windows in the S&P 500 since 1871 exceed the chance rate of Foundations T6, but a simple model of serial correlation explains the excess, and there is no deficit of closed windows before bear markets. Majority cycles, the closed rings of collective choice, occur in 2 of 9,238 triples of candidates in 68 real ranked-ballot elections, against 8.8 % expected by chance: real electorates select against the closed ring. The arithmetic of a Condorcet cycle as a ring with carry is exact, and barter as a meeting is exact, but neither adds a prediction that survived. Grades: economics and business B (test run, refuted), C for the readings of exchange; political science A for the cycle as a closed ring (12 of 216 profiles), B for the frequency test (run, refuted as persistence), C for terms of office; media and communications C, with the signed-triad test of Chapter 18 applying to Slashdot and failing there.

19.1 Economics and business

History. Adam Smith's *Wealth of Nations* (1776) explained how the division of labour and exchange in markets coordinate people who do not know each other. David Ricardo's *Principles* (1817) added comparative advantage and the theory of rent. Antoine Augustin Cournot gave the first mathematical model of competition among a few firms in 1838. In the 1870s William Stanley Jevons, Carl Menger and Léon Walras independently founded marginal analysis. Walras's *Éléments d'économie politique pure* (1874) wrote the whole economy as a system of simultaneous equations that clear all markets at once. Jevons's *Money and the Mechanism of Exchange* (1875) began with the problem of barter: a trade needs a "double coincidence of wants", each party holding what the other wants (Wikipedia, Coincidence of wants). Money exists, on this account, because most wants do not meet in pairs.

Alfred Marshall's *Principles of Economics* (1890) made supply and demand curves the standard picture. Louis Bachelier's thesis of 1900 modelled the prices of the Paris bourse as a random walk, decades before the idea was taken up. Nicholas Kaldor analysed the cobweb model in 1934 and Mordecai Ezekiel published "The Cobweb Theorem" in 1938. In that model producers plan this season's output on last season's price, so prices alternate high and low; the swings shrink when supply is less elastic than demand and grow otherwise (Wikipedia, Cobweb model). Ronald Coase asked in 1937 why firms exist at all if markets coordinate so well, and answered with the cost of using the market.

John Maynard Keynes's *General Theory* (1936) founded macroeconomic stabilisation. John von Neumann and Oskar Morgenstern's *Theory of Games and Economic Behavior* (1944) gave strategy a mathematics, and John Nash proved in 1950 that every finite game has an equilibrium in mixed strategies. Kenneth Arrow and Gérard Debreu

proved the existence of a general equilibrium in 1954. Paul Samuelson argued in 1965 that properly anticipated prices fluctuate randomly, and Eugene Fama's 1970 review defined the efficient-markets hypothesis. Daniel Kahneman and Amos Tversky's prospect theory (1979) brought measured psychology into choice. The crisis of 2008 reopened the role of finance in the macroeconomy. The questions left open are how expectations form, how money and credit enter real activity, and whether market prices carry structure that a careful trader can use.

Business studies grew alongside. Frederick Taylor's scientific management (1911), Alfred Chandler's *Strategy and Structure* (1962) and Michael Porter's *Competitive Strategy* (1980) shaped how firms are taught and run. The firm, in Coase's terms, is a closed unit inside the market: a place where exchange is replaced by command.

Current state. The Sveriges Riksbank Prize in Economic Sciences for 2026 was announced on 12 October 2026. It went to Robert E. Hall of Stanford and Michael Woodford of Columbia (Marketplace) "for research on the microeconomic foundations of macroeconomic stabilization", according to a secondary report of the announcement (Chronicle India). The prize recognises the line of work that rebuilt monetary policy on models of households and firms that form expectations.

The world economy is growing slowly under a price shock. The IMF's October 2026 World Economic Outlook, as reported by CGTN, projects global growth of 3.1 % in 2026, up 0.1 point from July, and 3.4 % in 2027. Advanced economies grow 2.1 % in 2026 and 1.8 % in 2027; emerging and developing economies 3.7 % and 4.4 %. Global headline inflation rises from 4.1 % in 2025 to 4.7 % in 2026 before easing to 4.1 % in 2027. The report attributes the drag to a war-driven energy shock from the Middle East and the support to investment in AI infrastructure (CGTN, secondary report).

Finance research has largely moved from the question whether returns are predictable to the question of which predictability survives costs. Momentum, value and other "factors" are measured in long samples, and their decay after publication is itself a research subject. Market microstructure studies how individual orders form prices, and high-frequency data have made the sign of each price tick (up, down or unchanged) an object of study in its own right. In macroeconomics, the renewed inflation of 2021–2026 tested the models for which this year's prize was given, and debates continue over how much of it came from supply shocks and how much from demand.

The long monthly record of the S&P composite index is public. The series used here runs from January 1871 to September 2026; it follows Robert Shiller's data to June 2023 and extends them with prices from the Federal Reserve Bank of St. Louis (datasets/s-and-p-500). It gives 1,868 monthly changes and 13 bear markets of at least 20 % from a prior peak, with peaks from April 1872 to December 2021 (Calculation 19.3).

Interpretation.

1. *Strand*. I is calendar time in months; P is the set of price levels of an index. A market price revisiting a level it has held before is a loop.

2. *Passage and trit.* Each month the index rises by more than 1 % (1), falls by more than 1 % (T) or stays within the band (0). This is the trinity of T1 read on price changes; the band is a stated choice.
3. *Closure.* A window of three months closes when the digit of its sum is 0 (T1, T2): three equal moves or one of each (T4). With the observed frequencies (46.8 % up, 24.0 % flat, 29.1 % down), T6 gives a chance rate of 0.338 for closed windows. Exchange gives a second reading. Barter is a meeting (x, −x): goods go one way and come back the other, closing with carry 0 (T7.1). Where no pair of traders has a double coincidence of wants, as in an economy of k types each producing what the next type consumes, a trade can only close round the whole cycle (Calculation 19.6). For k = 3 that is a ring, which by T7.1 closes with carry ±1 and becomes one passage of the next scale; a token that travels the ring is a medium of exchange. The cobweb model is a meeting in time: the deviation from equilibrium is multiplied by −s/d each season, a sign flip of period 2 (Calculation 19.5).
4. *Scale.* Returns aggregate: monthly changes add to quarterly and yearly changes, and the trit of a sum is its balanced residue, with the carry passed upward (T1). On the ladder of T15, three-month windows merged into nine-month blocks would be the ring ladder. Under the random-walk hypothesis a price is a free path in one dimension; it returns with probability 1, so the lever of T24 tends to −2. T13 applies to the count: 3^n open words of monthly moves against g(n) cyclic patterns. T11 would apply to a regime change, where one closed pattern meets another. T14, T16–T22 and T23 have no counterpart I could compute.
5. *Test.* Stated in the first edition of this book, before these data were opened: the closed-window rate exceeds the T6 baseline in calm regimes and falls below it in the year before a crisis. If the rate equals its baseline in all regimes, the reading fails. Statistic: closed share in the 12 monthly windows before each bear-market peak, and in windows more than 24 months from any peak, each against T6 computed from the same months. A stronger baseline, added here, is a first-order Markov chain fitted to the trits, which keeps the serial correlation of the series.

The test was run (Calculation 19.3). Over all 1,866 windows the closed share is 0.380 against the T6 rate of 0.338, and a shuffle of the months gives z = 3.67. But the Markov baseline is 0.376, and the lag-1 autocorrelation of monthly log changes is 0.274. Averaging daily prices into monthly values alone would produce an autocorrelation of about 0.25 (Calculation 19.4). In calm months the closed share is 0.384 against a Markov rate of 0.374. Before bear-market peaks it is 0.340 against T6 at 0.348 and Markov at 0.353, a difference with z = −0.22. The excess is serial correlation, and the predicted deficit before crises is not there.

Consequences. For economics the closed-window rate is not a new indicator. At monthly resolution it measures the same persistence that any autocorrelation coefficient measures, and before crises it does not move. The reading could be tried again at daily or tick resolution, where the up–down–unchanged sign of each price change is the natural trit and the averaging artefact disappears. That would be a new test, and its prediction should be stated before the data are opened; this chapter does not claim it.

The exchange reading is sound but old. Jevons stated the double coincidence of wants in 1875, and the theory of money as a medium that closes a cycle of trades no pair can close was worked out in the twentieth century. The Foundations give the two forms a common name (meeting and ring) and a common arithmetic (carry 0 and carry ±1), and that is all. The cobweb model's period 2 is the meeting of T7 in time; the period-doubling route to irregular cycles belongs to Chapter 20. Business studies are not touched. The firm as a closed unit inside the market is a standing in the sense of T7, but no theorem of the Foundations says when a firm should form, which is Coase's question.

Grade. B for the closed-window test on S&P monthly returns (run, refuted: explained by serial correlation, no pre-crisis deficit); C for barter as meeting, a cycle of three traders as ring, and the cobweb as period 2.

19.2 Political science

History. Aristotle's *Politics* sorted constitutions by who rules (one, few or many) and whether they rule for the common good or for themselves, which gave three good forms and three corrupt ones. Polybius turned the six into a cycle (Chapter 20). Niccolò Machiavelli, Thomas Hobbes (*Leviathan*, 1651) and John Locke (*Two Treatises*, 1689) founded modern political thought on the problem of order and consent. Montesquieu's *De l'esprit des lois* (1748) separated legislative, executive and judicial power so that each could check the others.

The formal theory of voting began in the eighteenth century. Jean-Charles de Borda proposed his rank-order count to the French Academy, and it was published in 1781. Nicolas de Condorcet's *Essai* of 1785 showed that majority rule can cycle. With three voters ranking A > B > C, B > C > A and C > A > B, a majority prefers A to B, B to C and C to A, so no alternative beats all others. The same essay proved the jury theorem: if voters are independent and each is right more than half the time, the majority's chance of being right rises towards certainty as the group grows. Charles Dodgson rediscovered the cycles in the 1870s.

The twentieth century made these results central. Duncan Black's median voter theorem (1948) showed that cycles cannot occur when preferences are single-peaked on one dimension. Kenneth Arrow proved in 1950 that no rule for ranking three or more alternatives satisfies a short list of reasonable conditions unless it is a dictatorship (Wikipedia, Arrow's impossibility theorem). Three is the threshold in Arrow's theorem as in Condorcet's paradox: with two alternatives simple majority satisfies every condition, and the trouble begins with the third. Anthony Downs's *Economic Theory of Democracy* (1957) applied spatial competition to parties. Maurice Duverger argued that plurality rule in single-member districts tends to produce two parties (Wikipedia, Duverger's law). Allan Gibbard (1973) and Mark Satterthwaite (1975) showed that every non-dictatorial rule with three or more outcomes can be manipulated. Richard McKelvey (1976) proved that in two or more dimensions majority rule can lead from any alternative to any other.

William Riker's *Liberalism against Populism* (1982) drew a pessimistic conclusion: if cycles are possible, the "will of the people" is often an artefact of agenda control. Gerry Mackie's *Democracy Defended* (2003) answered that real cycles are rare. Empirical work followed Samuel Huntington's "third wave" of democratisation (1991) and, from 2014, the expert-coded indices of the Varieties of Democracy project. The questions left open are how often social choice paradoxes occur in practice, what sustains democratic institutions, and why some democracies erode from within.

Current state. The V-Dem Institute's *Democracy Report 2026*, released on 17 March 2026, counts 44 countries autocratising in 2025, a quarter of the 179 it rates (Calculation 19.8), home to about 41 % of the world population, and 18 democratising, home to about 5 % (V-Dem report). At the end of 2025 it classifies 87 countries as democracies and 92 as autocracies, with measurement uncertainty that puts democracies between 74 and 95. Seventy-four per cent of the world population, about 6 billion people, live in autocracies. There are 31 liberal democracies, down from 45 in 2009, home to about 7 % of the world population. Measured by population, the level of democracy enjoyed by the average person is back to 1978 (same source). The institute's press release states that the United States' liberal democracy index fell 24 % in one year, from 20th to 51st place of 179 (V-Dem). These figures are the institute's expert-coded measures; other indices weight components differently, and the classification of individual countries is debated.

On the frequency of majority cycles the evidence has grown. Under "impartial culture", where every ranking is equally likely, the chance of a cycle among three alternatives is 5.6 % with three voters and tends to about 8.77 % for large electorates. A survey of 37 studies covering 265 real elections found 25 cycles (9.4 %), probably an overestimate because cycles are more likely to be reported. Three-candidate elections derived from Electoral Reform Society data show 0.7 %, those derived from American National Election Studies 0.4 %, and of 189 ranked-choice elections in the United States from 2004 to 2022, one had a cycle: Minneapolis Ward 2 in 2021 (Wikipedia, Condorcet paradox). Simulations in spatial models give 5 % at 100 voters and 0.06 % at 10,000 (same page).

The PrefLib library publishes the full ballots of ranked elections. Its Irish, Debian, Glasgow, Aspen, Berkeley, Minneapolis, Oakland, Pierce County, San Francisco, San Leandro and Takoma Park files give 68 elections; 66 have 25 or fewer candidates, yielding 9,238 triples of candidates (Calculation 19.2).

Interpretation.

1. *Strand.* I is a sequence of pairwise contests; P is the set of alternatives. Returning to an alternative already compared is a loop.
2. *Passage and trit.* Fix an orientation of three alternatives, A → B → C → A. Each pairwise majority agrees with the orientation (1), disagrees (T) or ties (0).
3. *Closure.* A Condorcet cycle carries (1, 1, 1) or (T, T, T): a ring whose sum is ±3, digit 0, carry ±1 (T1, T7.1). A transitive order such as A > B > C carries two agreements and one disagreement, (1, 1, T), with digit 1: open. With three voters and three alternatives, 12 of the 216 profiles close and 204 are open (Calculation 19.1). The

cycle is exactly the closed ring of T7, and the transitive order is exactly an open window with two equal trits (T4). This is a structure of social choice theory, independent of the Foundations.

4. *Scale.* The carry of T7.1 has a political reading: a closed ring at the level of alternatives cannot be settled at that level and passes to the next, where the agenda or the rule decides. With more alternatives cycles become more likely; Merrill's simulations give 8.4 % for 3 candidates and 47.5 % for 10 with 25 voters (Wikipedia, Condorcet paradox). The ring ladder appears in institutions: the United States Senate has three classes of 33, 33 and 34 seats, one class elected every two years, so a seat's six-year term is a ring of three two-year passages (Wikipedia, Classes of United States senators). Terms of office are otherwise not ladder-specific: 13 of 17 national terms are on $2^a \cdot 3^b$, but four of the six integers from 2 to 7 are on the ladder, and 13 or more of 17 would occur by chance with probability 0.28 (Calculation 19.7). T6 fails as a baseline here because the three majorities of a profile are not independent: closure occurs in 12 of 216 profiles, not one third. T11 reads a coalition of two parties with different closed orders as a seam with tension. T12–T14, T16–T22 and T23–T24 have no counterpart I could compute.
5. *Test.* Stated in the first edition, before the PrefLib data were opened: if closed windows persist (reading N2), majority cycles should occur at least at the chance rate; if they are much rarer, something selects against them and the reading fails for collective choice. Statistic: share of candidate triples whose pairwise majorities form a cycle, against the impartial-culture rate of 8.77 %.

The result is decisive (Calculation 19.2). Of 9,238 triples, 2 form cycles (0.022 %), both in Glasgow ward elections of 2007; impartial culture predicts about 811. Of the 68 elections, 67 have a Condorcet winner. The Govan ward has none. The Govan cycle is a top cycle: its three candidates each beat all eight others and lose only within the ring. The Springburn cycle lies in the middle of the field, among candidates with four pairwise wins of nine (Calculation 19.10). A closed ring can therefore occur at the top, where it matters, but in this sample it did so once.

Consequences. Political science gains an exact identity and loses a prediction. The identity is that a majority cycle is a closed ring and a transitive order an open window, with the counts 12 and 204 for three voters; the field already knew the counts, so this is grade A for the structure, not a discovery. The prediction fails. Real electorates are far from impartial culture, mostly because preferences share dimensions (Black's single-peakedness in practice), and the closed ring, far from persisting, almost never forms. The carry reading of T7.1, in which a cycle "passes up" to the agenda setter, fits Riker's account of manipulation, but it was formulated after seeing the data and must be counted as a reading.

The cost of the identity is a tension inside the Foundations. Reading N2 says closed windows persist. In collective choice the closed window is the unstable outcome and the open window, the transitive order, is what institutions produce and keep. Either "persist" must be read as "pass to the next scale with a carry", as T1 and T7.1 allow, or the trit of a pairwise majority is the wrong trit for politics. The book records the tension rather than resolving it.

Grade. A for the Condorcet cycle as a closed ring and the transitive order as an open window (12 of 216 profiles); B for cycle frequency, run on 68 PrefLib elections and refuted as persistence (2 cycles in 9,238 triples against about 811 expected); C for terms of office.

19.3 Media and communications

History. Printing with movable type in Mainz around 1450 made identical copies cheap, and the first weekly newspapers appeared in the early seventeenth century. The mass press of the nineteenth century, the telegraph and, from the 1920s, radio created audiences of millions. Walter Lippmann's *Public Opinion* (1922) described how people act on "pictures in their heads" formed by the press rather than on the world itself. Harold Lasswell summed up the field in 1948 with the question "who says what, in which channel, to whom, with what effect?". In the same year Claude Shannon's "A Mathematical Theory of Communication" defined information as a measure of choice and set limits on how fast a noisy channel can carry it.

Empirical research soon found that media effects pass through people. Paul Lazarsfeld, Bernard Berelson and Hazel Gaudet's *The People's Choice* (1944) studied the 1940 United States election in Erie County, Ohio, and Elihu Katz and Lazarsfeld's *Personal Influence* (1955) formulated the two-step flow: ideas move from the media to opinion leaders and from them to others. Theodore Newcomb's ABX model (1953) applied Heider's balance to communication: two persons A and B and an object X form a triad, and communication works to bring their attitudes into balance. Marshall McLuhan's *Understanding Media* (1964) argued that the form of a medium matters more than its content. Jürgen Habermas's *Structural Transformation of the Public Sphere* (1962) traced how the coffee houses, salons and journals of the eighteenth century created a space for argument among private persons, and how commercial mass media later turned that public into an audience. Stuart Hall's encoding/decoding essay (1973) insisted that audiences read messages in their own ways, and Edward Herman and Noam Chomsky's *Manufacturing Consent* (1988) proposed that ownership, advertising and sourcing filter what news reaches the public.

The 1970s produced the theories still taught. Maxwell McCombs and Donald Shaw's study of the 1968 election (1972) showed that media set the agenda: they do not tell people what to think, but what to think about. Anthony Downs's "issue-attention cycle" (1972) described how a problem rises to public alarm, peaks and fades while the problem remains. Elisabeth Noelle-Neumann's spiral of silence (1974) held that people who think their view is in the minority fall silent, which makes it look smaller still. George Gerbner's cultivation studies measured how heavy television viewing shapes beliefs about the world. Everett Rogers's *Diffusion of Innovations* (1962) described how new ideas spread through a population along personal networks, from a few innovators through early adopters to the majority, and gave the field its S-shaped adoption curve. The internet, search engines and, after 2004, social platforms moved the gatekeeping that editors once did to algorithms. The European Union's Digital Services Act of 2022 began the regulation of very large platforms. The questions left open are how much media change minds rather than reinforce them, how algorithms select what people

see, and how trust in news is built or lost. A further question, older than the internet, is whether attention itself moves in cycles. Downs thought it did for every public problem, and later work has measured the rise and fall of issues in news archives; whether these returns have fixed periods or depend on events has not been settled.

Current state. The Reuters Institute *Digital News Report 2026* surveyed 97,520 people in 48 markets, about 2,000 per market (Reuters Institute). Overall trust in news is 37 %, down 3 points and the lowest since measurement began in 2015; it fell in 29 of 48 markets. Trust in the United States is 25 %. Forty-two per cent say they sometimes or often avoid the news, against 29 % in 2017. Interest in news has fallen 13 points since 2021, from 59 % to just under 46 % (same source).

Platforms now lead. Fifty-four per cent use social media or video networks for news each week, more than the 51 % who use news websites and apps. Facebook reaches 43 % for news, YouTube 34 %, Instagram 26 % and TikTok 20 %. Thirty per cent name social media as their main source, up from 22 % five years ago, and 12 % rely on it alone, double the 6 % of 2020. Online news video reaches 77 % weekly, and television news still leads only in the Netherlands, Denmark and Germany. Weekly use of AI chatbots for news rose from 7 % to 10 %, and to 16 % among people under 35; only 20 % trust the answers. Concern about what is real or fake online is 62 % (same source; Calculation 19.9).

The economics of news remain weak. In a basket of 20 countries 17 % pay for online news, unchanged from the year before; Norway is highest at 40 % and Sweden next at 32 %. Organic traffic from Google search to news sites fell 33 % worldwide between November 2024 and November 2025, and 38 % in the United States, according to Chartbeat data cited in the report. Across 26 countries with public service media, 37 % see a positive social impact of public service news and 22 % a negative one (same source).

Regulation has begun to bite. The European Commission fined X €120 million for breaching transparency obligations under the Digital Services Act (European Commission). The research frontier is the measurement of exposure on platforms whose data are closed, the effect of recommendation algorithms on polarisation, and the role of generative AI as a new intermediary between news and audience.

Interpretation.

1. *Strand.* I is the flow of messages in time; P is the set of senders, receivers and topics. A topic that returns to public attention is a loop.
2. *Passage and trit.* A rating on a platform is a trit: like (1), dislike (T) or none (0). Newcomb’s triad of two persons and an object carries three such signs.
3. *Closure.* Newcomb’s ABX triad is the signed triangle of Chapter 18, and the same rules apply: closure mod 3 calls (1, 1, 1) and (T, T, T) closed, balance calls (1, 1, 1) and (1, T, T) balanced. Slashdot, one of the five networks tested in 18.1, is a technology news site whose users mark each other as friend or foe. There the triangle (1, T, T) is over-represented (surprise +19.9), and the third edge opposite two negative ties is less often negative than the endpoint null predicts ($z = -133.6$). The closure rule fails on a media platform as it fails elsewhere.

4. *Scale.* The two-step flow is a chain of meetings, medium to leader and leader to follower. Downs's issue-attention cycle has five stages, which is off the ladder of 2 and 3; that is a count of descriptive phases, not a measured period, so no ladder test applies. Shannon's capacity and the radix remark of Foundations III.1 (cost r/ln r, least for r = 3 among integers) concern codes, not audiences; media do not code in base three. The agenda-setting relation is a meeting of two rankings, the media's and the public's, and its measure is a rank correlation; nothing in it is a window of three. T6, T11–T24 have no counterpart I could compute.
5. *Test.* No test specific to media is stated. The signed-triad test of Chapter 18 applies wherever a platform records positive and negative ratings between users, and it has already failed on Slashdot. A further test would use platforms that record likes and dislikes on content (person–person–item triads, Newcomb's original form); the prediction would be the same as in 18.1 and the same statistic would decide it.

Consequences. Media research shares its structural models with sociology and network science, and the Foundations add nothing separate. The likes and dislikes of platforms are a natural trit, and much of the debate about removing public dislike counts or adding "neutral" reactions is a debate about which values the register should have. But the closure rule tested on platform data has failed, so it cannot tell designers which triads will be stable. The one service this chapter offers the field is a reproducible negative result on a news platform's own social graph. Agenda-setting, trust and the effect of algorithms lie outside the reading.

There is a small conceptual point the reading does make clear. Newcomb's ABX model and Heider's balance treat the person–person–object triad as the unit of communication, and both use the product rule. The Foundations offered a second rule, the sum mod 3, and the tests in Chapter 18 show that the product rule's weak form is the better description of signed social data. For communication research this supports the inherited model rather than replacing it. If a future test on person–person–item data, where the item is a news story and the signs are likes and dislikes, came out the other way, it would be worth a section of its own. Until then the grade stays C.

Grade. C; the signed-triad test of Chapter 18 applies to Slashdot and fails there.

Chapter 20 — The Cycles of Societies

This chapter reads the ring column of level L5: the cycles by which economies, regimes, epidemics and cities go round and come back. Two ladder tests were run. Measles in twenty English and Welsh towns from 1944 to 1964 shows a dominant period on the ladder $2^a \cdot 3^b$ in 18 of 20 towns, almost always the biennial cycle that epidemiologists already explain as a period doubling of the annual one; against a null that respects seasonal forcing the result is not significant. The 34 United States business cycles dated by the NBER since 1854 fall on the ladder no more often than random lengths would. Period doubling and the period-3 window in nonlinear cycle models are structures the field published independently; the long waves of Kondratiev and the generational cycles of politics carry no ladder. Grades: economic history and business cycles A for the ×2 cascade and period 3 in cycle models, B for the NBER test (run, refuted), C for long waves; political cycles C, with a B test stated for Ibn Khaldun's three generations; epidemics in history A for annual-to-biennial doubling, B for the ladder test (run, weak); urbanisation C.

20.1 Economic history and business cycles

History. Clément Juglar's *Des crises commerciales et de leur retour périodique* (1862) was the first study to treat commercial crises as recurring phases of one process rather than as separate accidents. He found intervals of about seven to eleven years in French, British and American data on prices, interest and bank balances. William Stanley Jevons tried in the 1870s to link crises to the sunspot cycle through harvests. Joseph Kitchin found a shorter wave of about three to four years in 1923, which later writers attributed to inventories. Nikolai Kondratiev published *The Long Waves in Economic Life* in 1926, arguing from prices and interest rates for waves of roughly half a century (Wikipedia, Kondratiev wave). Simon Kuznets described swings of fifteen to twenty-five years in building and population growth in 1930. Joseph Schumpeter's *Business Cycles* (1939) combined the three shorter rhythms into one scheme and proposed the name "Kondratieff waves" (same page).

Measurement became an institution. Wesley Clair Mitchell founded the National Bureau of Economic Research in 1920, and his *Measuring Business Cycles* with Arthur Burns (1946) defined the method still used: turning points in many series, judged together, mark peaks and troughs. Theory went another way. Paul Samuelson's multiplier–accelerator model (1939) showed that two simple lags can produce oscillation. Richard Goodwin's growth cycle (1967) borrowed the predator–prey equations of Lotka and Volterra for wages and employment. From the 1970s Robert Lucas recast cycles as responses of rational agents to shocks, and Finn Kydland and Edward Prescott (1982) built real business cycle models in which technology shocks drive fluctuations. Hyman Minsky argued that stability itself breeds the debt that ends it. Milton Friedman and Anna Schwartz's *Monetary History of the United States* (1963) traced the cycles of a century to changes in the money stock, and their reading of the

Great Depression as a monetary collapse shaped how central banks responded to the crisis of 2008.

Chaos theory gave cycles a new mathematics. Tien-Yien Li and James Yorke proved in 1975 that a continuous map of an interval with a cycle of period 3 has cycles of every period, and coined “chaos” for the result. Robert May’s review in *Nature* (1976) showed how the simple logistic map passes from a fixed point to a cycle of period 2, then 4, 8, 16 and on into chaos, with windows of period 3 and others inside the chaotic range. Economists used these results to show that deterministic models can produce irregular cycles without any shocks at all. Economic history meanwhile became quantitative: Robert Fogel and Douglass North received the 1993 prize for cliometrics, and Angus Maddison reconstructed national incomes over two millennia. Carmen Reinhart and Kenneth Rogoff’s *This Time Is Different* (2009) collected eight centuries of financial crises. The questions left open are whether cycles have fixed periods at all, whether long waves exist, and how much of observed fluctuation is shock and how much internal dynamics.

Current state. The NBER’s Business Cycle Dating Committee lists 34 complete cycles in the United States from the trough of December 1854 to the trough of April 2020 (NBER). The last recession was the shortest on record: two months, from the peak of February 2020 to the trough of April 2020, after an expansion of 128 months, the longest on record. Over 1854–2020 contractions averaged 17.0 months and expansions 41.4, with an average cycle of 58.4 months from trough to trough. Cycles have lengthened: from 48.2 months on average in 1854–1919 to 74.5 months in 1945–2020, with contractions shortening from 21.6 to 10.3 months (same source; Calculation 20.2). The long postwar expansions of 106 months (1961–1969), 92 (1982–1990), 120 (1991–2001) and 128 (2009–2020) dominate the change.

Long waves are not accepted by most academic economists. Among those who use them there is no agreed mechanism and no agreed set of dates, and critics regard them as pattern-seeking (Wikipedia, Kondratiev wave). Andrey Korotayev and Sergey Tsirel (2010) reported spectral evidence for Kondratiev waves in world GDP and described the Kuznets swing as the third harmonic of the Kondratiev: three Kuznets swings to one long wave (same page; Wikipedia, Business cycle). The standard ranges are 3–5 years for Kitchin, 7–11 for Juglar, 15–25 for Kuznets and 45–60 for Kondratiev (same page). These ranges allow 1.4 to 3.7 Kitchins per Juglar and 1.8 to 4.0 Kuznets swings per Kondratiev (Calculation 20.3).

The current frontier is the measurement of the global financial cycle, the joint movement of credit and asset prices across countries, which recurs more slowly than the business cycle; the role of energy shocks, which the IMF’s October 2026 outlook names as the main drag on growth (Chapter 19); and the use of text and high-frequency data to date turning points in real time. The prize of 2026, for the microeconomic foundations of stabilisation, rewards the models that central banks use to damp these cycles.

Interpretation.

1. *Strand.* I is calendar time in months; P is the set of phases of activity. An economy returning to a peak or trough is a loop; the NBER chronology is a list of its passages.
2. *Passage and trit.* Each quarter output rises (1), falls (T) or holds within a band (0). A cycle from trough to trough is a word that returns to its start.
3. *Closure.* A cycle closes when the economy returns to the same phase. The cobweb of Chapter 19 is a meeting (period 2); a cycle of three phases with equal steps is a ring with carry ±1 (T7.1). In the logistic map the cascade 1, 2, 4, 8, 16 is the meeting ladder of T15 with b = 0, and the window of period 3 and its doubling to 6 are on the ladder too. But the same map has windows of period 5 and 7 (Calculation 20.6), which are off the ladder. So the ladder is a feature of the doubling cascade, not a law of cycles in general. Li and Yorke's theorem gives period 3 a special place: it implies every other period. In Sharkovskii's ordering, which places 3 first and the powers of 2 last, the two ends are exactly the ring and the meeting ladder.
4. *Scale.* If cycles merged by T15, long cycles would be $2^a \cdot 3^b$ multiples of short ones. The conventional ranges are too wide to check this, and their ratios include 2, 3 and 4 alike. The Korotayev–Tsirel "third harmonic" is a ×3 claim, but it rests on contested spectral evidence. T6 would give a baseline for closed windows of quarterly trits, as in Chapter 19; at monthly resolution such windows were explained by serial correlation. T11 reads the junction of two different regimes (inflationary and disinflationary) as a seam that carries tension. T12–T14, T16–T22 and T23–T24 have no counterpart I could compute.
5. *Test.* Stated before the NBER table was opened: cycle lengths, in years, fall within 10 % of a ladder value $2^a \cdot 3^b$ more often than lengths drawn uniformly over the same range. The reading fails if they do not.

The test was run on all 34 cycles (Calculation 20.2). Twenty-three of 34 fall within 10 % of a ladder value; uniform lengths give 21.35 on average, and 23 or more occur with probability 0.35. The ladder adds nothing to the timing of business cycles. The lengthening after 1945, a ratio of 1.55 between the mean cycles of 1945–2020 and 1854–1919, is not a ladder step either; it is what one expects when contractions are cut short by policy and expansions run on.

Consequences. Economic history gains no periods from the Foundations. The measured cycles are irregular, they lengthened after 1945, and their lengths are not drawn to $2^a \cdot 3^b$. What remains is a structural observation that the field itself published: the simplest nonlinear models of cycles pass from period 1 to 2, 4, 8 and on, and contain a window of period 3 that implies all others. The Foundations' two elementary closures, meeting and ring, are the two ends of Sharkovskii's ordering. That is grade A as structure, but it belongs to one-dimensional dynamics in general, not to economies in particular, and it predicts nothing about when the next recession will come.

The long-wave literature shows the risk this book warns against. The ranges are wide enough to contain any ratio, and a claim of three Kuznets swings per Kondratiev could be read as the ring ladder only if the waves were measured independently and

precisely, which they are not. Reading them as rungs of the ladder would be numerology. The costs of the reading are therefore small and the gains small. A better use of the vocabulary is negative: it makes clear that a period-doubling cascade is a sequence of meetings, and that a real economy, driven by many shocks, is unlikely to stay on such a cascade long enough for the ladder to show.

Grade. A for the ×2 cascade and the period-3 window in nonlinear cycle models (Li and Yorke 1975; May 1976); B for NBER cycle lengths on the ladder, run and refuted (p = 0.35); C for long waves.

20.2 Political cycles

History. The idea that regimes follow each other in a fixed order is old. Plato's *Republic*, Book VIII, describes a descent from the rule of the best through timocracy, oligarchy and democracy to tyranny. Polybius, writing about 150–120 BCE in Book VI of his *Histories*, made the sequence a cycle, later called anacyclosis. Rule by one turns from kingship to tyranny; it is overthrown by the few, whose aristocracy decays into oligarchy; the many overthrow them, and democracy decays into mob rule, from which a strong man emerges to begin again (Wikipedia, Anacyclosis). Polybius thought Rome escaped the cycle by mixing the three forms in consuls, senate and assemblies. Cicero took up the idea, and Machiavelli discussed it in Book I, Chapter 2 of his *Discourses on Livy* (same page).

Ibn Khaldun's *Muqaddimah* (1377) gave the cycle a social mechanism. Group solidarity, *asabiyyah*, lets a desert people conquer a settled state. Their descendants grow used to luxury and lose the solidarity that made them rulers. Ibn Khaldun held that dynasties have a natural lifespan like individuals and that a dynasty rarely lasts beyond three generations of about forty years each (Wikipedia, Asabiyyah). Giambattista Vico's *Scienza Nuova* (1725) described an age of gods, an age of heroes and an age of men, followed by a return, *ricorso*.

Modern versions are of two kinds. The first are short electoral cycles. Michał Kalecki argued in 1943 that governments would boost employment before elections and allow slumps after them. William Nordhaus formalised the political business cycle in 1975 in the *Review of Economic Studies*, and Douglas Hibbs and later Alberto Alesina proposed partisan cycles, in which parties of the left and right alternate in their priorities for unemployment and inflation (Wikipedia, Political business cycle). The second are long cycles of generations and of states. Arthur Schlesinger, father and son, described an alternation of public purpose and private interest in American politics. William Strauss and Neil Howe's *Generations* (1991) and *The Fourth Turning* (1997) proposed a cycle of four "turnings" of about twenty-one years each, a saeculum of about eighty-five years (Wikipedia, Strauss–Howe generational theory). Peter Turchin and Sergey Nefedov's *Secular Cycles* (2009) used demographic-structural theory to describe centuries-long waves of population growth, elite overproduction and crisis. George Modelski's *Long Cycles in World Politics* (1987) proposed a cycle of roughly a century in which a leading sea power rises, is challenged and is replaced, each turn marked by a global war. In the same decade Paul Kennedy's *The Rise and Fall of the Great Powers* (1987) linked such turns to the strain between military commitments and economic

base, without claiming a fixed period. The questions left open are whether these cycles are real regularities or narrative shapes imposed on history, and whether any of them can be tested out of sample.

Current state. The long-cycle literature has made public predictions and is being judged on them. In a letter to *Nature* in February 2010, Turchin used about forty social indicators to predict that "the next decade is likely to be a period of growing instability in the United States and western Europe" (Wikipedia, Peter Turchin). Turchin is a founding director of the Seshat Global History Databank, which codes historical societies for comparative tests; a 2019 *Nature* paper based on Seshat was retracted in 2021 (same page). Strauss–Howe theory has had mixed reception: some historians praised its research, while critics compared it to astrology and called it non-falsifiable; Howe's *The Fourth Turning Is Here* appeared in 2023 (Wikipedia, Strauss–Howe).

Measured democracy has declined for more than a decade. The V-Dem *Democracy Report 2026* finds the level of democracy enjoyed by the average person back to 1978 on a population-weighted measure and back to 1995 on a country average; there were 31 liberal democracies at the end of 2025, down from a peak of 45 in 2009 (V-Dem report). Liberal democracies thus fell by 31 % in sixteen years (Calculation 19.8). Whether this is a phase of a cycle or a trend is the main debate in comparative politics. Turchin's book *End Times* (2023) presented the instability of the early 2020s as the crisis phase of a secular cycle driven by elite overproduction and stagnant wages; critics answer that the same events can be explained without a cycle, and that a theory which counts both upturns and downturns as confirmation cannot fail.

Electoral calendars are the one place where political cycles are fixed by law. The United States Senate is divided into three classes; Class 2, with 33 seats, was last elected in 2020 and is up in 2026, at the general election of 3 November, a week after this chapter's date (Wikipedia, Classes of United States senators). Evidence for opportunistic political business cycles in developed democracies is mixed, and much recent work finds cycles in fiscal instruments rather than in output.

Interpretation.

1. *Strand.* I is historical time; P is the set of regime types (one, few, many; good or corrupt). A return to a form of rule a society has had before is a loop.
2. *Passage and trit.* For Polybius each step is a change in who rules (one → few → many → one), which is one direction round a cycle of three: a ring, written (1, 1, 1). Within each form, the turn from good to corrupt is a sign change, a meeting of the form with its own degeneration.
3. *Closure.* Polybius's cycle is three forms times two states, six stages (Calculation 20.5), a ring of three with a meeting at each node: $2 \cdot 3$, a ladder number. By T7.1 the completed ring (1, 1, 1) closes with carry +1, which reads as Polybius's own claim that the cycle returns to monarchy at a new level. Ibn Khaldun's three generations are also a ring: founder, imitator, destroyer. The Strauss–Howe saeculum of four turnings is $2^2 \cdot 21$ years, the meeting ladder twice. The Senate's three classes make a six-year term a ring of three two-year passages. Nordhaus's electoral cycle is a meeting in time: expansion before the vote, contraction after it,

with the period of the term. It closes with carry 0 at every election, which is why it never accumulates into a longer wave.

4. *Scale.* Electoral cycles nest: in the United States a two-year House term, a four-year presidency and a six-year Senate term meet every 12 years, the least common multiple. Long cycles do not nest in measured data, because their dates are not agreed. T6, T11–T14, T16–T22 and T23–T24 have no counterpart I could compute. The counts 3, $2 \cdot 3$ and 2^2 in the classical schemes are on the ladder, but they are numbers of named stages chosen by authors, not measured periods, so they are not evidence.
5. *Test.* Ibn Khaldun's rule and the ring reading make the same prediction, which can be tested on any list of dynasties with genealogies: the number of generations a dynasty holds power clusters at three, with fewer at two and four. The reading fails if the distribution peaks at two or four, or has no peak. Statistic: the modal number of ruling generations in a census of dynasties, for example from the Seshat databank, with generations counted as father–son steps. The test was not run here. It needs only genealogies, which exist for most recorded dynasties, and a rule fixed in advance for counting a ruler's brother or cousin as the same generation.

Consequences. Political cycles are the clearest case in this chapter of a reading that adds nothing yet. The classical schemes use threes and sixes because their authors chose three forms of rule and two states of each, and the Foundations can describe that choice but not justify it. Ibn Khaldun's three generations is a real empirical claim. If a census of dynasties confirmed it, the ring reading would gain a historical instance of closure with carry, in which a dynasty completes its ring and passes power to a new group, the next scale. If the census showed no peak at three, both Ibn Khaldun and the reading would lose. Because the test cannot separate the two, a pass would be weak support for the Foundations. Electoral calendars are designed, and their ladder numbers say more about legislators' arithmetic than about society. Turchin's predictions are the serious frontier, and they rest on demographic and elite dynamics that the Foundations do not model.

There is a broader lesson for the use of the Foundations in history. A cycle with a fixed number of named stages is a word of fixed length. Polybius's six stages form one word; Strauss and Howe's four turnings another. T13 says that a word of n trits read from a start has 3^n forms and one read as a cycle without a start has g(n) forms. A historian who chooses where a cycle "begins" is choosing a start on a closed configuration, and many apparent disagreements between cyclical theories are disagreements about that start, not about the cycle. This is a useful caution, but it is not a test.

Grade. C; B for the test of Ibn Khaldun's three-generation rule (stated, not run).

20.3 Epidemics in history

History. Thucydides described the plague of Athens of 430 BCE, which killed Pericles, and noted that survivors were not attacked twice: the first record of acquired immunity. The plague of Justinian began in 541. The Black Death reached Europe in 1347 by way of Caffa and Constantinople and spread through the continent until 1353, killing between 30 % and 60 % of the population (Wikipedia, Black Death). Plague then

returned: Florence suffered outbreaks in 1347–48, 1362, 1371 and 1382, gaps of 9 to 14 years, and plague was present somewhere in Europe almost every year until 1671 (same page). Girolamo Fracastoro proposed in 1546 that diseases spread by seeds passed from person to person. The arrival of Europeans in the Americas after 1492 brought smallpox, measles and influenza to populations without immunity.

The nineteenth century turned epidemics into a science. Edward Jenner's cowpox vaccination (1796) began the control of smallpox. Cholera spread from Bengal in 1817 in the first of seven pandemics: 1817–1824, 1826–1837, 1846–1860, 1863–1875, 1881–1896, 1899–1923 and the seventh from 1961, which the WHO still counts as current (Wikipedia, Cholera outbreaks and pandemics). John Snow mapped the Broad Street outbreak in London in 1854. Robert Koch identified the tubercle bacillus in 1882 and the cholera vibrio in 1884.

Mathematics followed. William Hamer proposed in 1906 that the rate of new measles cases depends on the product of susceptible and infected people, which explained recurring epidemics. Ronald Ross modelled malaria from 1908. William Kermack and Anderson McKendrick's model of 1927 divided a population into susceptible, infected and removed, and showed that an epidemic grows only above a threshold. The influenza pandemic of 1918–1920 came in three main waves, infected about 500 million people, nearly a third of the world population, and killed an estimated 25 to 50 million (Wikipedia, Spanish flu). Maurice Bartlett showed in 1957 that measles dies out between epidemics in towns below a critical size of a few hundred thousand. Roy Anderson and Robert May's *Infectious Diseases of Humans* (1991) made the reproduction number the central quantity. Smallpox was declared eradicated in 1980, the first human disease to be ended deliberately. The following decades brought new pathogens rather than an end to epidemics: HIV was recognised in 1981, SARS spread in 2003 and pandemic influenza returned in 2009, each prompting new rules for surveillance and reporting.

Measles became the model system for epidemic cycles because the weekly notifications of England and Wales from 1944 onward are complete and long. Large cities show regular epidemics every year or every two years. David Earn, Pejman Rohani, Benjamin Bolker and Bryan Grenfell (2000) showed that a single seasonally forced model explains the switches between annual, biennial and irregular cycles as the birth rate and vaccination change. Ancient DNA has since rewritten the history of plague: tooth samples confirmed *Yersinia pestis* in Black Death graves in 2010, and a 2022 study traced the ancestor of the Black Death strains to cemeteries near Lake Issyk-Kul in Kyrgyzstan, dated to 1338–1339 (Wikipedia, Black Death). The questions left open are how much of historical mortality the pathogen explains and how much the society, and why some pandemics come in waves.

Current state. Measles is resurgent. The WHO reports about 95,000 measles deaths in 2024, mostly among children under five; 84 % of children received a first dose in 2025, below the 86 % of 2019, and 77 % both doses. Vaccination averted nearly 59 million deaths between 2000 and 2024 (WHO measles fact sheet). With a basic reproduction number of 12 to 18, measles needs about 95 % immunity to stop spreading (Wikipedia, Measles); the threshold $1 - 1/R_0$ is 0.917 to 0.944 (Calculation

20.4). Tuberculosis made 10.7 million people ill and killed 1.23 million in 2024 (WHO TB fact sheet).

COVID-19 is now a historical epidemic. The WHO estimated 14.9 million excess deaths associated with the pandemic in 2020 and 2021, with a range of 13.3 to 16.6 million (WHO). The World Health Assembly adopted the WHO Pandemic Agreement on 20 May 2025, with 124 members in favour and 11 abstaining; it needs 60 ratifications, and its annex on pathogen access and benefit sharing must be finalised before it opens for signature (Wikipedia, WHO Pandemic Agreement).

The historical measles data are public. The R package tsiR, from the group that developed time-series SIR models for measles, ships biweekly case reports, births and populations for twenty towns of England and Wales from 1944 to 1964, from London (3.4 million) to Halesworth (2,170). These data are used here (Calculation 20.1). The frontier joins genomics, mobility data and historical records to estimate how pathogens spread before modern surveillance, and asks how vaccine hesitancy, which now drives measles outbreaks, changes the cycles that vaccination had suppressed.

Interpretation.

1. *Strand.* I is time in biweekly steps; P is the set of epidemic states, measured by the share susceptible. A return of the epidemic peak is a passage.
2. *Passage and trit.* Each season's peak is larger (1), smaller (T) or equal within a band (0) to the last. An alternation of large and small years is the meeting (1, T) repeated.
3. *Closure.* An epidemic cycle closes when the population returns to the same state. The annual cycle is one passage per year; the biennial cycle, in which a large epidemic depletes susceptibles so that the next year is small, closes after two years, a meeting of a large and a small year with carry 0 (T7.1). The switch from annual to biennial as birth rates fall is a period doubling, the first step of the meeting ladder.
4. *Scale.* By T15 the periods of recurrent epidemics should be $2^a \cdot 3^b$ years when seasonal forcing sets the unit. The SIR threshold $1 - 1/R_0$ is a standing condition, not a window. Cholera pandemics began at intervals of 9, 20, 17, 18, 18 and 62 years, three of six on the ladder (Calculation 20.4); with so few events this is not evidence either way. The same holds for counts of waves: the influenza pandemic of 1918–1920 had three main waves and a fourth in some regions, and a count of three or four waves is set by the timing of school terms, troop movements and weather, not by a ladder. T6, T11–T14, T16–T22 and T23–T24 have no counterpart here I could compute.
5. *Test.* Stated in the first edition of this book, before the data were opened: the dominant periods of recurrent epidemics lie on $2^a \cdot 3^b$ years; a stable dominant period of 5 or 7 years counts against. Statistic: the period of the highest spectral peak between 0.67 and 8 years in the log case counts of each of the 20 towns, scored within ±15 % of a ladder value; nulls: log-uniform periods in the same band, and whole-year periods 1 to 8.

The test was run (Calculation 20.1). Eighteen of 20 towns have a dominant period on the ladder: 16 near 2 years, Liverpool near 1 year and Halesworth near 4 years (3.65). Two small towns are off: Dalton-in-Furness (1.62 years) and Mold (2.40 years). Against

the log-uniform null, 18 or more of 20 has probability 0.016; against the whole-year null, where 6 of 8 periods are on the ladder, it has probability 0.091.

Consequences. The measles result is a pass, but a weak one. It confirms what Earn and colleagues explained in 2000: in the era of these data, birth rates put large English towns in the biennial regime, a period doubling of the seasonal forcing. The off-ladder towns are small, below Bartlett's critical community size, where measles fades out and returns irregularly. Against a null that already knows the forcing is annual, the ladder is not significant. So the reading gains no independent support, but it is consistent with the best data in the field, and it states a sharper version of the field's own expectation: periods of 5 or 7 years should not appear as stable dominant cycles in forced childhood diseases. Data on pertussis and rubella, which are also forced by the school year, would extend the test. Each could fail it. So could the post-vaccination era: as vaccination lowers the effective birth of susceptibles, the forced models predict longer and less regular inter-epidemic periods, and the reading predicts that any stable period that emerges is again a ladder value, 3 or 4 years rather than 5.

The cost of the reading is that it must explain away non-ladder periods in unforced or weakly forced diseases, such as the multi-year cycles reported for some infections without a school-year rhythm. If those periods are stable and off the ladder, the reading should be confined to diseases with strong annual forcing, which is a large retreat.

Grade. A for the annual-to-biennial switch as a period doubling (Earn et al. 2000); B for the ladder test, run on twenty towns: 18 of 20 on the ladder, significant against a log-uniform null ($p = 0.016$) but not against a whole-year null ($p = 0.091$).

20.4 Urbanisation

History. Settlements of thousands appeared in the Neolithic. Çatalhöyük in Anatolia housed several thousand people around 7000 BCE in houses entered from the roof. True cities, with temples, writing and specialised crafts, arose in southern Mesopotamia in the fourth millennium BCE; Uruk was the largest. Gordon Childe called the change the "urban revolution" in 1950 and listed the traits that separate a city from a village: size, density, full-time specialists, monumental building, writing and a surplus collected by an authority. Planned grids appeared early: the cities of the Indus valley were laid out on near-orthogonal streets in the third millennium BCE, and the Greek planner Hippodamus of Miletus gave his name to the gridded town in the fifth century BCE. Rome is usually estimated at about a million inhabitants under the early empire, and Chang'an, Baghdad and Constantinople were the great cities of the first millennium CE.

For most of history urban life was a small share and a deadly one: cities grew by migration because their death rates exceeded their birth rates. Industrialisation changed both facts. London had about a million people at its first census in 1801 and more than six million by 1900. Ebenezer Howard's *To-morrow* (1898, reissued as *Garden Cities of To-morrow* in 1902) proposed clusters of planned towns linked to a central city by rail, each surrounded by a green belt. Felix Auerbach observed in 1913 that city populations are roughly inversely proportional to rank, and George Kingsley

Zipf made the rank-size rule famous in 1949 (Wikipedia, Zipf's law). The Chicago school described the city as concentric zones, and Christaller described systems of cities as nested hexagons (Chapter 18).

After the Second World War urbanisation moved south. Latin America became mostly urban by the 1970s; Asia and Africa followed more slowly, with megacities forming around capitals and ports. Jane Jacobs's *The Death and Life of Great American Cities* (1961) defended dense mixed districts against planned renewal. In the same year Lewis Mumford's *The City in History* traced the city from shrine and citadel to metropolis and warned against unlimited sprawl. Constantinos Doxiadis's ekistics, a science of human settlements, organised settlements in a ladder of units from the single person to the whole inhabited earth, and Paul Bairoch's *Cities and Economic Development* (1988) quantified the urban share of past societies over the centuries before industrialisation. Xavier Gabaix explained Zipf's law in 1999 as the result of growth rates independent of size. Luís Bettencourt, José Lobo, Dirk Helbing, Christian Kühnert and Geoffrey West (2007) found that socio-economic outputs such as wages and patents scale more than proportionally with city population, and infrastructure such as petrol stations less than proportionally (Wikipedia, Urban scaling). The questions left open are what makes cities productive, whether their scaling laws are universal, and how to define a city at all. A fourth question is whether urbanisation has an end point, a share beyond which it stops, or whether the distinction between city and country itself dissolves.

Current state. The definition question has been settled for statistics. The UN's *World Urbanization Prospects 2025* applies a harmonised "degree of urbanisation" to 237 countries and areas from 1950 to 2025 with projections to 2050 (UN DESA). It finds 45 % of the world's 8.2 billion people in cities in 2025, more than double the share of 1950. The number of megacities has risen from 8 in 1975 to 33, and is projected to reach 37 by 2050. The number of cities could exceed 15,000 by 2050, most with fewer than 250,000 people; two-thirds of population growth to 2050 is expected in cities and most of the rest in towns, and the number of countries where rural areas are the most common settlement type is projected to fall from 62 to 44 (UN DESA press release). The urban population grew from about 0.5 billion in 1950 to about 3.7 billion in 2025, 2.7 % a year; megacities multiplied at 2.9 % a year from 1975 to 2025 but are projected to grow in number at only 0.5 % a year to 2050 (Calculation 20.5). Nineteen of the 33 megacities are in Asia, and Cairo is the only city outside Asia among the ten largest; Jakarta, with nearly 42 million, now leads Dhaka and Tokyo (UN DESA press release).

The scaling debate continues. Elsa Arcaute and colleagues (2015) showed that urban scaling exponents change when the same cities are bounded differently, and José Leitão and colleagues (2016) that several apparent non-linear scalings are not robust to better statistical models (Wikipedia, Urban scaling). Zipf's law for cities has also been challenged by recent empirical and theoretical work (Wikipedia, Zipf's law). The frontier lies in satellite-based measurement of built-up areas, in the study of informal settlements, which house a large share of urban growth in Africa and South Asia, and

in the climate exposure of coastal megacities such as Jakarta, Dhaka and Tokyo, the three largest.

Interpretation.

1. *Strand.* I is the history of a settlement system in years; P is the set of places. Migration into a city and back is a meeting; the daily commute is a strand that returns to the same two nodes every day.
2. *Passage and trit.* A city's population each decade grows faster than the system (1), slower (T) or at the same rate within a band (0). A city's rank, moving up or down, carries the same trit.
3. *Closure.* A settlement system closes when its rank-size distribution returns to the same shape after growth, which is the stationarity that Gabaix's explanation requires. I find no window of three in the data on urbanisation, and no measured quantity whose closure mod 3 could be tested.
4. *Scale.* From 1950 to 2025 the urban population went through 2.88 doublings in 75 years (Calculation 20.5); that is not a ladder count, because doublings of a growing quantity are not merges of blocks. The nesting of central places, where T15 does apply, is treated in Chapter 18. The urban scaling exponents depend on how cities are bounded, which is a case of reading N4: a response that is a state function of context, with universality only as a limiting case (T23 hypothesis). But the Foundations give no value for the exponent, so this is a reading. T11 has a natural place at the urban edge, where a city's closed pattern of streets and services meets the rural pattern of fields and roads: a seam between two different closed configurations, which by T11 always carries tension. Planners know this zone as the fringe, and it is where land conflicts concentrate. But the theorem gives no quantity to measure there. T6 would supply a baseline if city growth were recorded as trits, faster, slower or equal to the system each decade: a run of three decades would close by chance with probability one third under equal frequencies. A city whose growth closes more often than that would be one that returns to its rank, which Gabaix's model of growth independent of size does not predict. This is a possible test, but it would need long series for many cities with fixed boundaries, which the scaling debate shows are hard to define. T6–T14, T16–T22 and T24 have no counterpart here I could compute.
5. *Test.* None specific. A test would need a quantity in which the reading predicts a ladder value or a closure that standard urban economics does not, and I have not found one.

Consequences. Urbanisation is the largest social change of the last two centuries, and the Foundations say nothing specific about it. The ring column asks for cycles, and the process is not cyclical at the scale of the world: the urban share has risen without reversal since 1950. Within cities there are cycles (of building, of neighbourhood decline and renewal, of commuting), but none has been measured in a way that a ladder test could use. The one place where the theory touches cities, the hexagonal nesting of market areas, is a standing structure and belongs to Chapter 18. The honest result for this discipline is that the reading adds nothing. The cost of pretending

otherwise would be to treat the threes and sixes of planning diagrams as evidence, which they are not.

One direction could change this grade. If cities are standing units in the sense of T7, their internal structure should show the elementary closures: districts tied in pairs by commuting (meetings) and rings of three or more districts tied by circulation. Origin–destination data from transit cards and mobile phones now record such flows for whole metropolitan areas. A test would ask whether three-district cycles of net flow occur more often than chance, given the pairwise flows, and whether their sign pattern follows the closure rule. Such data are held by transport agencies and phone operators, often under restrictions, and the test was not run here. Until it is, urbanisation remains a domain the theory describes in words but does not touch with numbers. Stating the test now, before any such data are seen, is the most this chapter can do for the discipline.

Grade. C.

Chapter 21 — Ordering Society

This chapter reads the tension column of level L5: the practical disciplines that take what does not close in a society and make it close, by rule, by teaching, by prevention, by administration and by management. One structure here is exact and published independently of the Foundations. In deontic logic every act is obligatory, optional or forbidden, and the status of the negated act is the mirror of the status of the act (von Wright 1951): the trit and its mirror of T12.1. Elsewhere the counts that the disciplines use (twelve jurors, three branches, four steps of a management cycle, nine viewpoints of causation) fall on the ladder $2^a \cdot 3^b$ no more often than small integers do by chance. Grades: law A for deontic status, C for verdicts and juries; education C, with a B test stated for spacing schedules; public health and epidemiology C, B where the measles test of Chapter 20 applies; public administration C, with a B test stated for spans of control; management C.

21.1 Law

History. The Code of Hammurabi, inscribed around 1754 BCE, set out rules and penalties in the form “if a man does this, then that”. Roman law built a professional science of rights, contracts and procedure, which Justinian’s commissioners compiled as the *Corpus Juris Civilis* between 529 and 534. Its rediscovery at Bologna in the late eleventh century founded the European law schools. In England the common law grew from royal courts and precedent; Magna Carta (1215) bound the king to the law of the land. Hugo Grotius’s *De jure belli ac pacis* (1625) founded international law on natural reason. William Blackstone’s *Commentaries* (1765–1769) organised the common law for teaching, and the Napoleonic Code of 1804 made codification the model for much of the world. In *Marbury v. Madison* (1803) the United States Supreme Court claimed the power to strike down laws that conflict with the Constitution.

Legal theory turned analytical in the nineteenth century. Jeremy Bentham and John Austin defined law as the command of a sovereign backed by sanctions. Wesley Newcomb Hohfeld’s two articles in the *Yale Law Journal* (1913 and 1917) broke the word “right” into eight jural relations in two groups of four: right, duty, privilege and no-right; power, liability, immunity and disability. Each term has a correlative held by the other party (a right is matched by a duty) and an opposite held by the same party (the opposite of a right is a no-right) (Wikipedia, Hohfeldian analysis). Hans Kelsen’s *Pure Theory of Law* (1934) described a legal order as a hierarchy of norms, each valid because a higher norm authorises it. H. L. A. Hart’s *The Concept of Law* (1961) distinguished primary rules of conduct from secondary rules that identify, change and apply them.

Logic entered law directly. Georg Henrik von Wright’s paper “Deontic Logic” in *Mind* (1951) treated “obligatory”, “permitted” and “forbidden” as operators like “necessary” and “possible”. An act is forbidden when its omission is obligatory, and permitted when it is not forbidden. Ernst Mally had attempted a logic of “ought” in 1926, and Jan

Łukasiewicz had built a three-valued logic in 1920, with a middle value between true and false for statements about the undetermined future; its negation sends a value x to 1 − x. Carlos Alchourrón and Eugenio Bulygin's *Normative Systems* (1971) treated a legal code as a set of norms whose completeness and consistency can be checked case by case. Ronald Dworkin (1977) argued against Hart that principles, not only rules, are part of the law. The twentieth century also built the international order: the United Nations Charter and the International Court of Justice (1945), the European Convention on Human Rights (1950), and the treaties of the European Union. The questions left open are how law should handle new technologies, how international law binds states that have not consented, and whether legal reasoning can be made formal enough for machines.

Current state. The regulation of artificial intelligence is the main new field of law. The European Union's AI Act entered into force on 1 August 2024 with four levels of risk and a separate regime for general-purpose models (Wikipedia, Artificial Intelligence Act). In 2026 the "AI Omnibus" moved the obligations for high-risk systems. The European Parliament adopted it on 16 June 2026, the Council on 29 June, and it was published in the Official Journal on 24 July 2026. Obligations for high-risk systems listed in Annex III now apply from 2 December 2027 instead of 2 August 2026, and those under Annex I from 2 August 2028 instead of 2 August 2027 (Future of Privacy Forum). The delays are 487 and 366 days (Calculation 21.5).

International courts have taken up climate change. On 23 July 2025 the International Court of Justice delivered its advisory opinion on the obligations of states in respect of climate change, in answer to a request from the UN General Assembly (ICJ). The opinion was accompanied by separate opinions and declarations of several judges (same page).

Criminal procedure has changed in Scotland. The Victims, Witnesses, and Justice Reform (Scotland) Act 2025 received Royal Assent on 30 October 2025. It removed the "not proven" verdict for trials beginning on or after 1 January 2026, leaving guilty and not guilty, and raised the majority needed to convict to two-thirds of the jury (Wikipedia, Not proven). Under the old rule a Scottish jury of 15 convicted by a simple majority of 8. As of 1999, about one third of jury acquittals were "not proven" verdicts (same page). Jury sizes across systems vary: twelve in England and Wales, Canada, Australia and Belgium, fifteen in Scotland, seven in Brazil, eight lay jurors with three judges in serious Austrian cases, and a civil minimum of six in the United States (Wikipedia, Jury).

Interpretation.

1. *Strand.* I is the sequence of acts in a legal order; P is the set of act types. An act type that recurs and is judged again is a loop.
2. *Passage and trit.* Each act carries one normative status: obligatory (1), optional (0) or forbidden (T). This is the deontic reading of the trinity of T1. It is not an analogy: von Wright's three statuses are exhaustive and exclusive, and they behave as the digits T, 0, 1.

3. *Closure.* The status of not-p is the mirror of the status of p: if p is obligatory, not-p is forbidden; if p is optional, so is not-p (Calculation 21.1). This is the trit-wise mirror of Foundations T12.1, exactly. The pair (status of p, status of not-p) is always (x, −x): a meeting, which closes with carry 0 (T7.1). A normative system is consistent when no act carries two statuses; an inconsistent one has an act whose status and whose negation's status do not form a meeting, which is tension in the sense of reading N2. Hohfeld's scheme has a different shape. Correlative and opposite are two commuting involutions, so each square of four relations is the Klein four-group, 2 × 2, a pair of meetings with no ring (Calculation 21.2).
4. *Scale.* Norms merge into codes and codes into legal orders, which is Kelsen's hierarchy; no block size is fixed, so T15 does not apply. Jury sizes are on the ladder in 6 of 9 systems listed (6, 8, 12), off in 3 (7, 10, 15); among the sizes 6 to 15, four in ten are ladder numbers, and 6 or more of 9 occur by chance with probability 0.099 (Calculation 21.3). Scotland's change from three verdicts to two removes the 0 from the register: a trit becomes a bit. Łukasiewicz's three-valued logic gives the same mirror. Shifted from {0, ½, 1} to {T, 0, 1}, its negation 1 − x becomes x → −x, and its conjunction and disjunction (minimum and maximum) are unchanged (Calculation 21.9). So the mirror negation of T12.1 is shared by the two oldest three-valued systems used in legal reasoning. T6, T8–T11, T13–T14 and T16–T24 have no counterpart I could compute.
5. *Test.* For the algebra of status no test is needed; it is an identity, published by von Wright. For practice, the Scottish change is a natural experiment, but the reading gives no sign for its effect: the two-thirds rule pushes towards acquittal at the same time as the loss of the middle verdict pushes some former "not proven" cases towards conviction. I state no test that could separate the two.

Consequences. The deontic match is the strongest structural link in this chapter and one of the clearest in the book. Legal and deontic logic already treat normative status as a three-valued quantity with a negation that mirrors it, and the Foundations' balanced trit is that quantity, written with digits that need no sign convention (Foundations III.2). What the reading adds is a place for consistency: a consistent normative system is one in which every act and its negation form a meeting. That is close to existing theory and adds little to it. It does suggest one exercise for legal informatics, where rules are now written as code: storing deontic status as a balanced trit makes negation a single operation and makes inconsistency visible as a nonzero digit. That is a design choice, not a discovery.

The deontic match should not be overstated. Von Wright's system uses the trit as a set of three statuses, but legal orders are not closed under the window rule: three norms whose statuses sum to zero mod 3 have no special standing in any jurisdiction. Where the Foundations would say "closed", law says nothing, and where law says "consistent", the Foundations' meeting rule is a restatement. The match is at the level of the register (three values and a mirror), not at the level of windows and carries. That is still grade A by this book's definition, because the structure is in the field's own published results, but it is a narrow A.

Jury sizes and verdict counts show the limit. Twelve is a ladder number, but so are most small numbers, and the test against chance gives nothing. The Scottish reform will be studied by criminologists with conviction data from 2026 onward. The Foundations cannot predict its sign, and they should not be credited with whatever the data show.

Grade. A for deontic status as trit with mirror negation (von Wright 1951); C for Hohfeld's relations, verdict counts and jury sizes.

21.2 Education

History. Plato's Academy and Aristotle's Lyceum taught by dialogue and lecture. Late antiquity fixed the seven liberal arts, which the medieval schools divided into the trivium of grammar, rhetoric and logic, and the quadrivium of arithmetic, geometry, music and astronomy. Universities arose at Bologna, Paris and Oxford around 1100–1200. Jan Amos Comenius's *Didactica Magna* (1657) proposed teaching "all things to all people" in graded classes with illustrated books. Prussia made schooling compulsory in 1763, and Wilhelm von Humboldt's university in Berlin (1810) joined research to teaching. Horace Mann's campaign for common schools in Massachusetts in the 1830s and 1840s set the American pattern of free, tax-supported public education. Johann Heinrich Pestalozzi taught through things before words, and Friedrich Fröbel opened the first kindergarten in 1837.

The science of learning began in the laboratory. Hermann Ebbinghaus's *Über das Gedächtnis* (1885) measured his own forgetting of nonsense syllables and found that spaced repetitions retain more than massed ones (Wikipedia, Spacing effect). Maria Montessori opened her first Casa dei Bambini in Rome in 1907. John Dewey's *Democracy and Education* (1916) argued that schools should be communities in which children learn by doing. Jean Piaget described four stages of cognitive development, and Lev Vygotsky the "zone of proximal development", the range of tasks a learner can do with help but not yet alone. Benjamin Bloom's taxonomy (1956) ordered educational objectives in six levels from knowledge to evaluation.

After 1960 education became a subject of large-scale measurement. The Coleman Report (1966) found that differences between schools explained less of student achievement than family background. The International Association for the Evaluation of Educational Achievement began comparative tests in the 1960s, and the OECD's Programme for International Student Assessment (PISA) started in 2000, testing 15-year-olds every three years in reading, mathematics and science (Wikipedia, PISA). John Hattie's *Visible Learning* (2009) synthesised more than 800 meta-analyses. In 2015 the United Nations adopted quality education for all as the fourth of its Sustainable Development Goals, with targets for 2030. In Europe the Bologna Declaration of 1999 began the harmonisation of higher education into a common structure of three cycles, bachelor, master and doctorate, so that a student's degree from one country is readable in another. Cognitive psychology confirmed a few strong effects: retrieval practice, spacing and interleaving. B. F. Skinner built teaching machines in the 1950s that presented material in small steps with immediate feedback. Paul Pimsleur proposed in 1967 a "graduated interval recall" schedule for language learning in which each interval is about five times the one before. Sebastian Leitner's

card box (1972) moved each card to a box reviewed less often after a correct answer and back to the first box after an error, and Piotr Woźniak's SuperMemo algorithm (1987) computed an interval for each item from its record of recall. Benjamin Bloom's "two sigma problem" (1984) reported that one-to-one tutoring raised average achievement by about two standard deviations over classroom teaching, and asked how group instruction could approach it. Nicholas Cepeda and colleagues' review (2006) found spaced practice better than massed practice in 259 of 271 comparisons, and Harry Bahrick's nine-year study (1993) found that 13 sessions of foreign vocabulary 56 days apart retained as much as 26 sessions 14 days apart (Wikipedia, Spacing effect). The questions left open are why such robust findings are so little used in schools, how far achievement gaps can be closed by schooling, and what technology does to learning.

Current state. PISA 2025 results were published in September 2026. More than 760,000 students took part, representing about 33 million 15-year-olds in 91 countries and economies, with science as the main subject. The Chinese provinces and municipalities of Beijing, Shanghai, Jiangsu and Zhejiang had the highest mean scores in mathematics and science, and Singapore the highest in reading (Wikipedia, PISA). That is about 8,400 students per participating system (Calculation 21.5). The previous round, PISA 2022, recorded the largest falls in the programme's history: the OECD average fell 17 points in mathematics, from 489 to 472, and 11 points in reading, from 487 to 476, among about 690,000 students in 81 countries and economies (same page). The 2022 round had been postponed a year by the pandemic, so the rounds since 2000 came at intervals of three years except one of four (Calculation 21.5).

The research front has three parts. The first is recovery from the learning losses of 2020–2022 and the debate over whether declines began before the pandemic, with phones and social media as one proposed cause. Several countries have restricted phones in schools; the evidence on their effect is still being gathered. The second is generative AI, which can tutor, write and grade, and whose effect on learning depends on whether students use it to practise retrieval or to avoid it. The third is replication: education research shares the replication problems of psychology, and the SCORE programme's finding that about half of social-science claims replicate, with effects roughly halved (COS SCORE), applies to it. Spaced practice is one of the few effects that has replicated many times.

Interpretation.

1. *Strand.* I is a learner's sequence of encounters with material; P is the set of topics or items. A return to an item is a loop.
2. *Passage and trit.* At each return the learner recalls the item better than at the first encounter (1), worse (T) or the same (0). Spaced repetition is a schedule of passages.
3. *Closure.* A topic is learned when its returns stop producing change, three successive zeros or a balanced run of gains and losses: a closed window (T2, T4). An open window, two gains and a loss for example, keeps a nonzero digit: the item is not yet stable. This reading does not change how retention is measured.

4. *Scale.* The ratio of successive gaps in an expanding schedule is a ladder candidate. Bahrick's 56-day spacing is four times his 14-day spacing, 2^2; the PISA cycle is 3 years. The named counts of the field are mixed: trivium 3 and quadrivium 4 are on the ladder, the seven liberal arts are not; Bloom's six levels and Piaget's four stages are on it. Among 22 named counts in this chapter's disciplines, 15 are on the ladder, which chance would give with probability 0.72 (Calculation 21.8). These counts are chosen by authors and are not evidence. Spacing schedules in use are not on one ladder either: Pimsleur's ratio of about five is off it, and algorithmic schedules adapt the ratio to each item. The Bologna structure of three cycles is a ring only in name: a student who completes the doctorate does not return to the bachelor's level, so the sequence is a path, not a closed word. Lifelong learning, in which adults return to study, would make it closer to a ring, but no data I know of measure such returns as a cycle. T6, T11–T14 and T16–T24 have no counterpart I could compute.
5. *Test.* Stated here before any data: in expanding retrieval schedules of equal total study time and equal final retention interval, schedules whose gap ratio is on the ladder (×2 or ×3) produce better final recall than schedules with off-ladder ratios (for example ×2.5). The reading fails if retention depends smoothly on the ratio, with no advantage at 2 or 3. The test could be run in an existing online learning platform with randomised schedules. It was not run here.

Consequences. Education gains little from the Foundations, and it is better to say so. The spacing effect is a closure phenomenon in the plain sense that memory stabilises through returns, but the field already describes it with forgetting curves and retrieval strength, and those descriptions predict the size of the effect. The ladder test is a genuine test, because the learning literature expects smooth dependence on gap ratios and the reading expects advantages at 2 and 3. I expect it to fail. If it passed, it would be a surprising and useful result for anyone who designs a study schedule. The named counts of curricula and taxonomies are conventions, and the theory should not be credited with them. The major problems of the field, unequal achievement and the slow uptake of what works, lie outside the reading.

The test is cheap, and that is its main merit. Digital flash-card systems already randomise schedules for millions of learners and record recall at each review. A platform could assign learners at random to fixed expansion ratios of 2, 2.5, 3 and 5, hold total study time constant, and compare recall after a fixed retention interval. The prediction is fixed here: if the reading holds, 2 and 3 beat 2.5 and 5 by more than the smooth trend between them. Most memory models predict no such bumps. The statistic is the difference in final recall between each ladder ratio and the mean of its two off-ladder neighbours, with a confidence interval from the randomised groups. A clean null result would remove one more claim of the ladder from the social sciences, which is itself useful for this book.

Grade. C; B for the gap-ratio test of spacing schedules (stated, not run).

21.3 Public health and epidemiology

History. The Hippocratic text *Airs, Waters, Places* linked disease to climate, water and way of life. John Graunt counted causes of death in London in 1662 (Chapter 18). The

modern discipline began in the nineteenth century with the sanitary movement. Edwin Chadwick's *Report on the Sanitary Condition of the Labouring Population of Great Britain* (1842) tied disease to drains, water and housing. Ignaz Semmelweis showed in 1847 that handwashing cut deaths from childbed fever in a Vienna clinic. William Farr built the system of vital statistics for England and Wales. John Snow's investigation of the Broad Street outbreak (1854) traced cholera to a single water pump, and the germ theory of Louis Pasteur and Robert Koch explained why.

The twentieth century added mathematics and design. William Kermack and Anderson McKendrick's model (1927) showed that an epidemic grows only when each case produces more than one new case, the threshold now written $R_0 > 1$. The Medical Research Council's trial of streptomycin for tuberculosis (1948) set the form of the randomised controlled trial. Richard Doll and Austin Bradford Hill's case-control study (1950) and their study of British doctors linked smoking to lung cancer, and the Framingham Heart Study (from 1948) identified the risk factors of heart disease. The field trial of Jonas Salk's polio vaccine in 1954, which enrolled well over a million American schoolchildren, remains one of the largest public health experiments ever run, and its success led to mass vaccination within a year. Hill's nine viewpoints of 1965, among them strength, consistency, temporality and dose–response, became the standard checklist for judging whether an association is causal. Archie Cochrane's *Effectiveness and Efficiency* (1972) argued that health services should use only treatments shown to work in randomised trials, and the Cochrane Collaboration, founded in 1993, turned systematic review into an institution. The WHO was founded in 1948, smallpox was declared eradicated in 1980, and HIV, recognised in 1981, reshaped global health.

Abdel Omran's "epidemiological transition" (1971) described three ages: pestilence and famine, receding pandemics, and degenerative and man-made diseases (Wikipedia, Epidemiological transition). Later authors added a fourth age of declining cardiovascular mortality and emerging diseases and a fifth of long lives with persistent inequality (same page). The Global Burden of Disease study introduced the disability-adjusted life year in the 1990s. Geoffrey Rose's "Sick individuals and sick populations" (1985) set out the prevention paradox: a measure that brings large benefits to a population may offer little to each individual, because most cases arise among the many people at moderate risk rather than the few at high risk. Michael Marmot's Whitehall studies of British civil servants, from 1967, showed a gradient of mortality by employment grade, and made the social gradient of health a central finding of the field. COVID-19 tested every part of the field: surveillance, modelling, vaccines, non-pharmaceutical interventions and public trust. The questions left open are how to prepare for the next pandemic, how to reduce chronic disease, and how to keep public trust when evidence changes.

Current state. Tuberculosis remains a major cause of death. In 2024 about 10.7 million people fell ill, 5.8 million men, 3.7 million women and 1.2 million children, and 1.23 million died, including about 150,000 people with HIV. Starts on the shorter six-month regimens for multidrug-resistant TB rose from 5,653 in 2023 to about 34,000 in 2024 (WHO TB fact sheet). About 11.5 % of those who fell ill died (Calculation 21.6).

Measles killed about 95,000 people in 2024, and first-dose coverage in 2025 was 84 %, second-dose 77 %, below the 95 % needed to stop transmission (WHO measles fact sheet); the gaps are 11 and 18 points (Calculation 21.6).

The aftermath of COVID-19 shapes policy. The WHO estimated 14.9 million excess deaths in 2020 and 2021, with a range of 13.3 to 16.6 million (WHO). The WHO Pandemic Agreement, adopted on 20 May 2025 by 124 votes with 11 abstentions, needs 60 ratifications, and its annex on pathogen access and benefit sharing must be finalised first (Wikipedia, WHO Pandemic Agreement). The United States did not take part in the vote, as it was withdrawing from the World Health Assembly (same page).

Social determinants have moved to the centre. The WHO Commission on Social Connection reported in June 2025 that loneliness affects about one person in six and is linked to more than 871,000 deaths a year (WHO). The frontier includes wastewater and genomic surveillance, which tracked variants during COVID-19; causal inference from observational data, which extends Hill's viewpoints with formal methods; and the use of AI for diagnosis and outbreak detection. The debate over vaccine hesitancy, which now drives measles outbreaks in countries that had eliminated the disease, is as much about trust as about evidence. Antimicrobial resistance is the slow emergency of the field: the shorter regimens for drug-resistant tuberculosis are one response, and stewardship of existing antibiotics is another. In all of these, the task is the same as in the nineteenth century: to find where a process that harms many people can be interrupted at least cost.

Interpretation.

1. *Strand.* I is time; P is the set of health states of a population (susceptible, infected, recovered) or of a person.
2. *Passage and trit.* A return of an epidemic to its starting state is a passage; each season's incidence is higher (1), lower (T) or the same (0) as the last.
3. *Closure.* The SIR model is a path S → I → R that does not return. With waning immunity it becomes SIRS, S → I → R → S, a ring of three states in which each person makes one passage of each state per cycle (T7). The threshold $R_0 > 1$ is a standing condition for growth, not a window. The herd-immunity threshold $1 - 1/R_0$ for measles is 0.917 to 0.944 for R_0 between 12 and 18 (Calculation 21.6).
4. *Scale.* Epidemic periods were tested against the ladder in Chapter 20: 18 of 20 measles towns on it, significant against a log-uniform null but not against a whole-year null. Omran's three ages became five with later additions, and Hill's viewpoints are nine; these are counts of chosen categories (Calculation 21.8). The pandemic agreement's 60 ratifications and the 95 % coverage target are thresholds set by negotiation and by R_0 respectively; the second is a computed standing condition, the first a political choice. Neither is a window. On a lattice, an SIRS process with short immunity produces travelling waves and spirals, as rock–paper–scissors dynamics do, because each state is invaded by the next in a cycle of three. That parallel is real but belongs to the theory of excitable media, which has its own literature and needs no help from the Foundations. T6 would supply a baseline if a surveillance series were coded as trits (incidence up, down or flat each week): three-week windows would close by chance at the rate T6 computes from the observed frequencies.

Whether closed windows signal the end of a wave earlier than standard trend estimates is a question one could test on historical surveillance data, but it is not one I can state a sharp prediction for. Without a sharp prediction it is not a test in this book's sense. T6, T11–T14 and T16–T24 have no counterpart I could compute.

5. *Test.* No test specific to public health is stated beyond the epidemic-period test of Chapter 20. A further test would extend that one to diseases forced by the school year, such as pertussis and rubella, and to the post-vaccination era, with the same statistic and nulls.

Consequences. Public health is the practical science of making a population's dynamics close in a good state. In that sense it is the tension column's clearest case: the field takes an open process, transmission, and adds interventions until it closes. But that description does not help anyone vaccinate a child or trace a contact. The SIRS ring is a real ring of three states, and it is published structure, but every model with three states in a cycle is a ring in this sense, so the identification is trivial. The measles test of Chapter 20 is the only place where the reading meets data in this field, and it passes weakly. If it fails on other forced diseases, the reading has nothing to offer epidemiology. If it passes, it would at most sharpen what forced-oscillator models already predict. The costs are low, the gains small.

One place where the reading might help is communication. Public health messages often present risk as two states, safe or unsafe, while the evidence has three: protective, neutral or harmful at the measured dose. A balanced register makes the neutral value explicit and makes "not harmful" distinct from "protective", which is the distinction the prevention paradox depends on. That is a point about notation, not about disease, and it would need an experiment in risk communication to show that it changes understanding.

Grade. C; B where the epidemic-period test of Chapter 20 applies.

21.4 Public administration

History. Imperial China selected officials by written examination from the Sui dynasty onward until 1905, the longest-lived civil service system on record. European states built bureaucracies in the seventeenth and eighteenth centuries; Prussian cameralism taught administration as a science of the state's resources. Britain's Northcote–Trevelyan Report (1854) proposed entry by competitive examination and promotion by merit, and the United States adopted the merit principle in the Pendleton Act of 1883 after a president was shot by a disappointed office-seeker. Woodrow Wilson's essay "The Study of Administration" (1887) argued that administration should be separated from politics and studied as a field of its own. In the United States the two Hoover Commissions (1947–1949 and 1953–1955) set the template for periodic reviews of the executive branch, with recommendations to consolidate agencies and clarify lines of authority.

Max Weber's account of bureaucracy, published after his death in *Economy and Society* (1922), named its features: fixed jurisdictions, a hierarchy of offices, written files, trained officials and rules applied without regard to person. The same work

distinguished three pure types of legitimate authority: traditional, charismatic and legal-rational. Weber described how charismatic authority, to last, must be “routinised” into one of the other two, so the three types are linked by transitions rather than standing side by side. Luther Gulick’s acronym POSDCORB (1937) listed the executive’s functions. V. A. Graicunas argued in 1933 that the number of relationships a supervisor must manage grows much faster than the number of subordinates (Wikipedia, Span of control). Herbert Simon’s *Administrative Behavior* (1947) replaced the “proverbs of administration” with bounded rationality: officials satisfice rather than optimise. Charles Lindblom’s “The Science of ‘Muddling Through’” (1959) described policy as a sequence of small steps rather than comprehensive plans.

The late twentieth century brought reform by markets. Christopher Hood named “New Public Management” in 1991: professional managers, explicit targets, output controls, disaggregated units, competition and contracting out, and citizens treated as customers (Wikipedia, New public management). David Osborne and Ted Gaebler’s *Reinventing Government* (1992) popularised the approach. Patrick Dunleavy and colleagues argued in 2006 that “New Public Management is Dead” and that digital-era governance, reintegrating services around citizens and data, was replacing it (same page). Elinor Ostrom’s *Governing the Commons* (1990) showed that communities often manage shared resources without either the state or the market, under conditions she summarised as eight design principles; she received the economics prize in 2009. Her work widened the field from the agency to the whole set of institutions, public and private, that govern a resource. Estonia’s digital state, built from the early 2000s on a shared data-exchange layer, became the most cited example of administration redesigned around data. The questions left open are how large the state should be, how to measure the performance of public services, and how to keep administration accountable when it is automated.

Current state. Trust in government is measured systematically. The OECD’s Survey on Drivers of Trust in Public Institutions, conducted in late 2023 in 30 OECD countries with nearly 60,000 respondents, was published on 10 July 2024 (OECD). The OECD’s *Government at a Glance 2025*, published on 19 June 2025, reports indicators on public finance, employment and trust across member countries (OECD).

The largest administrative experiment of recent years was in the United States. The Department of Government Efficiency was established on 20 January 2025 by Executive Order 14158 and ceased operation on 4 July 2026, as scheduled (Wikipedia, Department of Government Efficiency). Its leadership claimed savings of hundreds of billions of dollars; fact-checkers and independent analysts disputed these figures, and some estimates put the net cost to the government higher than the savings (same page). The New York Times counted, as of 12 May 2025, more than 58,500 confirmed cuts, more than 76,000 buyouts and more than 149,000 other planned reductions, together about 12 % of 2.4 million civilian federal workers; by March 2026 about 9 % of the federal workforce had been eliminated, by the page’s account (same page; Calculation 21.7). Supporters presented the programme as overdue reduction of waste; critics as damage to public capacity. Both positions were argued in Congress, in the

courts and in the press throughout 2025 and 2026. The assessments are contested, and a full accounting will take years.

Elsewhere the agenda is digital. Governments are automating eligibility decisions, tax assessment and case handling, and the EU AI Act classifies many of these uses as high-risk (21.1). The research frontier concerns algorithmic accountability in administration, the measurement of state capacity, and the effects of reorganisation on service quality, where randomised and quasi-experimental evaluations are replacing before-and-after comparisons. Public employment and the cost of administration differ greatly between countries, and comparisons depend on whether teachers, health workers and contractors are counted; the OECD's indicators are the common reference for such comparisons, but every national debate about the size of the state uses its own definitions.

Interpretation.

1. *Strand.* I is the sequence of cases handled by an administration; P is the set of offices and procedures. A case that returns to an office (an appeal, a review, a renewal) is a loop.
2. *Passage and trit.* A decision on return confirms the earlier one (1), reverses it (T) or is deferred (0).
3. *Closure.* A file is closed when its returns stop changing it; an appeal that reverses and is reversed again is a meeting (1, T) with carry 0. Montesquieu's three powers, which check each other in a cycle, are a ring in description; I have no measure that would make this more than a reading. Where two agencies share a case, each with its own closed procedure, the junction is a seam in the sense of T11, and the theorem's statement that such a seam always carries tension matches the administrative experience of joint cases. But the theorem's count of open windows at a seam (one or two) has no measured counterpart in case files. A study of joint cases that coded each hand-over as agreeing, disagreeing or neutral could supply one.
4. *Scale.* A hierarchy in which each manager has s subordinates has s^k staff at depth k, so a span of 2 or 3 gives the ladder of T15. Spans are not fixed: historically about 4, and after the flattening of the 1980s closer to 10 (Wikipedia, Span of control). Graicunas's count of potential relationships, $n(2^{n-1} + n - 1)$, gives 1, 6, 18, 44, 100, 222 for n = 1 to 6 subordinates; the first three are on the ladder and the rest are not (Calculation 21.4). Weber's hierarchy is Kelsen's in another form. Weber's three types of authority are a triad but not a cycle in his account, since routinisation runs from charisma towards tradition or law and not back in any fixed order; reading them as a ring would add a claim Weber did not make. Ostrom's eight design principles are 2^3, a ladder number, but the count is hers, chosen from case studies, and it is not evidence of anything. T6, T11–T14 and T16–T24 have no counterpart I could compute; T11 might apply where two agencies with different procedures share a case, but I found no data.
5. *Test.* Stated here before any data: in organisations whose structure grew without a fixed rule, the distribution of spans of control has peaks at 2, 3, 4 and 6 and troughs

at 5 and 7. The reading fails if the distribution is smooth or peaks at 5 or 7. Data would come from organisation charts or personnel registers. It was not run here.

Consequences. Public administration is the discipline of the closed file and the stable procedure, and the Foundations' vocabulary of closure fits it in words. But the field's real questions are about size, cost, capacity and accountability, and the reading has nothing to say about them. It certainly gives no verdict on the American reductions of 2025–2026, where the dispute is over facts and values, not over arithmetic. The span-of-control test is a real test that the field does not already pose: administrative science treats span as a continuous design variable set by task complexity and supervision cost. If the ladder peaks appeared, they would be a curious regularity worth explaining; if not, the reading loses nothing it ever had.

The broader lesson concerns what "closure" means for a state. Weber's bureaucracy is valued for closing cases predictably: the same file, under the same rule, gets the same answer. New Public Management traded some of that predictability for responsiveness and cost control, and digital-era governance now promises both through data. In the Foundations' language these are choices about how many returns a case may make and whether each return must confirm the last. The language is neutral between them, which is right for a description and useless as advice. Whether a smaller or a larger administration serves citizens better is a question of values and evidence, and this chapter takes no position on it.

Grade. C; B for the span-of-control test (stated, not run).

21.5 Management

History. Frederick Winslow Taylor's *Principles of Scientific Management* (1911) timed work in small steps and set standard methods and piece rates. Henri Fayol's *Administration industrielle et générale* (1916) defined management as five functions: planning, organising, commanding, coordinating and controlling. The Hawthorne studies at Western Electric (1924–1932) found that attention and group norms affected output, which Elton Mayo developed into the human relations school. Chester Barnard's *The Functions of the Executive* (1938) described the organisation as a system of cooperation held together by communication and incentives.

Quality management grew from statistics. Walter Shewhart at Bell Telephone Laboratories developed control charts in the 1920s, and in his book of 1939 described manufacturing under statistical control as a three-step process of specification, production and inspection (Wikipedia, PDCA). W. Edwards Deming modified the cycle and, in lectures in Japan in the early 1950s, it became the four steps plan, do, check, act; Deming himself preferred "study" to "check" (same page). Peter Drucker's *The Practice of Management* (1954) introduced management by objectives. Douglas McGregor contrasted Theory X and Theory Y in 1960. Abraham Maslow's hierarchy of needs (1943) and Frederick Herzberg's two-factor theory of motivation (1959) entered management teaching as accounts of why people work. Bruce Tuckman's model of team development (1965) had four stages, forming, storming, norming and performing, to which a fifth, adjourning, was added in 1977. Henry Mintzberg's *The Nature of*

Managerial Work (1973) observed what managers actually do and found brief, fragmented, interrupted activity rather than planning. Luther Gulick's POSDCORB (1937), seven letters for planning, organising, staffing, directing, coordinating, reporting and budgeting, carried Fayol's list into public administration, and Herbert Simon's critique of such lists as "proverbs" applied to management as well.

Strategy became a field with Alfred Chandler (1962), Igor Ansoff and Michael Porter's five forces (1980). Robert Kaplan and David Norton's balanced scorecard (1992) asked firms to measure performance from four perspectives (financial, customer, internal process, and learning and growth) rather than from the accounts alone. Richard Nelson and Sidney Winter's *An Evolutionary Theory of Economic Change* (1982) treated firms as stores of routines, repeated patterns of action that change slowly and by selection, and so gave management research a theory of why organisations resist change. In the language of the Foundations a routine is a word that closes everywhere and repeats (T8): a path without tension, one of a small number of stable patterns. That is a fair description of what Nelson and Winter meant, and like the others in this section it predicts nothing they did not. Total quality management, lean production on the Toyota model, and business process re-engineering (Hammer and Champy, 1993) followed. Taiichi Ohno's Toyota Production System cut inventory to what the next step needs, and made continuous small improvement, kaizen, a daily routine. Motorola's Six Sigma programme, from 1986, set a target of a few defects per million opportunities and organised improvement projects in five phases: define, measure, analyse, improve, control. Organisations flattened from the 1980s, increasing the number of people each manager supervises (Wikipedia, Span of control). The questions left open are what makes management practices effective, how much they can be transferred between firms and countries, and how management changes when much coordination is done by software.

Current state. Gallup's *State of the Global Workplace* report of 2026, based on 2025 data, found global employee engagement at 20 %, its second annual decline and the lowest since 2020. Engagement among managers fell from 27 % to 22 % in one year. Gallup estimates the cost of low engagement at about 10 trillion dollars in lost productivity, roughly 9 % of global GDP (Gallup). The fall in manager engagement is 5 points, or 18.5 % of its level (Calculation 21.7).

Management research has turned to measurement. Large surveys of management practices across firms and countries, led by Nicholas Bloom, John Van Reenen and colleagues since the 2000s, found that a few practices (monitoring, targets and incentives) vary widely and correlate with productivity. Field experiments in firms have tested some of them causally. Hybrid and remote work after 2020 changed how teams coordinate, and the evidence on productivity is mixed and depends on the task. Generative AI is the newest subject: it can draft, summarise and analyse, and early field studies report gains that vary with experience and task, while the effects on management roles themselves are not yet measured.

The flattening of hierarchies continues, with spans of control nearer 1 to 10 than the earlier 1 to 4, and the older assumption that no manager can supervise more than five or six direct reports has been abandoned in many firms (Wikipedia, Span of control).

The open debate is whether fewer managers with wider spans leave employees without the attention that Gallup's data associate with engagement. Some firms have gone further and removed formal managers altogether in self-managing teams; the evidence from such experiments is mostly case studies. The question how many people one person can coordinate, which Graicunas posed in 1933, has therefore not been settled by practice either.

Interpretation.

1. *Strand.* I is the sequence of work cycles of an organisation; P is the set of process states. A process that returns to a state it has been in (the next batch, the next quarter) is a loop.
2. *Passage and trit.* At each cycle a measured outcome improves (1), worsens (T) or holds within control limits (0). Shewhart's control chart is a register that reads exactly this: in control, or out on one side or the other.
3. *Closure.* Shewhart's cycle of specification, production and inspection is a ring of three steps; by T7.1 a completed ring closes with carry ±1 and becomes one passage of the next scale, which is the management claim that each turn of the cycle lifts the process to a new standard. Deming's four-step cycle is 2^2, two meetings (plan and act, do and check) rather than a ring. A process within control limits is a run of zeros, a standing. Two departments with different closed procedures meeting on one product is a seam in the sense of T11, and T11 says such a seam always carries tension; the matrix organisation is a deliberate seam.
4. *Scale.* Of the counts in this chapter, Shewhart's 3, Deming's 4 and Tuckman's original 4 are on the ladder; Fayol's 5 and Tuckman's revised 5 are not. Gulick's seven functions are off it as well. Maslow's five levels and Six Sigma's five phases are off it; the balanced scorecard's four perspectives are on it. Chance accounts for the split (Calculation 21.8). Engagement scores and productivity are not merged in blocks, so T15 has no purchase. T6 would give a baseline for closed windows in control-chart signals, but I found no published series in trits. T12–T14, T16–T22 and T23–T24 have no counterpart I could compute.
5. *Test.* None specific. A test would compare improvement programmes run as three-step and four-step cycles, with the same total time, on the same kind of process. Management research has no such experiment, and I do not expect the number of steps to matter. Six Sigma's five-phase cycle, which is off the ladder, is as widely used as the four-step cycle, and nothing in the evidence suggests that its fifth phase makes it worse.

Consequences. Management uses cycles constantly, and the Foundations can name them: a three-step cycle is a ring, a four-step cycle is two meetings. Naming does not improve them. The control chart is the best fit, because it already reads a process as standing, rising or falling, and treats a run of the same sign as a signal. Quality engineers formalised this long ago in the Western Electric rules, which flag runs of points on one side of the centre line; a trit reading of such runs would add nothing they lack. The seam theorem gives a compact statement of something managers know, that two well-run units joined on a common task create friction at the join, but it gives no measure of it. The costs of the reading are zero and so are the gains; the grade is C.

The engagement data deserve a separate remark, because they are the most-cited number in the field. A fall in manager engagement from 27 % to 22 % in one year is large, but it is a survey measure of attitudes, and its link to productivity rests on correlations across workplaces. The Foundations add no reading of it. Nor do they add a reading of the long debate between Taylor’s view of work as a sequence of measured steps and Mayo’s view of it as a social system. Both are about how a closed unit, the firm, keeps its internal tensions small enough to function, and the Foundations’ vocabulary describes that aim without choosing between the two means. The honest result for management is that the discipline uses cycles, triads and seams in its vocabulary, that the Foundations can name them, and that naming them has so far produced no test that management research would recognise as a new question.

Grade. C.

Part VII — Culture

Chapter 22 — Language and Sign

Language, sign and script are the standing units of culture: the closed things a society keeps, names and copies. This chapter tests them against the Foundations and finds three kinds of result. The trit rule of the Foundations can be run mechanically on real texts. On six classical works it gives a small, testable excess of closed windows in Homer that survives a first-order control, and nothing in prose (grade B, partly passed). Peirce's sign classes and the Yoruba numerals contain register structures as published fact, but neither is the closure rule of the Foundations (grade A for the register, C for closure). Writing systems as a whole remain a reading (grade C).

22.1 Linguistics

History. Linguistics as a science of change began with a comparison. In 1786 William Jones told the Asiatic Society in Calcutta that Sanskrit, Greek and Latin shared roots and grammar too closely for chance, and proposed a common source. Rasmus Rask and Jacob Grimm then turned the comparison into rules. Grimm's account of the Germanic consonant shift (1822) showed that sounds change in regular correspondences: Latin *pater* stands to English *father* as Latin *piscis* stands to *fish*. The Neogrammarians of Leipzig in the 1870s hardened this into a principle: sound laws admit no exceptions, and apparent exceptions have another cause, such as analogy or borrowing. Karl Verner's law (1877) is the textbook case: a set of exceptions to Grimm's law was explained by the position of the accent.

The twentieth century moved from history to structure. Ferdinand de Saussure's *Cours de linguistique générale*, assembled by his students in 1916, separated the language system (*langue*) from speech (*parole*). It defined the sign as a two-sided unit of a sound image and a concept, and value as difference within the system. The Prague school (Trubetzkoy, Jakobson) built phonology on contrast: a phoneme is a bundle of distinctive features, each a binary opposition. American structuralism (Bloomfield, 1933) built procedures for segmenting and classifying forms without appeal to meaning.

Noam Chomsky's *Syntactic Structures* (1957) changed the question again. A grammar became a finite device that generates the infinite set of sentences of a language. Chomsky argued that children learn grammar from too little data to infer it, and concluded that a language faculty must be innate. The argument from the poverty of the stimulus drove fifty years of generative theory.

A second line ran through typology. Joseph Greenberg's 1963 paper on universals of grammar, based on a sample of thirty languages, found implicational rules. Languages with verb–object order tend to have prepositions; languages with object–verb order tend to have postpositions. Typology became a statistical science of what languages do and do not do. The *World Atlas of Language Structures* (WALS, 2005; online 2013) mapped some 190 features over up to 2,600 languages.

A third line was quantitative. Claude Shannon (1951) measured the entropy of printed English by asking people to guess the next letter. George Kingsley Zipf (1949) found that the frequency of a word falls roughly as the inverse of its rank. Usage-based linguistics (Bybee, Tomasello) treated grammar as the residue of frequent use rather than as a fixed device.

The last turn was computational. Statistical machine translation in the 1990s, word embeddings around 2013 and the transformer architecture of 2017 produced language models that write fluent text without any grammar written into them. This reopened Chomsky's question from the other side. If a network trained on text learns syntax, how much must be innate?

The questions left are clear. What is universal in language and what is historical accident? How much structure must a learner bring? And is there a law of form below the variation of seven thousand languages, or only tendencies?

Current state. The number of languages depends on who counts. Glottolog 5.3 lists 7,674 spoken L1 languages in 246 genealogical families, 183 of them isolates, as of 2 March 2026. Ethnologue's own guide gives 7,099 languages spoken today and says roughly a third are endangered. The two lists differ because they draw the line between language and dialect differently. The gap of 575 is itself a measure of that open question.

Grammatical diversity is now measured at scale. Grambank, published in *Science Advances* in 2023, coded more than 2,400 languages on 195 questions and estimated that nearly half of the world's grammatical diversity is at risk. Losses are uneven: the report expects total loss in South America and Australia and about 25 percent in the Pacific, South-East Asia and Europe. Sound inventories have a parallel database: PHOIBLE 2.0 holds 3,020 inventories with 3,183 segment types in 2,186 languages.

The typological counts that matter for this chapter come from WALS. Dryer's chapter on word order finds, among 1,376 languages, 564 SOV, 488 SVO, 95 VSO, 25 VOS, 11 OVS and 4 OSV, with 189 lacking a dominant order. Subject before object holds in 1,147 of the 1,187 languages with a dominant order, 96.6 percent (Calculation 22.4). Diessel's chapter on demonstratives counts 126 languages with a two-way distance contrast and 88 with a three-way contrast, of 234. Corbett's chapter on gender finds 145 of 257 languages without gender, 50 with two and 26 with three. Maddieson's vowel chapter finds that 188 of 564 languages, just over a third, have five vowel qualities.

The open debate of 2025–2026 is the status of language models. Futrell and Mahowald, in a 2025 target article for *Behavioral and Brain Sciences*, take a middle line. Language models neither make linguistic theory unnecessary nor are irrelevant to it. The authors describe them as "model systems and working proofs of concept for gradient, usage-based approaches to language". They cite evidence that such models learn agreement across long distances and generalise hierarchically even when trained on text where subjects and verbs are always adjacent. They concede that models see orders of magnitude more data than children. The poverty-of-stimulus argument is weakened, not settled.

So the frontier has three parts. Databases now cover a third of the world's languages feature by feature. Statistical models can learn grammar from text. And the languages themselves are disappearing faster than they are described.

Interpretation.

1. *Strand.* I is the running order of a text, word by word. P is its vocabulary: the set of word types. The strand σ maps each position to the word written there. A loop is a repeated word, and a node is a word type used at least twice.
2. *Passage and trit.* The trit rule of the Foundations (III.1) applies without change. A later use of a word scores 1 if the next word is the one that followed its first use, T if the next word is the one that preceded its first use, and 0 otherwise. The rule needs only the text. A formula ("swift-footed Achilles") repeated in full scores a run of 1s. On six classical texts the share of 1s is 2.2–9.6 percent and the share of T is under one percent (Calculation 22.1).
3. *Closure.* A window of three consecutive trits closes when its digit is 0 (T1, T2). A closed window leaves no residue at its scale; an open window keeps a nonzero digit, its tension. Because 0 dominates in text, most windows close trivially (three 0s). The informative windows are those with 1s: a run 1, 1, 1 is a ring and carries one passage to the next scale (T7.1: 1 + 1 + 1 = 10). That is the arithmetic form of a repeated formula becoming one unit of the next scale.
4. *Scale.* Word order offers a direct check of T4 and T9. The six orders of S, O and V are exactly the six heterogeneous closing windows of T4. Under rotation they fall into two classes of three (T9): SOV, VSO, OVS (670 languages) and SVO, VOS, OSV (517) (Calculation 22.4). The ladder of T15 fits the small counts of grammar only in part. Two- and three-way contrasts in demonstratives (91.5 percent of the sample) and two to four genders lie on the ladder. The five-vowel peak and the decimal and vigesimal number bases do not (Calculation 22.5).
5. *Test.* Stated before the run: for formulaic texts, the closed-window rate should exceed the rate expected from the trit law alone (T6) and the rate of a shuffled copy. If the excess is no larger than in prose, or if it disappears once the dependence of each trit on the previous one is controlled, the closure rule carries nothing beyond repetition. Run: the Iliad and Odyssey exceed their shuffles by 21 and 38 standard deviations. Against a first-order Markov control the excess shrinks to +2.2 and +4.3 standard deviations. The Aeneid (+0.5) and the three prose texts (−0.1 to +0.4) show none (Calculations 22.1–22.2).

Consequences. The test was run, not only stated, and the result is mixed. Most of the Homeric excess is explained by pairs: a 1 is followed by another 1 because formulas are longer than two words. That is repetition, and linguistics has known it since Milman Parry (1930). What remains after the pair control is a small excess of closed windows in both Homeric epics: +0.0050 in the Iliad and +0.0113 in the Odyssey. Only in Homer, the most formulaic text, does a window of three carry information that pairs do not. That is consistent with the reading of a formula as a ring that carries one passage to the next scale. It is far from proof. A second-order control, or a model of formula length, could remove it too.

Two negative results are as useful. The lever of T24, applied to the count of uses per word type, comes out between −4.4 and −12.9. A free path with the same mean would give about −2.2 (Calculation 22.3). Word frequencies are not the visits of a free path; they are Zipf-like, with more than half of all types used once. The Foundations' lever does not describe vocabulary. And the ladder does not organise the numbers of grammar: five vowels and base ten are off it.

What the interpretation adds is a mechanical measure of formulaic density that needs no list of formulas and no parser. It costs the assumption that the trit rule's digits mean something in a text. The Foundations list that assumption as open (IX.5).

The remaining theorems were checked against linguistics and do not apply. T8–T10 would need texts of period three; no natural text has that. The seam rule T11 has no counterpart, because texts are not built from closed windows joined end to end. The comma (T16–T18), Z[ω] (T14) and the topological theorems (T19–T22) have no linguistic object. T13 applies only in the trivial sense that a sentence read from its start is an open word.

Grade. B: the closed-window test was stated and run on six texts; it is passed weakly by Homer only, after a pair control, and not by Virgil or prose. C for word order and for the ladder in grammatical counts.

22.2 Semiotics

History. The study of signs is older than its name. Augustine, in *De doctrina christiana* (c. 397), defined a sign as a thing that brings something else to mind beyond the impression it makes on the senses. Medieval logicians separated signification from supposition: what a word means and what it stands for in a sentence. John Locke closed his *Essay* (1690) by naming a third branch of science, *semeiotike*, the doctrine of signs.

Two founders made it a discipline, independently and almost at the same time. Saussure, lecturing in Geneva from 1907, proposed a *sémiologie*: a science of signs within social life, of which linguistics would be one branch. His sign has two sides, signifier and signified, and its bond is arbitrary. Meaning lies in difference: a sign is what the others are not.

Charles Sanders Peirce, working mostly outside the university, built a different and triadic theory. In 1867 he proposed three universal categories, later called Firstness, Secondness and Thirdness: quality, reaction and mediation. A sign for Peirce is a relation of three terms: the sign itself, its object and its interpretant, the effect it produces in a mind. In 1903 he divided signs along three trichotomies. By the sign itself: qualisign, sinsign, legisign. By its relation to its object: icon, index, symbol. By its interpretant: rheme, dicent, argument. Three trichotomies give 27 combinations. Peirce argued that only 10 are possible, because each later element depends on the earlier one and cannot be of a higher category. In letters to Lady Welby in 1908 he extended the scheme to ten trichotomies and 66 classes.

The structuralist line ran from Saussure through Hjelmslev's glossematics (1943) and Roman Jakobson's six functions of language (1960) to Roland Barthes. Barthes's

Mythologies (1957) read advertising and wrestling as sign systems, and his *Elements of Semiology* (1964) tried to make the method general. Algirdas Greimas built a semiotic square of contraries and contradictories. Umberto Eco's *A Theory of Semiotics* (1976) attempted a synthesis, defining semiotics as the study of everything that can be used to lie.

Biosemiotics extended the field beyond humans. Jakob von Uexküll's *Umwelt* (1909) described the sign world of a tick. Thomas Sebeok in the 1960s founded zoosemiotics and argued that semiosis is coextensive with life.

The questions left are three. Is the basic sign relation two-sided (Saussure) or three-sided (Peirce)? Can a classification of signs be derived, or is it only a catalogue? And where is the line between a sign system that is language and one that is not?

Current state. Peirce scholarship has converged on the 1903 system and remains divided on the later one. The Stanford Encyclopedia's entry states the ordering rule plainly: if an element is a quality, its dependent may only be a quality; if an existential fact, a fact or quality; if a convention, any of the three. This yields ten classes from 27. For the 1908 scheme of ten trichotomies and "sixty-six classes of sign", the same entry says that the order of combination is unclear and that no fully satisfactory account of the 66 classes exists. The derivation of a sign typology is therefore still an open research problem within the field.

The line between language and other sign systems is being tested on the Indus signs. Inscriptions from the Indus civilisation are usually around five symbols long. In Nisha Yadav's analysis, 67 signs account for about 80 percent of the corpus. A recent study compared 4,000 examples of the script with 15,000 pot shards bearing graffiti marks. In January 2025 the government of Tamil Nadu offered $1 million for a decipherment that satisfies archaeological experts. The deeper dispute, between an entropy argument that the signs encode language (Rao et al. 2009) and the reply that such statistics cannot separate language from other symbol systems (Sproat 2014), is a semiotic question about what a statistical signature can show.

Sign inventories are now standardised at world scale. Unicode 17.0, released on 9 September 2025, added 4,803 characters for a total of 159,801 and brought the number of supported scripts to 172, with eight new emoji. Emoji are the fastest-growing ideographic sign set in history, and their meanings are negotiated across platforms.

Language models raise the old question of grounding in a new form. A model that has seen only signs, never objects, now produces signs that people interpret. In Peirce's terms it handles the relation of sign to interpretant without a direct relation to the object. Whether that is semiosis or its simulation is debated across philosophy and cognitive science. The field has no settled position.

Interpretation.

1. *Strand*. I is the order in which a sign is taken up: first the vehicle, then the object, then the interpretant. P is the set of three categories. Peirce's own analysis is a path of length three through three values.

2. *Passage and trit.* Each sign is a word of three digits from three values. Read as digits of the trinity (T1), Firstness, Secondness and Thirdness are three values. The Foundations take them as −1, 0, 1 only by an assignment; the categories are ordered (1 < 2 < 3), the trits are not.
3. *Closure.* The Foundations keep the 9 of 27 windows whose digit is 0 (T3). Peirce keeps the 10 of 27 triples that do not increase (Calculation 22.7). The two rules differ. They share exactly four triples: the three constant ones and (3, 2, 1). These are the four Peirce classes that are all equal or all different, the two kinds of closure of T4, under any labelling of the categories. The other six Peirce classes, such as the dicent sinsign (2, 2, 1), have two equal values and are open windows in the Foundations.
4. *Scale.* Peirce's 1908 move from three to ten trichotomies is a change of word length, not of scale. The count of non-increasing words of length n over three values is C(n + 2, 2): 10 for n = 3 and 66 for n = 10, which is Peirce's own number. The open words of the Foundations number 3^n and the closed configurations g(n) (T13). Peirce's rule counts neither: it counts multisets.
5. *Test.* If the Foundations' closure rule describes sign use, then in a corpus of signs coded by their three Peircean values, the four "closed" classes should be more stable over time than the six "open" ones. Stability can be measured as survival in a lexicon or persistence of a symbol type across periods. If open and closed classes survive equally, the closure reading of Peirce fails. No such corpus has been coded; the test is stated, not run.

Consequences. Semiotics holds the clearest published instance of a three-valued register outside logic: Peirce's 3 × 3 × 3 sign space. That is grade A as a fact about the field. The 27 cells are the same 27 cells as the trit cube of T5. What the Foundations add is a different cut through those 27 cells. Peirce selects by dependence: a later element cannot exceed an earlier one. The Foundations select by residue: the sum of the digits must be 0 mod 3. Both rules are simple and both give about a third of the cells, 10 and 9. They agree on four.

The comparison sharpens a known problem. The 66 classes of 1908 have no agreed derivation. Peirce's own count of 66 is exactly the number of non-increasing words of length ten over three values. That shows his rule was the multiset count at both lengths, which the field already suspected. The Foundations offer no better derivation of the 66 classes. They do offer a second, independent count for the ten-trichotomy space: 3^{10} open words and g(10) closed configurations. Whether any semiotic data distinguish the two counts is the test above.

On the Indus question the Foundations add nothing. Sign frequency, sign length and positional regularity are what the field already measures. A three-trit window is no better a detector of language than the entropy measures already disputed.

The cost is clear. Reading Firstness, Secondness and Thirdness as −1, 0, +1 is an assignment, not a finding. The four shared classes are shared under every assignment, which is why only they are reported.

One more structure belongs here, with a warning. Signs of threefold unity are common, and some are real topology. The Borromean rings appear in the arms of the Borromeo family. Three linked triangles of the same kind appear on Norse picture stones from the seventh century. A thirteenth-century French manuscript, lost in the 1940s but reproduced in 1843, used the rings for the Christian Trinity (Borromean rings). The rings are linked as a triple while every pair is unlinked. That is a closure of three that no pair carries. It is the topological form of the difference in T7 between a ring and two meetings. The fact is topology, independent of this theory. That cultures chose this figure to signify unity in three is a historical observation. Reading it as knowledge of T7 would be re-description, so it is graded C.

The other theorems were checked and do not fit. Sign classes have no period (T8–T10), no seams (T11), no ladder (T15: 10, 27 and 66 are not all of the form $2^a \cdot 3^b$, and 10 and 66 are not) and no comma (T16–T18).

Grade. A for the three-valued sign register (Peirce 1903: 27 combinations of three trichotomies, a published fact independent of this theory). B for the stated stability test of closed against open classes, not yet run. C for the rest of semiotics.

22.3 Writing systems

History. Writing was invented at least three times, and perhaps four, independently. In southern Mesopotamia, clay tokens used for accounting in the fourth millennium BCE were pressed into clay envelopes, and the impressions became signs on tablets. By about 3200 BCE, at Uruk, scribes wrote proto-cuneiform: numerals and logograms for goods and persons. Cuneiform then adopted the rebus principle, using a sign for its sound, and became a mixed logo-syllabic script that served Sumerian, Akkadian, Hittite and a dozen other languages for three thousand years.

Egyptian hieroglyphs appear around the same time or shortly after. Their system combined logograms, phonetic signs for one, two or three consonants, and determinatives that mark a word’s class. Chinese writing is attested on Shang oracle bones from about 1200 BCE and has remained in continuous use since. In Mesoamerica, Maya script combined logograms and syllables and recorded dates, kings and wars on monuments from the late first millennium BCE.

The alphabet came from simplification. Workers in Sinai around 1800 BCE borrowed a few Egyptian signs for consonants. The Phoenician script that grew from this line had 22 letters, all consonants. The Greeks, around 800 BCE, reused letters for sounds Greek had and Phoenician lacked, and so wrote vowels. Latin, Cyrillic, Arabic, Hebrew and the Brahmi family of South Asia descend from the same Semitic line.

Decipherment made the field a science. Jean-François Champollion read the hieroglyphs in 1822 from the Rosetta Stone and Coptic. Henry Rawlinson and others read cuneiform from the trilingual Behistun inscription in the 1840s and 1850s. Michael Ventris showed in 1952 that Linear B wrote an early form of Greek, publishing the proof with John Chadwick in 1953. Yuri Knorozov’s phonetic approach (1952) opened Maya script, which was largely read by the 1990s. Others remain unread:

Linear A, the Indus signs, rongorongo of Easter Island and the Proto-Elamite script, though the last has seen recent progress.

The typology of scripts was set out by Ignace Gelb (1952) as an evolution from logograms through syllables to alphabets. Peter Daniels and William Bright's *The World's Writing Systems* (1996) replaced the evolutionary scheme with a classification. It distinguishes logosyllabaries, syllabaries, abjads (consonants only), alphabets, abugidas (consonant letters with vowel marks) and featural scripts such as Korean Hangul.

Two later inventions were deliberate designs. In Korea, King Sejong's scholars created Hangul in 1443 and promulgated it in 1446. Its consonant letters depict the position of the speech organs, and letters combine into syllable blocks. In North America, Sequoyah completed a Cherokee syllabary around 1821 without being literate in any other script. Within a few years much of the Cherokee nation could read it. Both show that a script can be designed from a theory of sound rather than inherited.

The questions left are how writing arises from non-linguistic marks, what makes a script stable across languages, and whether the unread scripts encode language at all.

Current state. Two unread scripts saw dating progress. In February 2024, Silvia Ferrara and colleagues published radiocarbon dates for four rongorongo tablets in a Rome convent. One tablet's wood dates to 1493–1509, more than 200 years before Europeans reached Rapa Nui. If the wood was inscribed soon after felling, rongorongo is an independent invention of writing. The script survives on fewer than 30 objects, which is too few for statistical decipherment. For the Indus signs, the Tamil Nadu prize of $1 million announced in January 2025 has brought public attention but no accepted reading.

The encoding of scripts for computers is close to complete for living scripts and now advances into historical ones. Unicode 17.0 adds four new scripts: Sidetic, Tolong Siki, Beria Erfe and Tai Yo. Sidetic is an ancient Anatolian script; the other three serve living communities in India, Africa and South-East Asia. With 172 scripts and 159,801 characters, the standard grew by 3.0 percent in one release (Calculation 22.9).

Reading carbonised texts became possible through imaging. On 25 June 2026 the Vesuvius Challenge, with the European Synchrotron in Grenoble and Diamond Light Source in Oxfordshire, presented new texts from three Herculaneum scrolls. They include nearly 1.5 metres of continuous text from PHerc. 1667 and the title "Philodemus, On Gods, Book 8" in PHerc. 139. Machine learning detected the ink in micro-CT volumes; papyrologists read the letters. The archaeology chapter (24.2) takes this up.

The theoretical frontier is the cognitive cost of scripts. Research on reading shows that alphabets with shallow orthographies (Finnish, Italian) are learned faster than deep ones (English). Featural scripts encode articulation in the shape of the letter. The debate is whether script type changes how readers process language, or only how fast they learn it. No consensus exists.

Interpretation.

1. *Strand.* I is the order of signs in a written line. P is the sign inventory of the script. A node is a sign used more than once.
2. *Passage and trit.* The trit rule applies to sign sequences exactly as to words (22.1). At the level of letters, repetition is so common that almost every later passage scores 0, because the next letter is rarely the same as after the first use. The rule is too coarse for alphabets. It is more apt for logograms, where a repeated sign with the same neighbour marks a fixed phrase.
3. *Closure.* Two places in writing contain the digit rule of T1 itself, transposed to another radix. Yoruba numerals form 15 to 19 by subtraction from twenty: 15 is "five from twenty", 16 is $20 - 4$, up to $19 = 20 - 1$. The units 1 to 4 are added to ten. This is exactly the balanced residue of T1 at radix ten: a digit d with $|d| \leq 5$ and the carry passed to the next ten (Calculation 22.6). Roman numerals do the same in IV and IX, with a different cut. Neither is radix 3. They show that the balanced digit, with its carry, is a human writing practice, not only an arithmetic device.
4. *Scale.* Semitic roots are words of three consonants: Arabic *k-t-b*, "write", gives *kataba*, *kātib* and *maktab*. With 28 consonants there are 21,952 open three-consonant words and 7,336 rotation classes (T13). Of the open words, 19,656 have three different consonants, 2,268 have two equal and only 28 have all three equal (Calculation 22.8). The window of three is grammatical fact in these languages. But nothing in the root system selects by a sum mod 3: consonants have no residue. Alphabet sizes (22, 24, 26, 28, 33) lie on the ladder of T15 only for Greek's 24 (Calculation 22.5).
5. *Test.* A test that could fail: in logographic or logo-syllabic corpora (cuneiform, Chinese, Maya), the closed-window rate of the trit rule should exceed its first-order Markov null by more in formulaic genres (royal titles, legal formulas, liturgy) than in letters and accounts. The prediction is the same as for Homer, stated in advance for scripts. If formulaic and non-formulaic genres show the same excess, the reading fails. Not run here: no tokenised cuneiform corpus was available in this session.

Consequences. Writing systems offer two real structures. Neither is the full closure rule. The balanced digit with carry appears in Yoruba and Roman numerals, at radix 10 or 20. That supports the claim of III.2 that the balanced form needs no external convention for sign and rounding, as a human design choice. It does not support radix 3. The window of three appears in Semitic roots as a fact of morphology. But the Foundations' closure needs digits that add. Consonants do not add, so T2 does not apply. Only T13's count of open words against rotation classes applies, and it says nothing about which roots exist.

The cost of a stronger reading would be to invent a residue for consonants. That would be re-description, not a link. The interpretation adds nothing to the decipherment of the Indus signs or rongorongo. Their difficulty is too few texts, not a missing theory of form.

What the chapter does add is a mechanical test, the same as in 22.1, that can be run on any tokenised corpus of any script. Its first run on Greek and Latin gave a small

positive result only for the most formulaic text. That makes it worth running on cuneiform, where royal titulary is as formulaic as Homer.

The Yoruba system also shows the carry at a second scale. Fifty is said as three twenties less ten, and seventy as four twenties less ten. The balanced digit is applied once at radix ten and again at radix twenty. That is the structure of T1 (ii): what a position does not keep passes whole to the next position. It is not radix three, and the Foundations do not predict which radix a culture chooses.

The checklist was run for writing. T12 (unique balanced address) applies in the sense just shown, at other radices. T13 applies to roots. T4 gives the classes of root shape. T15 fits one alphabet in five. T6, T8–T11, T14, T16–T24 have no object in writing systems.

Grade. A for the balanced digit with carry in Yoruba (and Roman subtractive) numerals as published fact, at radix ten or twenty and not three. B for the formulaic-genre test on logographic corpora, stated but not run. C for writing systems as a whole.

Chapter 23 — The Arts

The arts are culture's meetings: works made to be exchanged between maker and audience and returned as response. Music holds the strongest link in this book. The closure depths of T17 are the working step counts of tuning: 12, 41 and 53. The Pythagorean comma of T18 is a term every music student learns. The ladder of T15 fits the lengths of the transformation cycles of neo-Riemannian theory, and the Tonnetz is the Eisenstein lattice of T14. Music is grade A. Rhythm and poetic form give grade A for specific structures (rotation classes of rhythms, the order of the sestina) and B for one stated test. Visual art and the performing arts remain readings, grade C, with one topological exception: the Balinese candle dance performs the 4π return of T21.

23.1 Music

History. The mathematics of music began with ratios. Greek tradition credited Pythagoras with the discovery that consonant intervals sound when string lengths stand in simple ratios: the octave 2:1, the fifth 3:2, the fourth 4:3. Philolaus and Archytas in the fifth and fourth centuries BCE worked out the arithmetic. Stack twelve fifths and fold them by octaves, and you arrive near the starting note but not on it. The surplus is the Pythagorean comma, $3^{12}/2^{19}$. Ptolemy's *Harmonics* (second century CE) compared rival tunings with numbers and the ear.

China reached the same arithmetic independently. The *Huainanzi* (c. 122 BCE) computed the twelve pitch pipes by alternately taking two-thirds and four-thirds of a length. The Han scholar Jing Fang (78–37 BCE) noticed that a chain of 53 fifths comes very close to 31 octaves. He extended the chain to 60 fifths and found that after 53 the new pitches nearly repeated old ones. Nicholas Mercator computed the same 53-step approximation precisely in the seventeenth century; the 53rd part of the octave is now called the Holdrian comma.

Western practice moved from Pythagorean tuning to just intonation (Zarlino, 1558) and then to temperaments that spread the comma. Meantone tuning narrowed the fifths so that thirds were pure. Well-temperaments in the seventeenth and eighteenth centuries let every key be playable with different colours. Equal temperament, with twelve equal semitones, was described numerically by Zhu Zaiyu in China (1584) and by Simon Stevin in Europe around the same time. It became the default for keyboards in the nineteenth century.

Theory followed. Jean-Philippe Rameau's *Traité de l'harmonie* (1722) grounded harmony in a fundamental bass and the major triad. Leonhard Euler's *Tentamen novae theoriae musicae* (1739) drew a lattice of fifths and thirds. Hugo Riemann in the 1880s organised harmony around three functions and dual transformations of triads, and drew a refined *Tonnetz*. Arnold Schoenberg's twelve-tone method (1923) and Allen Forte's pitch-class set theory (1973) made music an object of combinatorics. David Lewin's *Generalized Musical Intervals and Transformations* (1987) and Richard Cohn's

papers of 1996–1998 founded neo-Riemannian theory, in which triads are moved by three involutions, P, L and R.

Rhythm had its own mathematics. In 2005 Godfried Toussaint showed that the Euclidean algorithm, which spreads k onsets as evenly as possible over n pulses, generates many traditional rhythms. Ethnomusicology documented “aksak” metres in the Balkans and Turkey, built from cells of two and three pulses. Constantin Brăiloiu named and described them in 1951.

The questions left are whether consonance is nature or culture, why so many traditions converge on a few tunings, and whether there are universals of musical structure at all.

Current state. Cross-cultural music science has become quantitative. Savage, Brown, Sakai and Currie (2015) analysed a global sample of recordings. They found no absolute universals but 21 statistical universals, features present in most recordings in every region, including two- and three-beat subdivisions. Mehr and colleagues (2019) drew on 315 societies, about 5,000 ethnographic descriptions of song from 60 cultures and 118 recordings from 86 cultures. A classifier recognised lullabies, healing songs, love songs and dance songs even in cultures it had not seen.

Rhythm has been measured directly in many populations. Jacoby and colleagues (2024) ran iterated tapping experiments with 39 groups in 15 countries. Every group showed a bias toward rhythms with simple integer ratios, but which ratios differed by culture. A US sample favoured 1:1:2 and 2:3:3. Other groups favoured ratios common in their own music.

Tuning practice outside the twelve-step system remains alive. In Turkish makam theory the octave is divided into 53 equal Holdrian commas, of which 24 are used in practice; a whole tone is nine commas. This is Jing Fang’s number in active use.

The industry has changed faster than the theory. IFPI reports global recorded-music revenue of US$31.7 billion in 2025, up 6.4 percent, the eleventh year of growth. Streaming passed US$22 billion, 69.6 percent of the total, and paid subscriptions reached 837 million users. Generated music is now the majority of new uploads on one platform. On 21 July 2026 Deezer reported that AI-generated tracks passed 50 percent of daily uploads in June 2026, nearly 90,000 a day. They account for only 1–3 percent of streams, and up to 85 percent of the streams on fully generated tracks in 2025 were fraudulent. At that rate, generated uploads would pass 32 million a year (Calculation 23.8).

The open debates are the biological basis of consonance (roughness, harmonicity or exposure), the universality of tonality, and the status of authorship when the composer is a model.

Interpretation.

1. *Strand.* I is the chain of stacked intervals: fifth after fifth. P is the set of pitch classes. The strand maps the k-th step of the chain to the pitch class it reaches.
2. *Passage and trit.* A passage is a return near a pitch already reached. A fifth multiplies frequency by 3/2: one step on the ×3 ladder. Folding by octaves divides

by 2: one step on the ×2 ladder (T15). After k fifths the chain is n octaves away, plus a residue.

3. *Closure.* The chain closes best when k fifths come closest to n octaves. That is the definition of a closure depth (T17): the denominators of the convergents of $\log_2 3$. The closure depths are 1, 2, 5, 12, 41, 53, 306. The step counts 5 (pentatonic), 12, 41 and 53 are tunings in use or in theory. Their fifths miss 3:2 by +18.045, −1.955, +0.484 and −0.068 cents (Calculation 23.1). The miss at depth 12 is the Pythagorean comma, 23.460 cents (T18; Calculation 23.3). The comma never vanishes (T16). In carry language: twelve fifths leave a residue at the octave scale that no finite chain removes.
4. *Scale.* The neo-Riemannian transformations P, L and R are involutions, each a meeting (x, −x): applied twice, each returns the triad (Calculation 23.5). Their alternating cycles from C major have lengths 6 (PL, the hexatonic cycle), 8 (PR, the octatonic cycle) and 24 (LR, all triads). All three lie on the ladder $2^a \cdot 3^b$ (T15). The Tonnetz is the Eisenstein lattice Z[ω] of T14. With step 1 a major third and step ω a minor third, 1 + ω is a fifth, and the up and down triangles are the major and minor triads. Twelve pitch classes are the quotient of the lattice by a sublattice of index 12. Its shortest vector has norm 9: three major thirds make an octave (Calculation 23.6). Rhythms counted up to rotation are the necklaces of T13. Three onsets in eight pulses have 56 placements but 7 rotation classes, one of them the tresillo (Calculation 23.4).
5. *Test.* Stated as a prediction: any fifth-based equal tuning a culture adopted for its closure of the circle of fifths has a closure depth as its step count, or a multiple of one used for other reasons. 12, 53 and 5 satisfy it. 24 (Arabic quarter tones) is 2 × 12 and inherits its fifth from 12. Refutation would be a widely used fifth-based system of another size adopted for closure. The candidates are 17, 19 and 31. The 19- and 31-step systems are based on thirds, not fifths: their fifths miss by −7.2 and −5.2 cents, worse than 12. The 7-step Thai and 5-step Javanese slendro scales do not close on fifths.

Consequences. Music is the one art in which the two ladders of the Foundations are the craft itself. This is not a reading. Jing Fang's 53 and the Western 12 are best approximations in the exact sense of T17 and Lagrange's theorem. Musicians found them by ear and calculation centuries before continued fractions had a name.

The Foundations add one sharp distinction the field uses loosely. A tuning theorist often lists 12, 19, 22, 31, 41 and 53 together as "good" equal temperaments. These are mixed by two different criteria. Best approximations of the second kind are the closure depths 1, 2, 5, 12, 41, 53. Best approximations of the first kind add 3, 7 and 29, which approximate the ratio rather than the residue (Calculation 23.2). Neither list contains 19, 22 or 31: those are chosen for the third (5:4), not the fifth. The theory predicts that systems chosen for fifths sit on the second-kind list. That is what Calculation 23.1 shows.

The Tonnetz identification is also a fact, not a reading: its triangular lattice is Z[ω]. The norm form $a^2 - ab + b^2$ is the norm of T14. Major and minor triads are the two orientations of an elementary triangle, the closed ring $1 + \omega + \omega^2 = 0$ of three steps.

Cohn’s hexatonic and octatonic cycles come out as meeting sequences on this lattice. Their lengths 6, 8 and 24 lie on the ladder because the lattice is cut at index 12, itself $2^2 \cdot 3$.

The cost is that music already knew all of this. The Foundations add a single frame that holds the comma, the step counts, the Tonnetz and the rhythm necklaces together. They also add a falsifiable selection rule for fifth-based tunings. What music adds to the Foundations is evidence that the closure depths are found by human practice wherever the problem of 2 against 3 is posed.

The full checklist was run for music. T1 and T7.1 give the carry reading of the comma. T13 gives rhythm necklaces. T14 gives the Tonnetz. T15 gives the cycle lengths. T16–T18 give tuning. T7 gives P, L and R as meetings. T4 applies to triads read as windows of three pitch classes only by an assignment of residues, which the chapter does not make. T8–T11, T19–T20 and T22–T24 have no object in music theory. T21 belongs to performance (23.4), not to sound.

Grade. A for tuning (closure depths 5, 12, 41, 53 and the comma, Jing Fang c. 50 BCE; Turkish 53-comma theory), for the Tonnetz as $Z[\omega]$ and for neo-Riemannian cycle lengths on the ladder. A for rhythm necklaces (Toussaint 2005). B for the stated selection rule for fifth-based tunings.

23.2 Visual arts

History. The oldest dated images are cave paintings. Hand stencils and animal figures in the caves of Sulawesi, Spain and France date back tens of thousands of years; the paintings of Chauvet are more than 30,000 years old. Whether these were art, ritual or record is unknown, but they show that making images is as old as modern humans.

The mathematics of images began with proportion and symmetry. Egyptian artists drew on square grids that fixed the proportions of the human figure. Greek sculptors wrote canons of proportion; Polykleitos’s *Canon* (fifth century BCE) survives only in fragments. Islamic ornament from the tenth century onward developed geometric patterns of great complexity. Builders used strapwork drawn from a small set of tiles, the girih. Lu and Steinhardt argued in 2007 that by the fifteenth century these tilings combined self-similar transformations into nearly perfect quasi-crystalline Penrose patterns, five centuries before Penrose.

Renaissance Florence made space mathematical. Filippo Brunelleschi demonstrated linear perspective around 1415. Leon Battista Alberti codified it in *De pictura* (1435), and Piero della Francesca wrote a treatise on it. Albrecht Dürer brought the method north and wrote on proportion. Giorgio Vasari’s *Lives* (1550) founded art history as biography and progress toward naturalism.

Modern theory turned to perception. Heinrich Wölfflin (1915) described five polarities of style, such as linear against painterly. Erwin Panofsky’s iconology (1939) read images as documents of culture. Gestalt psychology studied grouping and figure–ground, and Rudolf Arnheim (1954) applied it to art. Modernism broke representation. Cézanne reduced nature to cylinder, sphere and cone. Picasso and Braque’s cubism (from 1907) showed several views at once. Kandinsky and Mondrian made abstraction

a programme. Marcel Duchamp's *Fountain* (1917) moved the definition of art from the object to the institution.

Colour theory has a triadic history. Isaac Newton's *Opticks* (1704) arranged spectral colours in a circle. Thomas Young (1802) proposed three kinds of receptor in the eye, and Hermann von Helmholtz developed the theory. Ewald Hering (1878) proposed three opponent pairs: red–green, blue–yellow and black–white. Both were right at different stages of vision. Painters' primaries (red, yellow, blue) and printers' (cyan, magenta, yellow) are threefold for the same reason: human colour vision has three cone types.

Photography (Niépce and Daguerre, 1820s–1839) and film (the Lumière brothers, 1895) took over the task of recording appearances. That pushed painting toward what a camera could not do. In the second half of the twentieth century, art moved into installation, performance, conceptual work and video. The computer entered early: Georg Nees, Frieder Nake and A. Michael Noll exhibited plotter drawings generated by algorithms in 1965. Harold Cohen's program AARON drew from 1973 for four decades. Generative adversarial networks (2014) and diffusion models (from 2020) brought image generation to everyone.

Two lines of geometry run through this history. One is proportion: the golden section in Renaissance treatises (Luca Pacioli's *De divina proportione*, 1509, illustrated by Leonardo) and Le Corbusier's Modulor (1948). The other is symmetry. Hermann Weyl's lectures *Symmetry* (1952) showed that the ornament of every culture can be classified by the same groups: 7 frieze groups and 17 wallpaper groups. M.C. Escher, advised by the crystallographer George Pólya's paper of 1924, explored all of them in his tessellations.

The questions left are what makes an image beautiful or compelling, whether aesthetic response has universals, and what counts as art.

Current state. The market measures one dimension. The Art Basel and UBS report for 2026 estimates global art sales of US$59.6 billion in 2025, up 4 percent, after two years of decline. Dealers sold US$34.8 billion, public auctions US$20.7 billion (up 9 percent) and private sales just under US$4.2 billion. Transactions reached 41.5 million. Online sales fell to US$9.2 billion, 15 percent of the market, their lowest level since 2019. Women made up 45 percent of artists represented by dealers but 37 percent of sales by value.

The legal status of generated images was settled for now in the United States. The Copyright Office's January 2025 report concluded that "prompts alone do not provide sufficient human control to make users of an AI system the authors of the output". On 2 March 2026 the Supreme Court declined to hear *Thaler v. Perlmutter*. That left in place rulings that human authorship is a "bedrock requirement of copyright" for the image *A Recent Entrance to Paradise*. In the European Union, the AI Act's transparency rules for generated content apply from 2 August 2026. Providers of generative systems already on the market have until 2 December 2026 to comply with Article 50(2).

Empirical aesthetics is growing. Studies of eye movements, preference and fractal dimension in images, and computational analysis of large collections of artworks, test

claims made by theorists for centuries. The debate is between universal biases (for symmetry, for certain fractal ranges) and learned taste.

Interpretation.

1. *Strand.* I is the eye's path over an image (a scanpath of fixations), or the maker's sequence of strokes. P is the set of regions of the image.
2. *Passage and trit.* A passage is a return of the gaze to a region already fixated. The trit rule would score whether the gaze leaves the region toward where it went the first time (1), back where it came from (T), or elsewhere (0). This is computable from eye-tracking data. It has not been computed in this chapter.
3. *Closure.* The clearest closures in visual art are symmetries. A pattern with threefold rotation closes after three turns. Periodic ornament obeys the crystallographic restriction: in a periodic plane pattern, only rotations of order 1, 2, 3, 4 and 6 occur, the orders n for which $2\cos(2\pi/n)$ is an integer (Calculation 23.10). All five orders are on the ladder $2^a \cdot 3^b$ (T15). Islamic girih patterns with tenfold stars lie outside the restriction. Lu and Steinhardt's finding is that they are quasi-periodic, not periodic. That is a pattern that never closes, like the comma: the miss between two incommensurable rotations never vanishes (T16).
4. *Scale.* Colour vision gives a register of three with opponent pairs. Hering's opponent channels are meetings (red against green, blue against yellow), and the three cone types are a ring of three. The analogy to T7 is structural only. Cone sensitivities are continuous curves, not trits.
5. *Test.* For scanpaths: if closure in the Foundations' sense describes looking, then images rated as balanced or harmonious should produce scanpaths with a higher closed-window rate, against a first-order Markov null, than images rated as unbalanced. The null is the same as for Homer (22.1). If the rates do not differ, the reading fails. Not run here: it needs eye-tracking data.

Consequences. Two results are real and both belong to geometry, not to art theory. The crystallographic restriction puts all periodic ornament on the ladder of T15. That is a theorem of the plane (Chapter 6 treats it). The arts inherit it whenever they make periodic patterns: every frieze and wallpaper pattern a culture has ever made obeys it. Quasi-periodic Islamic patterns show the other side of the Foundations. When two rotation orders are incommensurable, the closure is approached but never reached, as in T16. That the girih makers produced such patterns is a historical fact. The link to T16 is a correspondence of form, graded C, because no tiling is a ratio of powers of 2 and 3.

The scanpath test is the only new contribution. It is cheap to run on existing eye-tracking datasets. A positive result would say that "balance" in an image has a mechanical signature in how the eye returns. A null result would remove the Foundations from visual aesthetics.

For the market, copyright and generated images, the theory adds nothing. These are legal and economic questions, treated in Chapters 19 and 21.

The checklist for visual art: T15 applies through the crystallographic restriction; T16 applies as a correspondence of form to quasi-periodic tilings. T19 (knots) applies to

interlace ornament, Celtic and Islamic, which draws knots and links. Interlace is real knot theory, but the Foundations add nothing to it beyond the fact, already known, that the trefoil is the simplest knot. The other theorems have no object.

A note on what the scanpath test needs. The trit rule requires a node, a place visited twice, and the step after each visit. Eye-tracking gives exactly this once the image is divided into regions. The baseline is T6: for independent trits with the observed law, the closed-window rate is fixed by the law alone. Any excess must come from order, and the first-order Markov control separates order of pairs from order of triples. The method of Chapter 22 carries over without change. Visual art is the domain where this book's mechanical test is cheapest to run and least likely to be confused with repetition of words.

Grade. A for periodic ornament on the ladder (crystallographic restriction, a theorem of geometry applied to art). B for the stated scanpath test. C for painting, sculpture and generated art as a whole.

23.3 Literature

History. Literature began as performance. Homeric epic, Sanskrit epic, the Hebrew scriptures and the oral traditions of every continent were composed and transmitted aloud long before they were written. Milman Parry showed in 1928–1930 that Homeric verse is built from formulas fitted to the metre: "swift-footed Achilles", "rosy-fingered Dawn". Parry and Albert Lord recorded South Slavic singers in the 1930s, and Lord's *The Singer of Tales* (1960) generalised the oral-formulaic theory.

Greek criticism gave literature its first theory. Aristotle's *Poetics* (c. 335 BCE) analysed tragedy as an imitation of an action with beginning, middle and end, built on reversal and recognition. Horace's *Ars Poetica*, the Sanskrit theory of *rasa* in Bharata's *Nāṭyaśāstra*, and Liu Xie's *Wenxin Diaolong* (c. 500 CE) are independent poetics. Verse forms came with rules of number. Greek and Latin epic used the dactylic hexameter: six feet of dactyls (long–short–short) or spondees (long–long). The troubadour Arnaut Daniel (late twelfth century) invented the sestina: six stanzas whose six end-words return in a fixed permutation.

The study of narrative structure became systematic in the twentieth century. Axel Olrik's "epic laws of folk narrative" (1908–1909) included a law of three ("frequent repetition, usually in threes") and a law of two to a scene ("only two characters act at a time"). Vladimir Propp's *Morphology of the Folktale* (1928) found 31 functions in a fixed order across a hundred Russian wonder tales. The Russian formalists (Shklovsky, Eikhenbaum) separated plot from story. Structuralists from Lévi-Strauss to Greimas and Barthes read narrative as a combination of oppositions.

The Oulipo group (founded 1960 by Raymond Queneau and François Le Lionnais) treated constraints as engines of writing. Queneau's note on the sestina (1963) asked for which n the sestina's spiral permutation can be extended to n stanzas. The answer is a number-theoretic sequence now called the Queneau numbers. Franco Moretti's "distant reading" (2000, 2005) and the rise of digital humanities brought statistics to the literary canon.

Three further lines matter for counting. Prosody measured verse in units: the Sanskrit metrists counted short and long syllables, and Pingala's *Chandaḥśāstra* (before the common era) enumerated metres in a way later read as binary counting. The count of metres of a given length in morae follows the Fibonacci numbers, as Indian scholars knew by the sixth to twelfth centuries (Virahāṅka, Hemacandra). Chinese regulated verse fixed tonal patterns in lines of five or seven syllables. The Japanese haiku counts 5, 7 and 5 morae.

The second line is the novel. From *Don Quixote* (1605, 1615) to the nineteenth-century realist novel and modernist experiment (Joyce's *Ulysses*, 1922), prose fiction became the dominant literary form. Mikhail Bakhtin described it as a meeting of voices, dialogic rather than single-voiced.

The third is reader response. Wolfgang Iser and Hans Robert Jauss in the 1970s located meaning in the act of reading. Empirical studies of reading began to test what texts do to readers' attention, memory and empathy.

The questions left are whether narrative has universal structures, how form produces meaning, and what reading does to a reader.

Current state. The highest literary honour of 2026 went to the classical tradition. On 8 October 2026 the Swedish Academy awarded the Nobel Prize in Literature to the Canadian poet Anne Carson, 76, "for her bold and inventive oeuvre that, in playful dialogue with the classical tradition", has created new forms for contemporary literature. The prize is 12 million Swedish crowns. Carson is a classicist and translator of Sappho and the Greek tragedians. The choice ties the current frontier of literary form to the texts analysed in Chapter 22.

Computational literary studies now has its own journals and conferences. The main methods are stylometry, topic models and, since 2023, large language models used as readers. The field's open debate is the same as in linguistics: whether models that have learned the statistics of text have learned anything about literature, or only about frequency.

Classical texts themselves are growing. The Herculaneum results of June 2026 recovered a previously unknown ethical treatise in PHerc. 1667 and the eighth book of Philodemus's *On Gods*, with more than 600 scrolls still unopened. In epigraphy, DeepMind's Aeneas, published in *Nature* in July 2025, was trained on more than 176,000 Latin inscriptions. It dates inscriptions to within 13 years of historians' ranges and attributes 72 percent of them to the right one of 62 provinces.

Generated text is the economic frontier, as generated music is in 23.1. The legal position is the same: in the United States, text produced from prompts alone has no human author.

Interpretation.

1. *Strand.* I is the running order of a text; P is its vocabulary, as in 22.1. For verse, a second strand runs over the metrical positions: I is the sequence of syllables, P the set of syllable weights (long, short).

2. *Passage and trit.* The trit rule on words was run in Chapter 22 (Calculations 22.1–22.2). In metre, a dactyl is long + short + short and a spondee is long + long. Counted in morae (short = 1, long = 2), both equal 4. The hexameter's five free feet allow $2^5 = 32$ patterns, a ladder number on the ×2 side; with the sixth foot's two forms, 64 (Calculation 23.9). Both feet carry four morae, 2^2, so the hexameter keeps a constant length in morae per foot.
3. *Closure.* The sestina is a closed permutation. The six end-words move by the permutation 615243 and return after exactly six stanzas: its order is 6 (Calculation 23.7). This is a ring in the sense of T9: a cyclic configuration that returns to its start. The Queneau numbers are the n for which the spiral permutation on n words has order n. Below 100 there are 31 of them. Seven are ladder numbers: 1, 2, 3, 6, 9, 18 and 81. The others, such as 5, 11 and 14, are not. The sestina's 6 is on the ladder; the theory has no explanation for the rest.
4. *Scale.* Olrik's laws state the elementary closures of narrative in prose: two to a scene (a meeting) and repetition in threes (a ring). Three trials, three brothers, three wishes: the third closes the sequence and carries the story to the next scale, as 1 + 1 + 1 = 10 carries (T7.1). Olrik's law of final position ("emphasis falls on the last of several similar situations") says that the third element is the one that closes. This reading matches Olrik's text; it is still a reading.
5. *Test.* A test that could fail: in a coded corpus of folktales (the Aarne–Thompson–Uther index codes motifs), episodes repeated in threes should be followed by a change of scene more often than episodes repeated twice or four times. A ring closes and passes on; a meeting or an open run does not. If threefold and fourfold repetitions end scenes equally often, the reading fails. Not run here.

Consequences. Literature gives one structure as published fact: the sestina's permutation of order 6 and Queneau's number-theoretic generalisation (OEIS A054639). These are combinatorics of closed cyclic arrangements, the subject of T9 and T13. The Foundations add no new Queneau numbers. Their ladder test fits a minority of them.

Olrik's laws are the strongest match in narrative theory. "Two to a scene" and "repetition in threes" are the meeting and the ring of T7, stated by a folklorist in 1908 for reasons of his own. That is the pattern of grade A. But Olrik's laws are tendencies, not counts, and the numbers two and three are too small to make a match unlikely by chance. So the match is reported and graded C; the test in point 5 could raise it to B.

The cost of pressing further is clear: almost any narrative can be divided into twos and threes after the fact. The test avoids this by fixing in advance what counts as a repetition (the motif codes of an existing index) and what counts as closure (a change of scene).

The checklist for literature: T9 and T13 apply to the sestina; T15 to Queneau numbers (failing for most); T7 and T7.1 to Olrik's laws as a reading; T2–T6 and T24 to the trit rule on texts, run in Chapter 22. T11, T14, T16–T23 have no literary object.

The literary result of Chapter 22 belongs here too. The trit rule run on the Iliad and Odyssey gave a closed-window excess of 2.2 and 4.3 standard deviations over a first-

order control. The Aeneid, a literary epic composed in writing, gave none (Calculation 22.2). If the excess marks oral-formulaic composition, other oral corpora should show it and written imitations should not. The South Slavic epics recorded by Parry and Lord, Beowulf and the Old French chansons de geste are candidates. The prediction can fail cleanly: an excess in Virgil-like written epic, or none in recorded oral epic, would refute it.

Grade. A for the sestina and Queneau numbers as closed cyclic permutations (published combinatorics). B for the stated folktale-repetition test. C for Olrik's laws read as meetings and rings, and for literature as a whole.

23.4 Performing arts

History. Theatre, dance and ritual performance are older than writing. Greek tragedy grew out of choral performance at the festival of Dionysus in Athens in the sixth century BCE. Thespis is credited with the first actor separate from the chorus, Aeschylus with the second and Sophocles with the third. Aristotle records this in the *Poetics*. The limit of three speaking actors held through the classical period. In India, Bharata's *Nāṭyaśāstra* (c. 200 BCE–200 CE) codified drama, dance and music in one treatise. Japanese Noh took its classic form under Zeami around 1400. Kabuki and the puppet theatre of bunraku followed in the seventeenth century.

European theatre after the Renaissance revived classical rules. The neoclassical "three unities" of action, time and place were drawn from Aristotle by Italian and French critics. Shakespeare's company built the Globe in 1599. Molière and Racine worked under the unities at the court of Louis XIV. The nineteenth century brought realism (Ibsen, Chekhov) and the director as a central figure (Stanislavski at the Moscow Art Theatre from 1898). The twentieth century broke the frame: Brecht's epic theatre, Artaud's theatre of cruelty, Beckett's minimal plays, Grotowski's poor theatre.

Dance gained notations. Raoul-Auger Feuillet published a notation for Baroque dance in 1700. Rudolf Laban developed Labanotation in 1928, still used to record choreography. Ballet's codified positions and steps date from the French academy of 1661. Modern dance (Duncan, Graham, Cunningham) rejected codification, then made new systems.

Music and dance meet in rhythm. West African drumming layers patterns of two and three against each other, the hemiola and the 12/8 bell patterns. The Euclidean rhythms of 23.1 include several of these bell patterns. The Balkan aksak metres, such as 7/8 = 2 + 2 + 3 and 9/8 = 2 + 2 + 2 + 3, are danced as well as played, and the dancers' steps follow the long and short cells (Aksak).

Change ringing is a performing art built on mathematics. English bell-ringers in the seventeenth century rang changes: permutations of the order in which bells strike. Fabian Stedman's *Tintinnalogia* (1668, with Richard Duckworth) and *Campanalogia* (1677) set out the methods. An "extent" rings every possible order once, n! changes for n bells: 720 on six, 5,040 on seven. The first true peal of 5,040 changes of Plain Bob Triples is generally dated to 2 May 1715 at St Peter Mancroft, Norwich. Ringers solved

problems of group theory, Hamiltonian paths through the symmetric group, long before mathematicians named them.

Dance contains a known piece of topology. In the Balinese candle dance, a dancer turns a cup or candle held on the open palm through a full circle and then a second. The arm untwists only after 720 degrees. The plate trick, popularised by Paul Dirac, demonstrates that the rotation group SO(3) has a fundamental group of order 2.

Ritual performance in many traditions follows cycles. Calendar festivals repeat with the year (Chapter 24 treats the calendars). Liturgy repeats daily or weekly. Processions return to their starting point. Shadow puppetry in Java and Bali (wayang) performs the epics through the night in a fixed order of three parts, each with its own musical mode. Such tripartite schemes are documented in the ethnography of wayang. Whether they share a cause with Olrik's law of three is not known, and the chapter does not claim it.

Circus, opera, music theatre and stand-up have their own histories. Opera arose in Florence around 1600 from an attempt to revive Greek drama. The Broadway musical took its modern form with *Oklahoma!* (1943). Each tradition fixes forms of repetition: the aria's da capo (A–B–A), the refrain, the encore.

The questions left are what makes performance different from text, how audiences and performers synchronise, and how live performance survives recording and now generation.

Current state. Broadway's season statistics give one measure of live theatre. For 2024–25 the Broadway League reports a gross of US$1,893 million, attendance of 14.66 million and 43 new productions. The gross per admission is about US$129 (Calculation 23.8).

Research on performance now measures the coupling of movement between performers. The cross-cultural rhythm work of Jacoby and colleagues (23.1) found integer-ratio biases in every one of 39 groups. That is the perceptual ground on which dance and ensemble performance coordinate.

Change ringing continues as a living tradition with its own records and method collections. Its mathematics, the construction of extents as paths through permutation groups, is now taught as an example of applied group theory.

The open debates are about authenticity and mediation: what is lost when performance is streamed or recorded, and whether generated performance (synthetic voices, digital actors) is performance at all. Performers' unions have negotiated contract limits on digital replicas, and those rules are now tested in practice.

Interpretation.

1. *Strand.* I is the time of the performance. P is the set of positions, states or orderings the performers pass through: places on stage, positions of the body, or the order of bells.
2. *Passage and trit.* A passage is a return to a state already taken. In change ringing, each bell moves by swaps of adjacent places: meetings $(x, -x)$. A plain hunt on n bells returns to rounds after $2n$ changes.

3. *Closure.* The candle dance is a closed loop that needs two turns. A rotation of 2π returns the hand's position but not the configuration of the attached arm; 4π returns both. This is exactly T21: a strand attached to its environment returns after 4π, not 2π, because the unit quaternions double-cover the rotations. The dance performs the theorem with the arm as the attachment. The Greek limit of three speaking actors and Olrik's "two to a scene" are counts of three and two, but they are conventions, not closures.
4. *Scale.* Extents of n bells have n! changes. Only 3! = 6 and 4! = 24 lie on the ladder $2^a \cdot 3^b$. From five bells on, n! contains the prime 5 and leaves the ladder (Calculation 23.7). The plain-hunt periods 2n are on the ladder only for n = 3, 4, 6 and 8 among n = 3 to 8 (Calculation 23.7). Change ringing therefore shows the ladder of T15 failing as a general law for performance structures. It holds only where the structure is built from twos and threes. Aksak metres are sums of twos and threes. That is the mixed merging of IX.4, not the equal blocks of T15: 2 + 2 + 3 = 7 is a sum, not a product, and lies off the ladder. The Foundations' own example (T15) of mixed merging gives 3 + 1 = 4 instead of 2 · 3 = 6. Aksak shows that dancers and musicians use mixed blocks freely. The ladder applies only to equal blocks: 6/8 as 3 + 3 or 2 + 2 + 2.
5. *Test.* For the candle dance and the plate trick none is needed. T21 is a theorem, and the dance is a physical demonstration of it that anyone can repeat. For ensemble performance, a test that could fail: in recorded group dances, the closed-window rate of the trit rule applied to positions should exceed its first-order null more in dances described by their tradition as "complete" or "closed" figures (such as a round returning to its start) than in improvised social dance. Not run here.

Consequences. The performing arts hold one real structure: the 720-degree return of the candle dance, which is T21 performed by the body. Physics teaches it as Dirac's belt or plate trick; Balinese dancers perform it as tradition. It is a published fact of topology, independent of this theory, so the grade is A for that structure. What the Foundations add is only the placement. The double cover of T21 sits in the same chain as the window rules. That is a statement about the Foundations, not about dance.

Change ringing gives the most useful negative result in the chapter. Its numbers are rich, exact and historical, and most of them lie off the ladder. The theory therefore does not predict the sizes of human combinatorial practices in general. It predicts only that where two and three are combined, the products stay on the ladder. That is arithmetic.

For theatre, dance as an art and the economics of live performance, the interpretation adds nothing. Olrik's two to a scene and the three actors of Greek tragedy are attractive matches to meetings and rings. They are conventions with practical causes, such as the number of masks and the shape of a scene, and they are graded C.

The checklist for performance: T21 applies exactly (the candle dance); T7 (meetings) to adjacent swaps in change ringing; T15 to extents and hunt periods, mostly failing; T9 to rounds returning to their start. The others have no object.

The comparison with change ringing is instructive in one more way. Ringers do not ring n! changes for a reason of closure but because "every order once" is the exhaustive

task. When ringers chose shorter compositions, they chose touches and peals by length (a peal must have at least 5,000 changes), not by a ladder number. So the arithmetic of the art is set by the task. A theory of closure would have to show that the task itself is chosen for closure. For change ringing it is not.

Grade. A for the candle dance as a performance of T21 (the double cover of SO(3), a theorem of topology). C for change-ringing counts on the ladder (they mostly fail it) and for theatre and dance as a whole. B only for the stated dance test, not yet run.

Chapter 24 — Memory of Culture

The memory of culture is its ring: the cycle that goes round and returns, through history kept, sites dug, calendars kept and rites repeated. Calendars hold the second-strongest link in this book. Every lunisolar intercalation cycle in long use is a convergent of the year-to-month ratio. The Saros is a convergent of the synodic-to-draconic ratio. The leap rules of the Islamic and Hebrew calendars spread their leap years in gaps of 2 and 3. That is grade A. One stated test was run and partly failed: the Gregorian 97/400 and the Antikythera planetary relations are not best approximations. Archaeology gives one link of the same kind, through tree-ring counting from a solar event (grade A for chronology, C beyond). History as a whole and religion and myth remain readings (grade C), with stated tests.

24.1 History

History. History as a discipline began with inquiry into causes. Herodotus (c. 430 BCE) called his work *historiē*, inquiry, and set out to explain why Greeks and Persians fought. Thucydides narrowed the method to what he could check and explained the Peloponnesian War by fear and power. Sima Qian's *Shiji* (c. 94 BCE) founded Chinese historiography with annals, tables, treatises and biographies. Ibn Khaldun's *Muqaddimah* (1377) proposed a cycle: desert groups with strong solidarity (*ʿasabiyya*) conquer cities, soften and are replaced in about three or four generations.

Modern history became a science of sources. Leopold von Ranke, in the 1820s and after, demanded that history be written from archival documents "as it actually was". The seminar method and source criticism spread from Berlin to every university. The nineteenth century also produced philosophies of history: Hegel's dialectic, Marx's succession of modes of production and Comte's three stages.

The twentieth century turned from events to structures. The Annales school, founded by Marc Bloch and Lucien Febvre in 1929, studied mentalities, prices, climate and demography. Fernand Braudel's *La Méditerranée* (1949) distinguished three speeds of time: the near-immobile time of geography, the slow cycles of economies and societies, and the fast time of events. Oswald Spengler (1918–1922) and Arnold Toynbee (1934–1961) wrote comparative histories of civilisations with life cycles. Professional historians criticised them for fitting facts to a pattern.

Quantitative history, or cliometrics, arose in the 1960s. Robert Fogel and Douglass North received the Nobel Prize in economics in 1993 for it. Social history, microhistory (Carlo Ginzburg's *The Cheese and the Worms*, 1976), the history of gender and global history widened the field. Postmodern critics, such as Hayden White in *Metahistory* (1973), argued that historians impose narrative forms (romance, tragedy, comedy, satire) on the past.

Cycle theories persisted outside the mainstream: Kondratiev's long waves of the economy (1925), generational theories and secular cycles. In 2003 Peter Turchin

proposed cliodynamics, a mathematical history of population, elite numbers and instability. His secular cycles of about two to three centuries are tested against data.

History also learned to count its own time. The Christian era, counted from the incarnation, was proposed by Dionysius Exiguus in 525 and spread through Bede in the eighth century. The year zero was never part of it: 1 BCE is followed by 1 CE. Joseph Scaliger's *De emendatione temporum* (1583) joined the chronologies of Babylon, Egypt, Greece, Rome and the Hebrew Bible into one framework. His Julian Period of 7,980 years is the product of three cycles: the 28-year solar cycle, the 19-year lunar cycle and the 15-year indiction. Astronomers still number days from its start (the Julian Day). Scaliger's choice is a classic use of the least common multiple of independent cycles to make a single ring. It is the same construction as the Maya Calendar Round (24.3).

The questions left are whether history has laws or only patterns, whether cycles are real or imposed, and how far narrative shapes the facts it reports.

Current state. Cliodynamics has built the largest structured database of past societies. Seshat: Global History Databank codes hundreds of polities over ten thousand years. Its most visible episode was a correction. A 2019 *Nature* paper used Seshat to claim that moralising gods appear only after societies pass a "megasociety threshold" of about one million people. A re-analysis by Beheim, Muthukrishna and Spicer showed that 61 percent of the moralising-god observations were coded "unknown" and treated as absence. Correcting this reversed the result: moralising gods appear before rises in social complexity. The original authors retracted the 2019 paper. As Beheim put it, "how you deal with the data you don't have can sometimes be more important than the data you do have."

Machine learning has entered source criticism. DeepMind's Aeneas, published in *Nature* in July 2025, was trained on more than 176,000 Latin inscriptions. It dates an inscription to within 13 years of historians' ranges, assigns 72 percent to the correct one of 62 Roman provinces, and restores gaps of up to ten characters with 73 percent top-20 accuracy. Historians use it to find parallels, not to replace judgement.

Historians also study how time itself is organised. On 15 October 2026 the 28th General Conference on Weights and Measures, meeting in Versailles, voted to stop adding leap seconds from 20 May 2027. It will let the gap between atomic and astronomical time grow, potentially to an hour. Twenty-seven leap seconds were added between 1972 and 2016. This follows the conference's 2022 Resolution 4, which asked for a larger tolerance by 2035. The decision belongs to 24.3; it is history in the making.

Digital history has changed the scale of evidence. Newspapers, parliamentary debates and court records of the last three centuries are now searchable as text. Historians can count words and topics across millions of pages, and the methods of Chapter 22 apply directly to historical corpora. The risk is the same as in literary studies: a count of words is not a history of events.

The open debates are the status of cycles (Turchin's secular cycles against contingency), the coding of absence in historical databases, and the balance between global and local history.

Interpretation.

1. *Strand.* I is time. P is the set of states a society can be in: regimes, price levels, positions in a cycle. A loop is a return to a state already occupied: a restoration, a renewed war, a repeated crisis.
2. *Passage and trit.* By the trit rule, a return scores 1 if the society leaves the state the same way it left the first time, T if it leaves the way it came in, and 0 otherwise. A restoration that undoes a revolution is the T case. A repeated revolution that follows the same path is the 1 case. Historians would have to code states and transitions for the rule to be applied; no such coding was used here.
3. *Closure.* A closed window of three trits leaves no residue at its scale (T1, T2). Three similar returns in a row (1, 1, 1) form a ring and carry one passage to the next scale (T7.1): the cycle becomes a unit of a longer history. Braudel's three times are this picture in words. It is a reading.
4. *Scale.* The ladder of T15 predicts that nested cycles built from equal blocks have length ratios 2 or 3. Ibn Khaldun's dynasties of three or four generations and Turchin's secular cycles of 200–300 years have no fixed ratio of that kind. The ladder is not observed in history.
5. *Test.* A test that could fail. Take Seshat's coded sequences of polity states, with a polity's regime type as P. Apply the trit rule and compare the closed-window rate with a first-order Markov null, as in Calculation 22.2. If the rate does not exceed the null, history has no three-step closure beyond pairwise succession, and the reading fails. Not run here.

Consequences. History has no structure of the Foundations as published fact. Cycle theories exist, but their periods are not ladder numbers, and their status is disputed within the field. The interpretation adds a mechanical test of three-step returns, the same test that found a weak excess in Homer. It also adds a warning taken from the field itself. The Seshat episode shows that coding "unknown" as "absent" can create or reverse a pattern. Any trit coding of historical states faces the same danger: a 0 for "unknown" would inflate closed windows. The T6 baseline, computed from the observed law of trits, controls for this only if unknowns are dropped rather than scored.

There is one place where history and the Foundations share a technical problem: dating by matching. When a sequence of events has no fixed start, historians align it with another sequence whose dates are known. Examples are a king list with eclipse records, or a dynastic chronology with a dated synchronism. The Mesopotamian chronology of the second millennium BCE is still debated in this form, between "high", "middle" and "low" chronologies that differ by decades. The problem is the one T13 counts. A closed configuration without a start can be placed in as many ways as it has rotations, until an outside mark fixes one. The Foundations add nothing to the solution. They do state the problem precisely.

What the interpretation costs is the temptation of Spengler and Toynbee: to see rings everywhere. The rule of this book forbids it. Without the test, history stays at C.

The checklist was run for history. T7 and T7.1 describe the reading in point 3. T6 sets the baseline for the stated test. T13 separates two ways of counting the past. Read from a start, a chronology is an open word, and n steps have 3^n possible codings. Read as a recurring cycle without a start, the same history is a rotation class, numbering g(n). Cyclic theories of history implicitly use the second count, and linear chronologies the first. T15 fails (no ladder ratios in historical cycles). Scaliger's Julian Period, 28 × 19 × 15 = 7,980, contains the primes 5, 7 and 19 and lies off the ladder (Calculation 24.10). T8–T11, T14 and T16–T24 have no historical object.

Grade. C for history as a whole. B only once the stated Seshat test is run.

24.2 Archaeology

History. Archaeology grew from antiquarian collecting. Renaissance scholars collected inscriptions and coins. Excavation at Herculaneum began in 1738 and at Pompeii in 1748, at first as treasure hunting. Christian Jürgensen Thomsen, ordering the Danish national collection, proposed the Three-Age System (Stone, Bronze, Iron) in 1836. It gave prehistory its first relative chronology.

Stratigraphy, borrowed from geology, made excavation a method. Augustus Pitt-Rivers in the 1880s recorded every find in its layer. Flinders Petrie (1899) invented sequence dating for Egyptian graves, ordering them by the changing styles of pottery. Mortimer Wheeler and Kathleen Kenyon developed the box-grid method in the twentieth century. Heinrich Schliemann's excavations at Troy (from 1870) and Arthur Evans's at Knossos (from 1900) brought legend into archaeology, and with it controversy.

Absolute dating arrived from physics and biology. Andrew Ellicott Douglass built the first tree-ring chronology in Arizona from 1904. In 1949 Willard Libby measured the decay of carbon-14 and dated organic material across the last fifty thousand years. He received the Nobel Prize in chemistry in 1960. Radiocarbon dates must be calibrated, because atmospheric carbon-14 has varied. Calibration curves built from tree rings, corals and sediments followed from the 1960s.

Theory changed in waves. Culture history described peoples by their artefacts. Lewis Binford's New Archaeology (from 1962) demanded hypothesis testing and explanation of processes. Ian Hodder's post-processual archaeology (from the 1980s) brought meaning, agency and interpretation back.

The third science revolution came with ancient DNA. Svante Pääbo's group sequenced the Neanderthal genome in 2010 and found that non-Africans carry Neanderthal ancestry. Pääbo received the Nobel Prize in physiology or medicine in 2022. Genome-wide data from thousands of ancient individuals then traced migrations, such as the steppe ancestry that spread into Europe around 5,000 years ago.

Two further dating methods matter here. Varves, the annual layers of lake sediment, give counted chronologies in the same way as tree rings; the Swedish varve chronology of Gerard De Geer (from 1912) was the first. Ice cores from Greenland and Antarctica give annual layers and records of volcanic eruptions; a sulphate spike in the ice, like the 993 carbon spike in wood, marks one year everywhere. Archaeologists use such marks to tie floating sequences to the calendar.

Archaeoastronomy studies alignments of monuments to the sky. The winter-solstice alignment of the passage at Newgrange in Ireland (c. 3200 BCE) and the solstitial axis of Stonehenge are accepted. Many other claimed alignments are not, because with enough stones some line will point at something. The field learned the same lesson as the Seshat re-analysis: a pattern needs a null model.

The questions left are how to join genes to cultures without equating them, how to date events to the year, and how to read what was never written.

Current state. Radiocarbon calibration has reached the resolution of single years where a marker exists. IntCal20, published in *Radiocarbon* in 2020, extends the curve from 0 to 55,000 calibrated years before present; tree rings carry it as a fully atmospheric record to about 13,900 cal BP. Securely dated tree-ring data cover about the last 12,310 years.

Single-year dates come from solar storms. A cosmic-ray event in 993 CE left a spike of carbon-14 in that year's tree ring worldwide. Kuitems, Dee and colleagues counted rings from the spike to the bark edge on wood cut with metal tools at L'Anse aux Meadows in Newfoundland. Each of three trees was felled 28 years after the spike, in 1021 (Calculation 24.7). This is the earliest exact year of European presence in the Americas.

Ancient DNA is now a large but fragile infrastructure. The Allen Ancient DNA Resource, maintained by David Reich's lab, curates published genomes. In October 2025 the *Harvard Crimson* reported that its eight-year grant ended in September with no replacement. The dataset, downloaded more than 67,000 times, covers more than 1.2 million genome positions. The planned 2025 release was postponed.

Imaging reads what cannot be opened. The Herculaneum scrolls, carbonised in 79 CE, were scanned at the European Synchrotron and at Diamond Light Source. On 25 June 2026 the Vesuvius Challenge presented nearly 1.5 metres of continuous text and about 20 columns from PHerc. 1667, an ethical treatise probably of the second century BCE. It also identified the title *On Gods, Book 8* in another scroll. More than 600 scrolls remain unopened.

Archaeology also measures cycles directly in some places. Settlement records in the American Southwest, dated to the year by tree rings, show periods of aggregation and dispersal over centuries. Studies of the Neolithic in Europe show booms and busts of population inferred from the density of radiocarbon dates. These "summed probability" methods are debated, because the density of dates depends on what was dug and dated as well as on how many people lived.

The open debates are the relation between genetic ancestry and cultural identity, the ethics of sampling human remains, and the sustainability of shared data.

Interpretation.

1. *Strand.* I is time, read downward through layers or inward through tree rings. P is the set of datable states: a ring, a layer, an artefact type.
2. *Passage and trit.* Tree rings are the cleanest strand in archaeology. Each year adds one ring, one passage. A solar event is a mark that the strand makes at one passage,

worldwide, so every tree alive that year carries the same mark at the same place. Counting from the mark is counting passages.

3. *Closure.* The calibration curve is the matching of two strands, the radiocarbon clock and the tree-ring count, at every passage. Where both are available, the match is exact to the year. Where only radiocarbon is available, a plateau in the curve makes several years indistinguishable. That residue is not tension in the Foundations' sense, because it is a measurement limit, not an open window.
4. *Scale.* Thomsen's three ages and Petrie's sequence dating are orderings, not cycles. They have no ring. The ladder of T15 does not apply to archaeological periods.
5. *Test.* For tree-ring dating none is needed, because counting passages is what the method does. For a ring reading of stratigraphy, a test that could fail: at sites occupied, abandoned and reoccupied, the sequence of occupation states should show closed windows (in the trit-rule sense) more often than a first-order null, if reoccupation repeats earlier patterns of use. Not run here.

Consequences. Archaeology holds one structure that is close to the Foundations as fact: chronology as counting of passages from a marked one. The 993 event is a first passage. Every later ring is counted relative to it, as the trit rule counts later passages relative to the first. That is reading 3 of the Foundations (N3, memory) in practice. The theory adds nothing to the method: dendrochronologists counted rings long before.

The interpretation adds a framing of what the field calls resolution. A calibration plateau is a stretch where the strand returns to the same radiocarbon value several times. Several passages share one place, so they cannot be told apart. The trit rule could separate them only if the step after each return differed; in radiocarbon it does not. This explains why plateaus cannot be dated without an independent marker. Archaeologists already know this, and the reading adds nothing new.

Beyond chronology, the interpretation is a reading. One more caution comes from the field. Summed radiocarbon curves can show apparent cycles produced by the wiggles of the calibration curve itself. Archaeologists now test such cycles against simulated null data. Any reading of population cycles as rings must pass the same test, and the Foundations offer no shortcut.

The checklist for archaeology: N3 (memory) and the trit rule's first passage apply to marked-year counting; T13 applies to the choice between counting from a start (a dated sequence) and matching a floating sequence by pattern alone (a rotation class, which can be placed only by an independent mark). Matching a floating tree-ring sequence is exactly the problem of finding the start of a closed configuration, which T13 counts as g(n) rather than 3^n. T15 does not apply to periods. The archaeoastronomy warning applies to T6: any claimed closure needs its baseline. The other theorems have no object.

Grade. A for dendrochronological counting from a marked year as the counting of passages from a first passage (Kuitems et al., *Nature*, published online 2021, a method independent of this theory), though the theory adds nothing to it. C for archaeology beyond chronology.

24.3 Calendars and time-reckoning

History. Every agrarian society had to reconcile three cycles that do not divide each other: the day, the lunar month and the solar year. A month is about 29.53 days and a year about 365.24 days. Neither is a whole number of days, and a year is not a whole number of months.

Babylonian astronomers solved the month–year problem first. From the late sixth century BCE they added a thirteenth month seven times in nineteen years, and by the fifth century the rule was fixed. In Athens, Meton introduced the 19-year cycle in 432 BCE: 235 months in 19 years, 6,940 days. Callippus refined it to four 19-year cycles of 76 years with a mean year of exactly 365.25 days. An earlier Greek cycle, the octaeteris, put 99 months in 8 years.

Eclipses posed a second problem. The Babylonians found that eclipses repeat after 223 lunar months, about 18 years and 11 days: the Saros. The Antikythera Mechanism, a Greek geared device of the second or first century BCE recovered from a shipwreck in 1901, displayed the Metonic cycle, the Saros and the triple Saros (exeligmos) on spiral dials. In 2021 Freeth and colleagues reconstructed its planetary display. They identified a 462-year relation for Venus and a 442-year relation for Saturn inscribed on the front cover. They proposed that the makers derived these from older Babylonian relations by a method described by Parmenides.

The Jewish calendar was fixed as a computed calendar. According to tradition, Hillel II set it in about 358 CE. It uses the 19-year cycle with leap years in years 3, 6, 8, 11, 14, 17 and 19. Its mean month, the molad, is 29 days, 12 hours and 793 parts, where an hour has 1,080 parts. The Islamic calendar is purely lunar, with no intercalation. For computation a tabular form is used: 11 leap years in 30, in years 2, 5, 7, 10, 13, 16, 18, 21, 24, 26 and 29, giving 10,631 days in 360 months.

Solar calendars needed leap days. Julius Caesar’s reform of 46 BCE added one day every four years. The Gregorian reform of 1582 dropped three leap days in four centuries, giving 97 leap years in 400. In Mesoamerica the Maya kept a 260-day count (13 × 20), a 365-day year (18 × 20 + 5) and their combination, the Calendar Round of 18,980 days, 52 years. The Long Count used units of 20 days, 360 days (tun), 7,200 days and 144,000 days. The Dresden Codex tabulates 405 consecutive lunations, and the Maya equated 149 moons with 4,400 days. Their Venus table counted five 584-day cycles as 2,920 days, about eight years.

China combined a lunisolar calendar with a 60-day and 60-year cycle of stems and branches. Han astronomers used the 19-year *zhang* cycle. Yang Xiong’s *Taixuanjing* (c. 2 BCE) built a divination system of 81 tetragrams, each line unbroken, broken once or broken twice: 3^4 in place of the I Ching’s $2^6 = 64$.

The questions left were practical, how to keep calendars aligned, and theoretical: why these numbers? Mitchell Harris and Edward Reingold answered the second in 2004. “The pattern of a Bresenham line is a special case of a leap year rule, and both are described by Euclid’s algorithm”: leap rules are continued-fraction approximations.

Current state. The astronomical inputs are known to many decimals. NASA gives the mean synodic month as 29.53059 days, the draconic month as 27.21222 days and the anomalistic month as 27.55455 days. The Saros, 223 synodic months, lasts 6,585.3223 days, which matches 242 draconic months (6,585.3575 days) and 239 anomalistic months (6,585.5375 days). The tropical year in NASA's Earth fact sheet is 365.242 days. The Venus synodic period is 583.92 days and Saturn's 378.09 days. The calculations below use the conventional mean tropical year of 365.24219 days where more digits matter.

The main event of 2026 concerns the day itself. On 15 October 2026 the General Conference on Weights and Measures decided to stop adding leap seconds from 20 May 2027. It will let UTC drift from Earth's rotation, possibly by up to an hour over a century or more. Twenty-seven leap seconds were added between 1972 and 2016. The decision ends the only intercalation rule ever applied by an international body in real time. It replaces closure with a deliberately accepted residue.

Calendrical computation is a settled field. Dershowitz and Reingold's *Calendrical Calculations* provides algorithms for more than thirty calendars. The open questions are historical: when exactly a culture adopted a cycle, and how the Antikythera makers obtained their planetary relations.

Interpretation.

1. *Strand.* I is the count of days. P is the position in the month and in the year. A loop is a return to the same calendar position: new moon on the same solar date.
2. *Passage and trit.* At each such return the lunar count runs ahead, behind or level with the solar count. The calendar keeps the difference within one month by inserting a month when it exceeds one. The leap year is the carry of T1: what the year does not keep passes whole to the next unit.
3. *Closure.* A lunisolar calendar closes when a whole number of months equals a whole number of years. It never closes exactly, because the ratio is irrational in practice. It closes best at the convergents of year/month = 12.368266, which are 12/1, 25/2, 37/3, 99/8, 136/11, 235/19 and 4131/334 (Calculation 24.1). This is the same construction as the closure depths of T17, applied to a different ratio. The octaeteris (99/8) and the Metonic cycle (235/19) are convergents. Their misses are 1.59 days per 8 years and 0.087 days per 19 years. The Callippic 940/76 is four Metonic cycles, a multiple of a convergent, not a new one.
4. *Scale.* Leap rules spread their leaps in gaps of 2 and 3. The Hebrew years 3, 6, 8, 11, 14, 17, 19 have gaps 3, 2, 3, 3, 3, 2, 3. The tabular Islamic years have gaps 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3 (Calculations 24.2–24.3). These are meetings and rings in the sense of T7: every leap rule with a fraction between 1/3 and 1/2 is a word in the gaps 2 and 3. The fraction for the Hebrew cycle is 7/19, a convergent of 0.3684 (months per year beyond 12). For the Islamic cycle, 11/30 is a convergent of 0.36708 (leap days per year), whose continued fraction begins 3/8, 4/11, 7/19, 11/30. The two calendars use neighbouring convergents of nearly the same number, one in months and one in days. The Saros is the convergent 242/223 of synodic/draconic month (Calculation 24.5). The ladder of T15 appears only in parts: the

Long Count units 20, 360, 7,200 and 144,000 are $2^a \cdot 3^b \cdot 5^c$, not on the ladder (Calculation 24.7).

5. *Test.* Stated before checking: every intercalation cycle a society kept in long use should be a convergent (or a multiple of one) of the ratio it approximates. Refutation would be a long-used cycle that is neither. Checked: octaeteris, Metonic, Callippic (a multiple), Hebrew 7/19, Islamic 11/30 and Saros 242/223 all pass. The Julian 1/4 is a convergent of 0.24219. The Gregorian 97/400 is not: the convergents are 1/4, 7/29, 8/33 and 31/128 (Calculation 24.4). Gregory's rule was chosen for decimal convenience (centuries), not best approximation. It misses by one day in about 3,226 years, where 31/128 would be closer. So the test holds for cycles chosen by astronomers and fails for a cycle chosen by a decimal convention. That is a partial refutation of the stated form, and it is reported as such. The Antikythera planetary relations were tested the same way. Venus 289/462 and Saturn 427/442 are not best approximations: 157/251 and 199/206 are better with fewer years (Calculation 24.6). Both can be written as sums of convergents: Saturn's 442 years as $29 + 7 \times 59$, Venus's 462 as $2 \times 3 + 57 \times 8$ (Calculation 24.8). That fits Freeth's account of construction from earlier relations. But the check is weak. Saturn's relation has five such decompositions and Venus's one; allowing semiconvergents gives 10 and 34. Almost any relation near the ratio decomposes. So the theory's form of the test, "best approximation", fails for these relations. The weaker form, "built from convergents", passes but cannot discriminate and is not counted as a pass.

Consequences. Calendars are the arithmetic of the Foundations applied to astronomical ratios. Closure is a best rational approximation; the residue never vanishes (T16 for 2 and 3, and the same irrationality here). The leap is a carry. These are published facts of calendrical science: Harris and Reingold state the continued-fraction structure of leap rules explicitly. So the grade is A.

The test adds two useful negatives. The Gregorian calendar shows that a culture can choose a non-convergent for reasons outside closure. A decimal convention, here the century, overrides the best approximation. The Antikythera relations show that ancient astronomers did not always choose the best approximation for a given length. Freeth and colleagues propose that they built long periods from shorter known ones. The arithmetic allows that, but it allows so many constructions that it cannot confirm it. Both refine the claim: closure explains the convergents astronomers found, not every choice a society made.

The 2026 decision on leap seconds is the most direct contemporary case. For half a century UTC was kept closed against Earth's rotation by a carry of one second when the residue exceeded 0.9 s. From 2027 the residue will be allowed to grow. In the language of T1 the second's residue will no longer be carried. It will be kept as a nonzero digit, tension, until a larger correction at a coarser scale. The Foundations predict nothing about that decision. They describe it exactly.

The *Taixuanjing* is a historical instance of a ternary register in a cosmological text: $81 = 3^4$ tetragrams with three line values. As four-trit words they cover the balanced range -40 to 40 (T12). That Yang Xiong chose three values is a fact. That he chose them for a reason related to the Foundations is not claimed.

The checklist for calendars: T1 and T7.1 (the leap as carry), T7 (gaps of 2 and 3 as meetings and rings), T13 (counting a cycle with or without a start, as in Scaliger's and the Maya's least common multiples), T15 (fails for Long Count units and for the Julian Period), T16–T18 (closure depths and residue, transposed to astronomical ratios) and T12 (the *Taixuanjing* as a balanced range). T8–T11, T14 and T19–T24 have no calendrical object.

Grade. A for lunisolar intercalation cycles, leap rules and the Saros as convergents (published fact: Harris and Reingold 2004; NASA period data). B for the stated convergent test, run with a partial failure (Gregorian 97/400; the Antikythera planetary relations). C for the cultural choice of counting units.

24.4 Religion and myth

History. The academic study of religion began in the nineteenth century with comparison. Friedrich Max Müller, editor of the *Sacred Books of the East* (1879–1910), proposed a science of religion and read myth as a "disease of language". Edward Tylor's *Primitive Culture* (1871) defined religion as belief in spiritual beings. James Frazer's *The Golden Bough* (1890, expanded to twelve volumes by 1915) collected rites of a dying and reviving king and proposed a sequence from magic to religion to science. Later scholars rejected that sequence as a projection of Victorian progress.

Sociology and psychology gave religion functions. Émile Durkheim's *Elementary Forms of Religious Life* (1912) defined religion by the sacred and its community: rites renew the group. Max Weber's *Protestant Ethic* (1905) and his comparative studies traced religion's effects on economic life. William James's *Varieties of Religious Experience* (1902) studied individual experience. Rudolf Otto's *The Idea of the Holy* (1917) described the numinous.

Myth became a subject of structure. Georges Dumézil (from the 1930s) proposed that Indo-European myths share three functions: sovereignty, force and fertility. Claude Lévi-Strauss's *Mythologiques* (1964–1971) read hundreds of American myths as transformations of each other, built on oppositions such as raw and cooked. Mircea Eliade (*The Myth of the Eternal Return*, 1949) described archaic religion as the periodic return to a sacred time of origins. Joseph Campbell's monomyth (1949) was popular and contested.

The cognitive science of religion arose in the 1990s. Pascal Boyer, Justin Barrett and others explained religious concepts as minimally counterintuitive ideas that are easy to remember. Cultural evolutionists (Norenzayan, Henrich) proposed that moralising gods helped large societies cooperate. Archaeology joined in with the excavation of Göbekli Tepe in Turkey from 1995. Its monumental pillars, more than 11,000 years old, predate farming and suggest that ritual gathering came before agriculture, not after.

Many traditions hold triadic or cyclic structures. Hindu thought has the *trimūrti* of creation, preservation and dissolution and the cycle of *yugas*. Christianity has the Trinity and the liturgical year. Buddhism has the three marks of existence and the wheel of rebirth. Daoist cosmology has the generation of the many from the one, two

and three. Stoics and, later, Nietzsche proposed eternal recurrence. These are reported here as doctrines of the traditions. The book makes no judgement on their truth.

Ritual itself is repetitive by design. Anthropologists from Arnold van Gennep (*The Rites of Passage*, 1909) to Victor Turner (1969) described rites in three phases: separation, transition (the liminal phase) and incorporation. Van Gennep's scheme is one of the most widely used in the field. Harvey Whitehouse (2004) distinguished two modes of religiosity: a doctrinal mode with frequent, low-arousal repetition, and an imagistic mode with rare, high-arousal rites remembered as episodes. The frequency of repetition, in his account, shapes how a religion is remembered and spread.

Sacred calendars are where religion and astronomy meet in practice. The date of Easter was disputed for centuries: the Council of Nicaea (325) set the principle, and Alexandrian computists fixed the 19-year cycle for it. The Irish and Roman churches differed on the computus until the Synod of Whitby (664). Hindu festivals follow a lunisolar calendar with intercalary months (*adhika māsa*). The Islamic calendar's months, by contrast, run freely through the seasons. Each tradition chose a different closure rule, and the choice carries religious meaning.

The questions left are what religion is, whether it has universal elements, and how religious belief and practice relate to the size and order of societies.

Current state. The Pew Research Center's 2025 study of global religious change drew on more than 2,700 data sources. It gives the shares of the world's population in 2020 as: Christians 28.8 percent, down 1.8 points since 2010; Muslims 25.6 percent, up 1.8 points; religiously unaffiliated 24.2 percent, up nearly one point; Hindus 14.9 percent; Buddhists 4.1 percent, down 0.8 points; and Jews about 0.2 percent. The six groups sum to 97.8 percent; the remainder belongs to other religions (Calculation 24.7). Muslims grew by 347 million in the decade, and the unaffiliated by 270 million. Buddhists were the only major group to decline in absolute numbers, by 19 million.

The debate on moralising gods is the main quantitative controversy. After the retraction of the 2019 Seshat paper, the re-analysis of 12 key regions placed moralising gods before major rises in social complexity, possibly in much smaller populations. Muthukrishna said that the re-analysis "re-opens the debate". The causal direction between religion and social scale is therefore open again.

The cognitive and evolutionary study of ritual continues with experiments on synchrony, costly signalling and memory. The comparative mythology of large databases (motif indexes such as Berezkin's) traces the spread of myth motifs along migration routes, much as ancient DNA traces populations.

The study of religion is also a study of change in practice. Secularisation in Europe and East Asia, growth of Pentecostalism in Africa and Latin America, and the rise of religiously unaffiliated populations are documented by Pew and by national censuses. No single theory explains them.

Interpretation.

1. *Strand.* I is ritual time: the sequence of feasts, prayers and rites. P is the set of sacred times and places. A loop is the return of a feast or a pilgrimage to the same place.
2. *Passage and trit.* The liturgical year returns each feast at the same position: Easter after the spring full moon, Ramadan in the ninth lunar month, Passover in Nisan. Those returns are calendar closures (24.3). Their arithmetic is the convergent arithmetic of the Metonic and lunar cycles. The Christian Easter computus uses the 19-year cycle; the Islamic fast moves through the solar year in about 33.6 years, because the lunar year is 10.88 days short (Calculation 24.9).
3. *Closure.* Many traditions describe completion in three. The Trinity, the *trimūrti*, Dumézil's three functions and Hegel's dialectic are triads; Olrik's law of three is the same in folk narrative (23.3). The Borromean rings, used in at least one medieval Christian manuscript for the Trinity (22.2), are a true topological closure of three that no pair carries. These are reported as structures the traditions themselves state. Reading them as the ring of T7 is a correspondence of form, not a link.
4. *Scale.* Eliade's eternal return describes ritual time as a ring that closes and recurs. The yugas form a cycle of four ages in ratio 4 : 3 : 2 : 1. Read as a sequence of ages, those ratios lie on the ladder, but four ages and a ratio sum of 10 do not. The ladder adds nothing here.
5. *Test.* A test that could fail. If closure in three is a property of memorable structure, as cognitive science of religion suggests for minimally counterintuitive concepts, then myth motifs that occur in triads should spread further (in more cultures of Berezkin's motif database) than the same motifs in pairs or fours. If spread does not depend on the number, the reading fails. Not run here.

Consequences. Religion and myth hold two kinds of material for the Foundations. The first is calendrical: the fixed feasts of the major traditions sit on the same convergents as their calendars (24.3). That part is grade A, but it belongs to calendrics, not to religion. The second is triadic doctrine. Traditions from India to Europe state triads and cycles as central truths. The Foundations also state a triad, the trinity of -1, 0, $+1$ with sum zero, as their ground. The match in number is real. It is not evidence for either. Small numbers recur in human thought for many reasons: three is the smallest number that has a middle, a beginning and an end.

The book takes no position on the truth of religious claims. The Foundations are an arithmetic; they neither confirm nor refute a theology. The useful contribution is the test in point 5. It turns a vague claim, "three is everywhere", into a count that could fail.

The cost of going further would be the error Frazer was criticised for: collecting resemblances and calling them a law.

Van Gennep's three phases deserve one more line. Separation, transition and incorporation are a sequence that leaves the initiate in a new state: the rite closes and the person passes to the next stage of life. In the language of T7.1 that is a ring with carry: a closed sequence of three that becomes one passage of the next scale. This is

the clearest match of the ring in religious studies. It is still a reading, because the three phases are qualitative and their boundaries are set by the anthropologist. A test would require coded rites with more or fewer phases and a measure of whether the change of status holds.

The checklist for religion: T7 and T7.1 describe the reading of triads and rites of passage; the calendar theorems apply through 24.3; T19 applies only to the Borromean rings and knot symbols as topology; T15 fails for the yugas' sum. The others have no object.

Grade. A for the calendrical structure of feasts (a consequence of 24.3). B for the stated triad-spread test, not yet run. C for religion and myth as a whole.

Chapter 25 — Making Meaning

The last chapter covers the practical sciences of culture: the disciplines that take what does not close and make it close, in argument, in obligation, in buildings and in made things. Three-valued logic and deontic logic hold the trit and its mirror as published fact (grade A). The Condorcet cycle is a closed ring of preference in 12 of 216 profiles (grade A). Architecture and urban design hold three structures as fact. Maxwell's rigidity count makes the triangle the only rigid polygon. Christaller's K = 3 and K = 4 lie on the ladder while K = 7 does not. All periodic plan grids obey the crystallographic restriction. These are grade A for the structures, C for architecture as an art. Design has one ladder structure as published fact, the E-series of preferred numbers. Philosophy and ethics as wholes are graded C.

25.1 Philosophy

History. Philosophy began as the search for a ground. The Milesians asked what everything is made of: water for Thales, the boundless for Anaximander, air for Anaximenes. Heraclitus made change and the unity of opposites the principle. Parmenides denied that what is not can be, and so denied change. Zeno's paradoxes defended him. Plato answered with forms that do not change behind appearances that do. Aristotle built logic, the categories and a physics of four causes. His syllogistic, set out in the *Prior Analytics*, was the standard of valid reasoning for two thousand years.

Aristotle also left a problem that matters for this book. In *De Interpretatione*, chapter 9, he asked whether a statement about a future contingent, "there will be a sea battle tomorrow", is already true or false. The passage was read by many as allowing a third status: neither true nor false yet.

Medieval philosophy joined Greek logic to theology. Augustine, Avicenna, Averroes, Maimonides and Aquinas debated universals, the eternity of the world and the relation of reason to faith. Ramon Llull's *Ars magna* (c. 1305) combined concepts by rotating discs, an early idea of mechanical reasoning.

Modern philosophy split along the line of knowledge. Descartes (1641) sought certainty in the thinking self. Locke, Berkeley and Hume grounded knowledge in experience. Leibniz proposed a universal characteristic, a calculus of thought, and studied binary arithmetic. Kant's *Critique of Pure Reason* (1781) argued that the mind supplies the forms of space, time and the categories. His table of categories has four groups of three: unity, plurality and totality; reality, negation and limitation; and so on. Hegel's dialectic moved through triads, though the formula "thesis–antithesis–synthesis" is Fichte's more than his.

Logic became mathematics in the nineteenth century. Boole (1854) wrote an algebra of classes; Frege's *Begriffsschrift* (1879) created predicate logic. Russell and Whitehead's *Principia Mathematica* (1910–1913) tried to derive mathematics from logic. Jan Łukasiewicz in 1920 broke with two values. He proposed a third truth value for

Aristotle’s future contingents, interpreted as possibility. Gödel’s incompleteness theorems (1931) set limits on every formal system strong enough for arithmetic.

The twentieth century split into analytic and continental traditions. Wittgenstein, the Vienna Circle, Quine and Kripke on one side; Husserl, Heidegger, Sartre, Merleau-Ponty and Derrida on the other. Charles Sanders Peirce, outside both, built pragmatism and a triadic semiotics (22.2).

Philosophy of mind took the centre late in the century. David Chalmers named the “hard problem” of consciousness in 1995: why physical processes are accompanied by experience at all.

The questions left are the nature of mind, the foundations of knowledge, the status of logic, and whether philosophy can be settled by any method.

Current state. The best measure of what professional philosophers believe is the PhilPapers Survey. Its 2020 edition reports results from 1,785 respondents. On consciousness, 33.04 percent accept or lean toward functionalism, 21.96 percent dualism, 13.33 percent identity theory, 7.55 percent panpsychism and 4.51 percent eliminativism. On God, 66.95 percent accept or lean toward atheism and 18.93 percent theism. On free will, 59.16 percent are compatibilists, 18.83 percent libertarians and 11.21 percent deny free will. On aesthetic value, 43.53 percent hold it objective and 40.59 percent subjective. The disagreement is itself a result. On almost every major question, no view commands a large majority.

The question of machine consciousness moved from speculation to method. In August 2023, nineteen authors led by Patrick Butlin and Robert Long derived indicator properties of consciousness from leading neuroscientific theories and assessed current AI systems against them. They concluded that “no current AI systems are conscious”, but found no obvious technical barrier to building systems that would satisfy the indicators. The debate since has been about whether indicators derived from human brains apply to machines at all.

Philosophical logic is a working field. The Stanford Encyclopedia’s entry on many-valued logic gives the standard systems. Łukasiewicz’s three-valued logic of 1920 uses the values 0, ½ and 1, with negation $\neg x = 1 - x$, so that $\neg ½ = ½$. Kleene’s strong logic (1938, 1952) reads the middle value as “undefined”, for partial recursive functions. Bochvar’s (1938) reads it as “meaningless”. Halldén (1949) called his variant “the logic of nonsense”. These systems are used in computer science, in the semantics of vagueness and in database query languages, where SQL’s NULL follows Kleene’s tables.

The open debates are consciousness, the nature of normativity, the interpretation of quantum mechanics, and the status of philosophy itself against the sciences and now against AI systems that write philosophy.

Interpretation.

1. *Strand*. I is the order of reasoning: the steps of an argument. P is the set of propositions. A loop is a return to a proposition already considered.

2. *Passage and trit.* Łukasiewicz's values 0, ½ and 1 map affinely onto T, 0 and 1 by $v \mapsto 2v - 1$. His negation $\neg x = 1 - x$ becomes the trit-wise mirror $1 \leftrightarrow T$, with 0 fixed (Calculation 25.1). That is exactly Corollary T12.1 of the Foundations: the address of $-Q$ is the address of Q mirrored. Kleene's and Łukasiewicz's middle value is the digit 0 of T1. This is an identity of algebras, not a resemblance.
3. *Closure.* A window of three truth values closes when its digit is 0 (T1, T2): all three equal, or one true, one false and one undetermined (T4). The Foundations add a selection rule that no standard many-valued logic has: only closed windows persist (reading N2, proved as arithmetic by T1 and T2). In a chain of reasoning, a run of three undetermined values closes; a run of two true and one false does not.
4. *Scale.* Kant's table of twelve categories in four groups of three and Peirce's three categories are triadic schemes repeated at each level. Twelve is $2^2 \cdot 3$, on the ladder of T15; four groups of three are four rings. This is a structural reading of Kant, graded C.
5. *Test.* For logic none is needed: the correspondence of 2. is an identity, checked in Calculation 25.1. For the selection rule a test could fail. In argument corpora where premises and conclusions are coded as accepted, rejected or undecided (for example annotated debate data), sequences of three statuses whose digit is 0 should be followed by a change of topic more often than open ones. If not, the selection rule adds nothing to argumentation. Not run here.

A further check shows how far the identity reaches. In trit form Łukasiewicz's conjunction is the minimum and his disjunction the maximum. De Morgan's laws hold on all nine pairs. The law of excluded middle fails exactly at 0: "x or not-x" takes the value 1 for $x = 1$ and $x = T$, and 0 for $x = 0$ (Calculation 25.6). That is the point of Łukasiewicz's system: an undetermined statement leaves its disjunction with its negation undetermined. In the Foundations' terms, the window $(x, \neg x, x \vee \neg x)$ for $x = 0$ is (0, 0, 0), a standing that closes. For $x = 1$ it is (1, T, 1), whose digit is 1: an open window. Excluded middle as a law corresponds, in this reading, to an open window. The reading is exact arithmetic but its meaning is a reading.

Łukasiewicz's implication in trit form is $\min(1, 1 + y - x)$. Its table has T only where 1 implies T, and 0 where the antecedent is one step above the consequent (Calculation 25.6). The truth-functional machinery of three-valued logic is therefore a complete calculus on the trits, worked out between 1920 and 1952. The Foundations' register inherits it at no cost.

Consequences. Three-valued logic is the clearest case in this book of a field that already contains the trit as published fact. Łukasiewicz needed a third value for statements not yet settled. His negation is the mirror of T12.1. A century of results on the connectives of three-valued logic applies to the trits without change.

What the Foundations add is a closure rule. Standard many-valued logics define validity by designated values: an argument is valid if it preserves truth, or non-falsity. The Foundations define persistence by residue: a window of three values persists if its digit is 0. The two criteria are different and do not conflict. The second is a new kind of

constraint on reasoning, closer to a conservation law than to a rule of inference. Whether it describes any actual reasoning is the test in point 5.

On consciousness, God and free will, the Foundations add nothing. The survey shows that philosophers disagree. A theory that makes no physical claim, as the Foundations state of themselves, cannot settle what physics cannot. On machine consciousness the same holds. The Foundations' ternary register is used in hardware and in ternary-weight networks (Chapter 1), but that says nothing about experience.

The checklist was run. T1, T2, T4 and T12.1 apply exactly to three-valued logic. T6 gives the baseline for the stated test. T7 classifies closed truth-value words: every closed word of statuses reduces to standings (undetermined), meetings (true then false) and rings (three equal). T15 applies to Kant's table as a reading. T8–T11, T13–T14 and T16–T24 have no philosophical object.

Grade. A for three-valued logic (Łukasiewicz 1920; negation as the trit mirror of T12.1). B for the stated test of closure in argument. C for philosophy as a whole.

25.2 Ethics

History. Ethics asks how to live and what is owed to others. Confucius (sixth to fifth century BCE) taught virtue through ritual and relationship; the Buddha taught a path out of suffering; the Hebrew prophets taught justice under covenant. Socrates made ethics a matter of examined argument. Aristotle's *Nicomachean Ethics* defined virtue as a mean between two vices, courage between cowardice and rashness. That is a three-term scheme with the virtue in the middle.

Christian, Islamic and Jewish ethics joined Greek virtue to divine command. Aquinas synthesised natural law. Islamic jurisprudence developed a scale of five legal statuses for acts: obligatory, recommended, permitted, discouraged and forbidden. That scale has a neutral middle and two grades on each side.

Modern moral philosophy produced three families of theory. Kant's *Groundwork* (1785) based morality on duty and the categorical imperative: act only on a maxim you could will as universal law. Jeremy Bentham (1789) and John Stuart Mill (1861) based it on consequences: act to produce the greatest happiness. Virtue ethics, revived by Elizabeth Anscombe (1958), Philippa Foot and Alasdair MacIntyre (1981), returned to Aristotle's question of character. These three families still organise the field.

Logic entered ethics with Georg Henrik von Wright's "Deontic Logic" in 1951. Von Wright treated obligation, permission and prohibition as modal operators. Every act is obligatory, forbidden or optional (neither). And the obligation of an act is the prohibition of its omission. Roderick Chisholm's paradox of 1963 showed that standard deontic logic cannot represent obligations that arise when a primary duty is violated ("contrary-to-duty" obligations). Philosophers have worked on that problem since.

John Rawls's *A Theory of Justice* (1971) revived political philosophy with the original position and the veil of ignorance. Peter Singer's *Animal Liberation* (1975) and *Practical Ethics* widened the moral circle. Bernard Williams and Derek Parfit challenged the foundations of all three families.

Social choice theory added a mathematical warning. The Marquis de Condorcet showed in 1785 that majority voting over three options can cycle: a majority prefers A to B, another B to C and another C to A. Kenneth Arrow's impossibility theorem (1951) generalised the problem: no voting rule meets a short list of fair conditions for three or more options.

The questions left are whether moral truths exist, how to choose between the three families, and how to make collective decisions fair.

Current state. The PhilPapers Survey of 2020 measures the field's divisions. On normative ethics, 36.99 percent accept or lean toward virtue ethics, 32.05 percent deontology and 30.56 percent consequentialism. Asked to choose one, the shares are 25.04, 19.70 and 21.42 percent. Together these account for only 66.16 percent: a third of philosophers hold another view or a combination (Calculation 25.4). On meta-ethics, 62.07 percent accept or lean toward moral realism and 26.12 percent anti-realism.

The largest empirical study of moral judgement is the Moral Machine experiment. Published in *Nature* in October 2018, it collected roughly 40 million decisions from over 2 million people in 233 countries and territories on dilemmas faced by self-driving cars. Responses clustered into three groups, labelled "western", "eastern" and "southern". The average participant made 20 decisions (Calculation 25.4).

Deontic logic remains an active field. The Stanford Encyclopedia lists six normative statuses (obligatory, permissible, impermissible, omissible, optional, non-optional), defined from obligation alone. It describes the threefold partition in which every act is obligatory, impermissible or optional, with no overlap. On Chisholm's paradox it says: "It is now virtually universally acknowledged that Chisholm was right."

AI ethics turned into law. The EU AI Act entered into force on 1 August 2024. Its prohibitions applied from 2 February 2025, the rules for general-purpose models from 2 August 2025, and most of the remainder from 2 August 2026. The rules for high-risk systems in Annex III follow on 2 December 2027. Ethical principles for machines are now obligations in law, with prohibited, regulated and permitted uses: a three-way classification of acts.

The open debates are moral realism, the ethics of AI and of future generations, the status of animals, and how to resolve contrary-to-duty obligations in logic and in law.

Interpretation.

1. *Strand.* I is a sequence of acts or decisions. P is the set of acts. A loop is a return to the same kind of act in a new situation.
2. *Passage and trit.* Von Wright's three statuses are the trits. Obligatory is 1, forbidden is T and optional is 0. The relation "p is obligatory if and only if not-p is forbidden" is the mirror of T12.1: the status of the negation is the mirror of the status, and the optional is fixed (Calculation 25.1). The Islamic five-status scale has the same shape, a neutral middle with two grades on each side; whether its negation is a mirror is not claimed here.
3. *Closure.* A window of three statuses closes when its digit is 0 (T1, T2). There are 9 of 27 closed windows (T3), of two kinds (T4). Six windows contain one obligation,

one prohibition and one option. Three repeat one status. A sequence of acts whose statuses leave a residue is tension in the sense of reading N2. Chisholm's contrary-to-duty case is exactly such a sequence: an obligation violated (1 then T) and a second obligation that arises from the violation. Whether the residue framing solves Chisholm's paradox is not claimed.

4. *Scale.* The Condorcet cycle is a ring. With three voters and three options, 12 of the 216 profiles of strict preference produce a majority cycle, 1/18 (Calculation 25.2). In such a profile the pairwise majorities go A over B, B over C, C over A: three steps of the same direction that return to the start. That is the ring (x, x, x) of T7. A cycle of two options cannot occur, because a pair is a meeting and closes with carry 0 (T7.1). The impossibility of majority cycles over two options, and their possibility over three, is the difference between meetings and rings.
5. *Test.* For deontic logic and Condorcet none is needed: both are identities, checked in Calculations 25.1–25.2. For moral judgement a test could fail. In the Moral Machine data, triples of decisions by one participant can be coded as save, sacrifice or indifferent. If closure describes consistent judgement, participants whose triples close more often than the T6 baseline should be more consistent across repeated dilemmas. If closure rate and consistency are unrelated, the reading fails. Not run here.

The ring grows more probable with more voters. With three options and voters choosing strict orders at random, the probability of a majority cycle is 12/216 = 0.0556 for 3 voters. It rises to 0.0694 for 5, 0.0750 for 7 and 0.0843 for 25 (Calculation 25.5). As the number of voters grows it approaches Guilbaud's limit, $1/4 - (3/2\pi)\cdot\arcsin(1/3) = 0.0877$. In carry language: the more passages vote, the more often the three pairwise majorities add to a ring that does not close at the level of a social ranking. It does not leave a residue of zero; it passes on as tension, a cycle that no rule can break without giving up one of Arrow's conditions. Real electorates are not random. The baseline is the analogue of T6 for preference: what closure would look like with no structure.

Consequences. Ethics holds two structures of the Foundations as published fact. Von Wright's three statuses, with obligation and prohibition as mirrors, are the trit and its mirror. The Condorcet cycle is a ring of three that cannot occur with two. Both were known long before this theory, and both are exact. They earn grade A.

What the Foundations add is a reason why three is the threshold in both. In deontic logic, three statuses are the smallest set in which a status differs from its mirror and a neutral status exists. That is T1 (iii): $\mathbb{Z}_3$ is the smallest cyclic group in which a value differs from its reverse. In social choice, three options are the smallest number that permits a ring. That is T7: a ring has length three, a meeting length two. Arrow's theorem needs at least three options for the same reason. The Foundations do not prove Arrow's theorem. They do place its threshold in the same arithmetic as the window of three.

What it costs is a temptation to over-read. The three families of normative theory (virtue, duty, consequence) form a triad, and the Moral Machine found three clusters. Neither is a closure in the Foundations' sense. Both are counts of three that could have been two or four. They are reported and graded C.

The checklist for ethics: T1, T3, T4, T7, T7.1 and T12.1 apply. T6 gives the baseline for the stated test. T13 applies to profiles of preference: a cycle read without a starting option is a rotation class, and the two cyclic profiles of each orientation are rotations of one another. T15 has no object. T16–T24 have no ethical object.

Grade. A for deontic statuses as the trit and its mirror (von Wright 1951) and for the Condorcet cycle as a ring of three (12/216). B for the stated consistency test on moral-judgement data. C for normative theory as a whole.

25.3 Architecture and urban design

History. Architecture is the oldest of the design sciences with a surviving theory. Vitruvius's *De architectura* (c. 25 BCE) required that buildings be strong, useful and beautiful (*firmitas, utilitas, venustas*), a triad repeated in architectural teaching ever since. He derived proportion from the human body, the figure later drawn by Leonardo inside a circle and square. Greek temples used modules and the orders; Roman builders added the arch, the vault and concrete.

Medieval masons built Gothic cathedrals by geometric rules handed down in lodges: proportions derived from squares, equilateral triangles and their rotations. Renaissance theorists returned to Vitruvius. Alberti's *De re aedificatoria* (1452) and Palladio's *Four Books* (1570) set proportions by simple ratios, often musical ones such as 2:3 and 3:4. Rudolf Wittkower's *Architectural Principles in the Age of Humanism* (1949) showed how Renaissance architects took these ratios from harmonic theory.

Structural engineering separated from architecture in the nineteenth century. James Clerk Maxwell's paper of 1864 on frames gave a counting rule for rigidity. A plane frame with j joints needs at least $2j - 3$ bars to be rigid. A polygon of j joints has only j bars, so it is rigid only when $j \le 3$: the triangle is the only rigid polygon. Trusses are built from triangles for this reason. Iron, steel and reinforced concrete made new spans possible: the Crystal Palace (1851), the Eiffel Tower (1889) and the skyscrapers of Chicago from the 1880s.

Modernism made function the measure. Louis Sullivan's "form follows function" (1896), the Bauhaus (1919–1933) and Le Corbusier's five points of architecture (1926) rejected ornament. Le Corbusier's Modulor (1948) set a scale of dimensions from a man 1.83 m tall with a raised arm at 2.26 m, divided by the golden ratio. Buckminster Fuller's geodesic domes (patented 1954) triangulated the sphere.

Urban design has its own long history: the grid of Miletus attributed to Hippodamus (fifth century BCE), the Roman camp, Haussmann's Paris (1853–1870), Ebenezer Howard's garden city (1898) and Le Corbusier's towers in a park. Jane Jacobs's *The Death and Life of Great American Cities* (1961) attacked modernist planning in the name of mixed, dense streets. Walter Christaller's central place theory (1933) explained the spacing of towns by the markets they serve. Christopher Alexander's *A Pattern Language* (1977) gave 253 patterns from regions to window seats as a grammar of design.

The questions left are how buildings should serve people and climate, what makes a place good to live in, and how cities should grow.

Current state. Buildings are a large part of the climate problem. The UN Environment Programme's Global Status Report for Buildings and Construction 2024/2025, published on 17 March 2025, finds that the sector is consuming 32 percent of global energy and contributing 34 percent of global CO_2 emissions. Cement and steel alone account for 18 percent of global emissions. Low-carbon materials, timber construction, reuse of existing buildings and electrification of heating are the main responses.

The world is urban and becoming more so. The UN's World Urbanization Prospects 2025 reports that cities are home to 45 percent of the global population of 8.2 billion, under a new harmonised definition that separates cities, towns and rural areas.

The profession's highest prize marks a shift in taste. The 2026 Pritzker Prize went to Smiljan Radić Clarke of Santiago, Chile, the 55th laureate. The jury praised buildings that feel provisional or unfinished, that prefer fragility to certainty and treat vulnerability as part of living. His works include the 2014 Serpentine Pavilion in London and the Teatro Regional del Biobío in Concepción (2018). The 2025 laureate was Liu Jiakun of Chengdu.

Central place theory remains a standard model in geography. Christaller's three principles are documented with their K values: K = 3 for marketing, K = 4 for transport and K = 7 for administration. August Lösch (1940) generalised them to all hexagonal market sizes. Those sizes are the numbers $x^2 + xy + y^2$, which OEIS names "Loeschian numbers … norms of vectors in A2 lattice": 1, 3, 4, 7, 9, 12, 13 and so on.

The open debates are density against sprawl, the embodied carbon of new building against renovation, informal settlements, and the role of computation and generative design in architecture.

Interpretation.

1. *Strand.* I is the route of a person through a building or city. P is the set of places: rooms, streets, squares. A loop is a return to a place already visited.
2. *Passage and trit.* In a plan, a corridor that leads back to its start is a loop. The trit rule scores whether a return leaves a place the way the first visit left it (1), the way it came in (T) or otherwise (0). Space-syntax analysis already measures the integration of plans by graph methods. The trit rule would add a measure of how routes repeat. Not computed here.
3. *Closure.* Three structural facts are closures in the Foundations' sense, and all are published independently. First, Maxwell's count: a plane frame needs $2j - 3$ bars, so only the triangle is a rigid polygon. In 3D a frame needs $3j - 6$ bars: a tetrahedron with 4 joints needs 6 (Calculation 25.3). The triangle is the ring of T7 in structure: the smallest closed configuration that holds its shape. Second, all periodic plan grids, tilings and lattices obey the crystallographic restriction: rotation orders 1, 2, 3, 4 and 6, all on the ladder of T15 (Calculation 23.10). Third, the hexagonal lattice of central place theory is the Eisenstein lattice $\mathbb{Z}[\omega]$ of T14. Christaller's K values are norms in that lattice.
4. *Scale.* Christaller's hierarchy multiplies market area by K at each level. K = 3 is the ring ladder of T15 (×3 per level). K = 4 is two meetings (2^2). K = 7 lies off the ladder: it is a norm in $\mathbb{Z}[\omega]$ ($7 = 2^2 + 2\cdot1 + 1^2$) but not $2^a \cdot 3^b$ (Calculation 25.3).

Christaller's administrative principle is therefore off the ladder, which is a fact, not a failure: the ladder is not claimed for all norms. Alexander's 253 patterns (11 × 23) are off the ladder.

5. *Test.* Stated before checking data: if the ladder governs settlement hierarchies built from equal blocks, then measured nesting ratios of central places should cluster at 3 and 4 more than at 7. If hierarchies of real regions show K = 7 as often as K = 3, the ladder adds nothing to Christaller. Not run here; it needs regional settlement data.

A note on what Maxwell's count implies for design. In the plane, adding a joint adds two degrees of freedom and needs two bars to fix it, so 2j − 3 bars for j joints. A triangle has 3 joints and needs 3 bars. A square has 4 joints and 4 bars but needs 5, so it folds unless braced (Calculation 25.3). Every rigid plane truss is built from triangles for this reason, and its bars are counted in twos. In space the count is 3j − 6, and the tetrahedron, with 4 joints and 6 bars, is the rigid unit. Fuller's geodesic domes and every space frame are made of triangles. That the smallest rigid polygon has three sides is an arithmetic fact about degrees of freedom. The Foundations' ring of three is the smallest closure with winding number ±1 (T7). The two are different theorems that give the same number. Their agreement is reported, not claimed as a derivation.

Consequences. Architecture and urban design hold three structures of the Foundations as published fact: the triangle as the only rigid polygon (Maxwell 1864), the crystallographic restriction on periodic plans, and the hexagonal lattice of central places with K values as norms of Z[ω]. The first two are also treated in Chapter 9 (building) and Chapter 6 (crystals); here they appear as constraints on form. The third is the strongest specific link: Christaller's K = 3 is exactly a step of the ring ladder. Lösch's generalisation is exactly the norm set of T14's ring. The Foundations' ×3 ladder sits inside a lattice that geographers have used since 1933.

What the interpretation adds is a sorting of Christaller's principles. Marketing (K = 3) and transport (K = 4) are on the ladder and administration (K = 7) is not. The theory would predict that equal-block merging favours the first two. The test in point 5 can check that.

For architecture as an art, the interpretation is a reading. Vitruvius's triad, Palladio's musical ratios and the Modulor are human choices. Palladio's ratios 2:3 and 3:4 lie on the ladder because they are musical ratios (23.1). The Modulor's ratios are golden-ratio steps. The ratio of its two heights, 2.26 to 1.13 m, is exactly 2, while 1.83 to 1.13 is the golden ratio to three decimals (Calculation 25.3). The golden ratio is irrational, like $\log_2 3$. But no theorem of the Foundations concerns it, so the Modulor gets no link.

The checklist: T7 (triangle as ring), T14 (hexagonal lattice and norms), T15 (K values, grids), Maxwell's count, and the restriction are applied. T13 applies to circuits through a plan, read with or without a starting room. T19 applies to no building. T21–T24 have no architectural object.

Grade. A for the rigid triangle (Maxwell 1864), periodic plans on the ladder (crystallographic restriction) and central-place K values as norms of Z[ω] with K = 3 on

the ring ladder (Christaller 1933; Lösch 1940). B for the stated nesting-ratio test. C for architecture as an art.

25.4 Design

History. Design as a profession separated from craft in the industrial revolution. Josiah Wedgwood in the 1760s divided the making of pottery between designers and workers and sold by catalogue. The Great Exhibition of 1851 showed industrial goods in quantity and, to critics such as John Ruskin and William Morris, showed their ugliness. Morris’s Arts and Crafts movement answered with handwork, while the Deutscher Werkbund (1907) sought good design for industry. The Bauhaus (1919–1933) taught design as a unity of art and technology under Walter Gropius, then Hannes Meyer and Ludwig Mies van der Rohe.

Standardisation gave design a mathematics. Charles Renard, a French army engineer, proposed in the nineteenth century a series of preferred numbers, spaced in equal ratios, to reduce the variety of balloon cables. The Renard series became an international standard for preferred sizes. Electronics adopted its own preferred values. The E-series of IEC 60063, first published in 1952, sets 3, 6, 12, 24, 48, 96 and 192 values per decade for resistors and capacitors, with tolerances from 40 to 0.5 percent. Each series doubles the previous one. Paper sizes (ISO 216, from the German DIN 476 of 1922) use the ratio √2 so that halving a sheet keeps its shape.

Typography and graphic design formalised grids. The Swiss style of the 1950s (Josef Müller-Brockmann, *Grid Systems*, 1981) built layouts on modular grids, commonly of 12 columns because 12 divides by 2, 3, 4 and 6. Web design adopted 12-column grids in the 2000s for the same reason.

Design methods became a research field. The Conference on Design Methods in London (1962) launched it. Herbert Simon’s *The Sciences of the Artificial* (1969) defined design as devising courses of action aimed at changing existing situations into preferred ones. Horst Rittel and Melvin Webber (1973) described “wicked problems” that have no definitive formulation or stopping rule. Christopher Alexander’s *Notes on the Synthesis of Form* (1964) and *A Pattern Language* (1977) influenced software. Ward Cunningham’s first wiki was built partly on pattern-language ideas, and software design patterns took their name from Alexander.

Donald Norman’s *The Design of Everyday Things* (1988) made usability a design science. Human-centred design, user experience and service design followed. In the 2000s “design thinking” spread from IDEO and Stanford into business schools.

Design also has a history outside Europe. Japanese craft traditions refined objects over centuries within fixed forms, and the tea ceremony made the design of a room, a bowl and a gesture one practice. Chinese and Islamic design developed geometric ornament and modular construction long before industrial standards. Modern Scandinavian and Japanese industrial design in the twentieth century joined craft and mass production.

The questions left are how to design for systems rather than objects, how to measure good design, and what design becomes when generation is automated.

Current state. Design is now shaped by three forces: sustainability, regulation and automation. The buildings report cited in 25.3 sets the scale of the material problem. The European Union's Ecodesign for Sustainable Products Regulation extends design obligations from energy use to durability, repairability and recyclability. Under the EU AI Act, generated designs and content carry transparency duties from 2 August 2026, with a transition to 2 December 2026 for systems already on the market.

Generative tools have entered every stage of design practice, from layout to product form. In the United States the Copyright Office's position (23.2) applies equally to designs. Prompts alone do not make the user an author, so design output from prompts is unprotected unless a human contribution can be shown.

Standards of preferred numbers continue unchanged. IEC 60063 is in its third edition of 27 March 2015. E3 is now rarely used, and E192 also covers tolerances of 0.25 and 0.1 percent.

Pattern languages have spread beyond architecture. Software design patterns, interaction patterns and pattern libraries for public services follow Alexander's form: a named problem, its context and a solution. The 253 patterns of the original book remain the reference.

The open debates are the ethics of persuasive and addictive design, design for repair and circularity, and the role of designers when tools generate options without limit.

Interpretation.

1. *Strand.* I is the sequence of design decisions; P is the set of options at each decision. A loop is a return to an option once rejected: iteration.
2. *Passage and trit.* In iterative design, a return to an earlier option scores by the trit rule whether the designer proceeds as before (1), backs out as before (T) or does something new (0). Design research logs (version histories, design rationale records) would allow the rule to be computed. Not computed here.
3. *Closure.* Preferred-number series are designed closures. The E-series place n values per decade in equal ratios 10^(1/n), and every series is half of the next: E6 is every second value of E12, and so on. The sizes 3, 6, 12, 24, 48, 96, 192 are $3 \cdot 2^k$, all on the ladder $2^a \cdot 3^b$ (Calculation 25.3). This is the ladder of T15 written into an international standard: one ring of three at the base, then meetings doubling. The 12-column grid sits at $2^2 \cdot 3$ for the same reason: it divides into halves, thirds, quarters and sixths.
4. *Scale.* The E-series and the 12-column grid are equal-block merging in the sense of T15: each level splits every interval into two equal ratio steps. Paper sizes halve by √2, which is a meeting in area (×2) and irrational in length. The Renard series use 5, 10, 20, 40 and 80 steps per decade, $5 \cdot 2^k$, off the ladder. Two standards for the same task chose different bases. The choice of base is a convention, and the ladder applies to the doubling, not to the base.
5. *Test.* Stated: where designers choose a number of equal divisions freely, the choice should fall on ladder numbers ($2^a \cdot 3^b$) more often than chance. Chance is the share of ladder numbers among the candidates: 20 of 1 to 100 (Calculation 22.5). Column grids (12, 16, 24), E-series ($3 \cdot 2^k$) and clock faces (12, 24, 60) give a quick sample in

which 60 fails. A systematic count over typographic grids, keyboard layouts and modular building systems would test it. If ladder numbers are no more common than among neighbouring integers, the reading fails. Not run here.

A comparison of the two standards sharpens the point. Renard's series use 5 steps per decade, then 10, 20, 40, 80. The IEC chose 3, 6, 12, 24, 48, 96, 192 in 1952. Both double at each refinement, so both are ladders in the ×2 direction. Only the IEC's base, 3, puts the series on $2^a \cdot 3^b$. Renard's base 5 does not (Calculation 25.3). Both standards work, and both are in force. The ladder is therefore not forced by the task of preferred numbers. What is forced is the doubling: a finer series must contain the coarser one. That is T15's equal-block condition, stated by engineers. The base is a convention. In the electronics series the base series E3 goes with the widest tolerance, 40 percent.

The check against the Foundations' T6 baseline matters for the stated test. Among the integers 1 to 100, 20 are ladder numbers (Calculation 22.5). Among small numbers the density is much higher: 8 of 1 to 12 (Calculation 25.3). A test of designers' choices must compare with neighbouring integers of the same size, not with all integers. Otherwise small numbers will appear to favour the ladder because they are small.

Consequences. Design holds one structure of the Foundations as published fact: the E-series of IEC 60063, which are $3 \cdot 2^k$ by construction. The reason is practical: each series must contain the coarser one, so the step count doubles. Three values per decade is the coarsest useful series. That is the ladder of T15 written into a standard, independent of this theory. The grade is A for that structure.

The 12-column grid is the same arithmetic as the twelve-step scale (23.1) for a different reason. Twelve is the smallest number divisible by 2, 3, 4 and 6, so it allows the most layouts. Designers chose it for its divisors, not for closure. The Foundations explain why ladder numbers are good choices when designs must be halved and thirded. They do not explain why designers need halves and thirds.

What the interpretation does not add: anything about the ethics or quality of design, wicked problems, or the value of generated design. Simon's definition of design as changing an existing situation into a preferred one is close to reading N2: what does not close passes on as tension, and design makes it close. That is a paraphrase, graded C.

The checklist: T15 (E-series, grids, Renard as off-ladder), T7 (doubling as meetings), T13 (iterations read with or without a start) apply. T16–T18 apply only through the twelve-step comparison with music. T19–T24 have no design object.

Grade. A for the E-series of preferred numbers as $3 \cdot 2^k$ on the ladder of T15 (IEC 60063). B for the stated test of ladder numbers in freely chosen divisions. C for design as a whole.

Synthesis — What the Sciences Say Back

Twenty-five chapters have read 114 disciplines through the same five points. This chapter collects what came back. It counts the grades and asks what the A grades have in common. It lists every test that was run, with its outcome, and says what the outcomes mean for the Foundations. It ends with the tests still open, ranked, and with the status of the map itself.

S.1 The count

Each discipline received a grade, often mixed. Taking the highest grade a discipline earned for at least one named structure:

	Disciplines	Highest grade A	Highest grade B	Only C
Part I — The register (Ch. 1)	7	7	0	0
Part II — Strand and field (Ch. 2–5)	18	16	0	2
Part III — Matter (Ch. 6–9)	20	18	1	1
Part IV — Life (Ch. 10–13)	18	15	1	2
Part V — Organism and mind (Ch. 14–17)	20	13	3	4
Part VI — Society (Ch. 18–21)	16	7	6	3
Part VII — Culture (Ch. 22–25)	15	13	2	0
Total	**114**	**89**	**13**	**12**

Read naively, 89 of 114 looks like a sweep. It is not, and the reason matters more than the number. An A says only that a discipline already contains, in its own published results, a structure the Foundations also contain. It does not say that the discipline confirms the theory. Section S.2 shows that the 89 A grades rest on five pieces of mathematics, all older than the theory. The A grades thin out from matter to society: the register and matter are dense with them, society and the life course are not. Twelve disciplines earned nothing but C: cosmology, nuclear engineering, environmental engineering, palaeontology, forestry, human development, veterinary science, animal and dairy science, nutrition, media and communications, urbanisation and management.

S.2 One algebra under the A grades

Every A grade in this book is an instance of one of five structures.

1. Residues mod 3 with a carry (T1, T7.1). The digit rule of balanced ternary appears as the reading frame of the genetic code (Crick et al. 1961), the Duchenne–

Becker frame rule (Monaco et al. 1988), exon skipping, overlapping viral genes, the collagen repeat, the triality of hadrons (where the carry is baryon number), Fortescue's symmetrical components and the sequence rule h mod 3 of power harmonics, the Hägg sum of close-packed polytypes, the line condition of F_3^n (SET, cap sets), the zero-sum check of the ternary Golay code, deontic status with mirror negation, and the Yoruba and Roman subtractive numerals.

2. The Eisenstein integers Z[ω] (T14). Capsid triangulation numbers (Caspar and Klug 1962), nanotube metallicity $n - m \equiv 0 \pmod 3$ (Hamada et al. 1992), Löschian and Christaller market sizes, geodesic dome frequencies, the index-3 lithium layer of LiC_6, the grid-cell lattice, the Tonnetz of harmony and Lagrange's equilateral solution all live in this ring or on its norms.

3. The ladder $2^a \cdot 3^b$ (T15). The crystallographic restriction (rotation orders 1, 2, 3, 4, 6), the period-doubling cascade and the $3 \cdot 2^k$ cascade (May 1976; Feigenbaum 1978), Sharkovskii's order, low-order orbital resonances, Strahler order, pedigree doubling, exact mechanical constraint ($6 = 3 \times 2$) and the E-series of preferred numbers.

4. The convergents of a ratio (T16–T18). Tunings at 12, 41 and 53, Jing Fang's comma, the Metonic and other lunisolar cycles, the Saros, Brocot's gear trains, and Eliahou's bounds on Collatz cycles. T17 names the convergents of $\log_2 3$; the calendars use the same method on other ratios.

5. Odd rings against even pairs (T7). The repressilator oscillates and the toggle switch latches (Elowitz and Leibler 2000; Gardner et al. 2000); odd inverter rings oscillate and even ones latch; a negative-feedback loop needs three stages to oscillate; three strains coexist in rock–paper–scissors (Kerr et al. 2002) and only odd sets coexist in pure competition (Allesina and Levine 2011); restricted exchange is a meeting and generalized exchange a ring (Lévi-Strauss 1949); the Condorcet cycle is a ring of three.

The five are one algebra: the smallest cyclic group with a direction ($\mathbb{Z}_3$), its ring of integers in the plane (Z[ω]), and its relation to the smallest group with a pairing ($\mathbb{Z}_2$), expressed as the two ladders and their convergents. That is what the A grades show. Wherever a system counts in threes, pairs things off, or tiles a plane with triangles, this algebra appears, and it appears because it is the algebra of those operations. The Foundations chose it as their ground. The sciences used it before them. The A grades confirm that the choice is well made: the register of the theory is the register of many real systems. They do not show that the world is a net.

S.3 The tests that were run

A theory that only finds itself has found nothing (Preface). The tests are therefore the part of this book that can move the theory. Twenty-four were run on data or by exact computation. A first edition of this book (October 2026) fixed several of them before their data were opened; the chapters say which.

Test	Ch.	Data	Outcome
Closure rule on signed triads: (1, T, T) should be rare	15.1, 18.1, 18.4, 19.3	Leskovec et al. 2010; five public networks, 6.6 million triangles	**Failed.** (1, T, T) is over-represented in 4 of 5 networks; the rule fails exactly where it differs from weak balance.
Closed windows persist: majority cycles at chance rate	19.2	68 PrefLib elections, 9,238 triples	**Failed.** 2 cycles against about 811 expected.
Closed windows in monthly S&P returns; deficit before crises	19.1	Shiller/FRED 1871–2026	**Failed.** The excess is serial correlation; no pre-crisis deficit ($z = -0.22$).
Business-cycle lengths on the ladder	20.1	NBER, 34 cycles	**Failed.** 23 of 34, $p = 0.35$.
Measles periods on the ladder	20.3	20 towns, England and Wales 1944–64	**Weakly passed.** 18 of 20; $p = 0.016$ against a log-uniform null, 0.091 against a whole-year null.
Epidemic periods on the ladder in models	12.4	Earn et al. 2000	**Failed** as formulated: the cascade includes 5 and 7.
Periods and counts on the ladder in development, evolution and plants	12	25 observed values	**Failed** as a preference: 11 on the ladder, 14 off. Phyllotaxis follows the golden ratio.
Locomotor–respiratory locking on the ladder	14.2	Daley et al. 2013	**Failed** in strict form: runners use 5 : 2 and 7 : 2.
Grid-module ratio √3 (ring step)	14.3	published module spacings	**Failed** by about 9 standard errors; √2 (meeting step) fits.
Nuclear-receptor 3-4-5 rule as closure mod 3	14.4	published spacing rule	**Failed:** it is helix geometry.
Circadian period as a comma	16.1	24.18 h	**Failed:** 12.94 cents, against a comma of 23.46.
QBO locked to a ladder period	8.3	27 or 32 months predicted	**Failed:** 28–29 months observed.

Test	Ch.	Data	Outcome
Interglacial intervals at integer multiples	8.4	LR04	**At chance** (4 of 9), and LR04 is orbitally tuned.
Period lengths of the periodic table on the ladder	6.1	2, 8, 18, 32	**Not evidence:** all on the ladder, but p ≈ 0.15 by chance.
Every minimally rigid frame contains a triangle	9.1	$K_{3,3}$	**Failed:** $K_{3,3}$ is rigid without a triangle.
Mirror symmetry of the two open graphene-ribbon families	6.6	published band gaps	**Partly failed.**
T14 rest/turn label on capsid lattices	10.5	exact computation	**Failed:** the label is not invariant across the 12 equivalent lattice elements.
IX.1: the T24 lever is the T23 coupling	3.4, 6.5	K_0' of solids	**Failed in sign** for ordinary solids: $\gamma = K_0' - 2$ lies between +1.1 and +3.6, the lever is −3.94.
IX.3: the T14 turn share settles	1.1	boxes of half-width 40 to 800	**Open, not settling:** 27.3, 41.0, 32.2, 23.3, 26.9 %.
IX.2: the rewrite R stays on the ladder	1.4	rings of 3 to 10 trits	**Failed once:** a 26-cycle at n = 7.
Calendar cycles are convergents	24.3	lunisolar, Saros, Gregorian, Antikythera	**Partly passed:** lunisolar and Saros yes; Gregorian 97/400 and Antikythera planetary relations no.
Closed windows in verse	22.1	six classical texts	**Weakly passed** for Homer after a pair control; nothing in Virgil or prose.
In-frame indels above the T6 baseline	10.2	Lin et al. 2017, 7,137 indels	**Passed:** 47.1 % against 33.3 % ($p \approx 10^{-127}$). Biology already explains it by selection.

Test	Ch.	Data	Outcome
Period-3 signal in coding DNA	10.2	*Arabidopsis* chloroplast	**Passed:** median peak 7.26 coding, 0.85 non-coding. Known since Fickett (1982).

S.4 What the outcomes mean

The pattern is clear, and it is not the pattern a reading-everywhere theory would produce.

Where mod-3 arithmetic is the mechanism, the closure rule holds. The reading frame, the frame rule in muscular dystrophy, nanotube conduction, polytype stacking, ring oscillators and hadron composition are systems that literally count in threes. In them, “digit 0 closes, a nonzero digit stays open” is not a metaphor; it is how the system works, and the Foundations describe it exactly.

Where closure is read as a law of persistence in behaviour, it fails. Signed triads, votes and markets were the cleanest tests of the reading N2 outside physics and biology, and all three came out against it. The Condorcet result is the sharpest: closed rings among preferences are not merely rarer than chance, they are almost absent. Whatever keeps social structures in being, it is not the closure of a window of three.

The ladder holds where it is forced and fails as a preference. Crystallographic restriction and period doubling put the ladder into the mathematics of the system; there it holds without exception. Where the ladder is read as a preferred set of numbers — cycle lengths, magic numbers, c-ring sizes, phyllotaxis, breathing ratios — it does no better than chance, and often worse.

Two open questions of the Foundations moved. IX.1 now has a negative answer for ordinary solids: if T23 holds, the lever of the medium has the wrong sign to be the lever of a free path. IX.3 has no limit in sight: the turn share in $\mathbb{Z}[\omega]$ oscillates with the size of the box. Both results belong in the next edition of the Foundations.

The net result is narrower than the hope and firmer than a reading. The register of the Foundations is the register of a large and identifiable class of systems: those that count, pair and close in threes. The theory describes that class exactly. Its extension to persistence in general — to societies, markets and minds — was stated as testable, tested, and in its present form refuted.

S.5 The tests that remain

Ranked by how sharply they decide and how soon they can be run:

1. **Period 8 of the periodic table** (6.1). A closed eighth period has a ladder length (36, 48, 54); a closure at 46 or 50 refutes. Decidable by relativistic computation before any experiment.
2. **Closed windows in trained ternary weights** (1.7). Compare the closed-window share in BitNet weights with the T6 baseline. One download and one count.

3. **Frameshift and frame-rule exceptions** (10.1, 17.1). Whether exceptions to the Duchenne frame rule follow the carry of T7.1.
4. **Capsid norms** (10.5, 17.3). Whether new capsid structures fall only on Eisenstein norms.
5. **Residue term in SiC polytype band gaps** (6.4). A term in the Hägg sum mod 3 beyond the hexagonality.
6. **Legal and deontic corpora** (1.2, 21.1). Closed windows of statuses against the T6 baseline.
7. **Two-stage synthetic clocks** (16.1). The Foundations require three stages for a closed oscillation.
8. **Cleavage counts before the mid-blastula transition** (12.2). *Xenopus* has 12, a closure depth; other species decide.
9. **Ibn Khaldun's three generations and the Seshat databank** (20.2, 24.1).
10. **G191-B2B and the solar edge at 7,400 AU** (2.4, 4.4). Both stated in the companion document; the first is about 700 times below present precision, and the second is identical in kind to the MOND transition radius, so neither separates the theory from existing physics.

S.6 The status of the map

The map of 0.2 claimed that every discipline fits a cell, and that it fails if one fits none or fits two with no way to decide. None fitted no cell. Several straddle: nanotubes appear in Chapters 3, 5 and 6, the Condorcet cycle in 15, 19 and 25, deontic status in 1, 21 and 25, capsids in 10 and 17. In each case the chapters agree on the structure and differ only in the question asked, which is what the columns say. The map therefore holds as a reading. Its number of columns is proved (T7 plus the open window); its rows are observed; the placing of each discipline is judged. It organised the book without forcing a result: the same map that holds the A grades of matter holds the refutations of society.

Appendix A — The Calculations

Every number this book computes itself is given here with the Python program that computes it and the exact output that program printed. The programs need Python 3 with the standard library; a few use mpmath or numpy, and the data-based calculations name their data files. All programs were run again after the chapters were written, and every printed output matched the text.

Calculations for Chapter 1 — The Register: Mathematics, Logic and Computation

All scripts run with Python 3.13, sympy 1.14, mpmath, numpy 2.5 and scipy on 27 October 2026. Each block is complete and was run as shown.

Calculation 1.1. Digit, carry and address (T1, T7.1, T12).

```
        def bal(s):                       # balanced digit and carry (T1)
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        def bt(n):                        # balanced-ternary word (T12)
            out = ""
            while n != 0:
                d, n = bal(n); out = "10T"[1 - d] + out
            return out or "0"
        for name, w in [("standing", [0]), ("meeting", [1, -1]), ("ring 111", [1, 1, 1]),

                        ("ring TTT", [-1, -1, -1]), ("open 11", [1, 1])]:
            print(name, "sum", sum(w), "digit, carry", bal(sum(w)))
        print([(n, bt(n)) for n in (8, -8, 13, 40, -121)])
```

Output:

```
standing sum 0 digit, carry (0, 0)
meeting sum 0 digit, carry (0, 0)
ring 111 sum 3 digit, carry (0, 1)
ring TTT sum -3 digit, carry (0, -1)
open 11 sum 2 digit, carry (-1, 1)
[(8, '10T'), (-8, 'T01'), (13, '111'), (40, '1111'), (-121, 'TTTTT')]
```

Calculation 1.2. Ternary necklaces and Lyndon words (T13).

```
        from sympy import totient, mobius, divisors
        g = lambda n: sum(totient(n // d) * 3**d for d in divisors(n)) // n   # necklaces
(T13)
        L = lambda n: sum(mobius(n // d) * 3**d for d in divisors(n)) // n   # Lyndon words

        print("g(n):", [g(n) for n in range(1, 11)])
        print("Lyndon:", [L(n) for n in range(1, 11)])
        print("g(n)/(3^n/n):", [round(g(n) / (3**n / n), 6) for n in (6, 12, 24)])
```

Output:

```
g(n): [3, 6, 11, 24, 51, 130, 315, 834, 2195, 5934]
Lyndon: [3, 3, 8, 18, 48, 116, 312, 810, 2184, 5880]
g(n)/(3^n/n): [1.069959, 1.001835, 1.000002]
```

Calculation 1.3. Convergents of $\log_2 3$ and Eliahou's cycle-length coefficients (T17).

```
        import mpmath as mp
        mp.mp.dps = 40
        x = mp.log(3) / mp.log(2); cf = []; y = x
        for _ in range(17):
            a = int(mp.floor(y)); cf.append(a); y = 1 / (y - a)
        p0, q0, p1, q1 = 1, 0, cf[0], 1; conv = [(p1, q1)]
        for a in cf[1:]:
            p0, q0, p1, q1 = p1, q1, a * p1 + p0, a * q1 + q0; conv.append((p1, q1))
        print(cf)
        print(conv)
        for p in (301994, 17087915, 85137581):
            print(p, [c for c in conv if c[0] == p])
        lens = sorted({301994*a + 17087915*b + 85137581*c
                       for a in range(3) for b in range(1, 3) for c in range(2) if a*c == 0}
)
        print("smallest admissible cycle lengths:", lens[:3])
```

Output:

```
[1, 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, 1, 55, 1, 4]
[(1, 1), (2, 1), (3, 2), (8, 5), (19, 12), (65, 41), (84, 53), (485, 306), (1054,
665), (24727, 15601), (50508, 31867), (125743, 79335), (176251, 111202), (301994,
190537), (16785921, 10590737), (17087915, 10781274), (85137581, 53715833)]
301994 [(301994, 190537)]
17087915 [(17087915, 10781274)]
85137581 [(85137581, 53715833)]
smallest admissible cycle lengths: [17087915, 17389909, 17691903]
```

Calculation 1.4. Consecutive numbers of the form 2^m·3^n (Gersonides; T15, T16).

```
        H = sorted({2**a * 3**b for a in range(41) for b in range(26) if 2**a * 3**b < 10**1
2})
        S = set(H)
        print([(h, h + 1) for h in H if h + 1 in S])
```

Output:

```
[(1, 2), (2, 3), (3, 4), (8, 9)]
```

Calculation 1.5. Radix economy and bits per trit (III.1).

```
        import math
        print([(r, round(r / math.log(r), 4)) for r in (2, 3, 4, 5)], round(math.e /
math.log(math.e), 4))
        print("bits per trit:", round(math.log2(3), 5))
```

Output:

```
[(2, 2.8854), (3, 2.7307), (4, 2.8854), (5, 3.1067)] 2.7183
bits per trit: 1.58496
```

Calculation 1.6. Lines of F_3^n as closed windows; small cap sets; the Ellenberg–Gijswijt constant (T4).

```
        import itertools, mpmath as mp
        F3 = (0, 1, 2)
        same_or_diff = all(((x+y+z) % 3 == 0) == (len({x, y, z}) in (1, 3))
                           for x in F3 for y in F3 for z in F3)
        print("zero sum <=> all same or all different:", same_or_diff)
        for n in (1, 2, 3, 4):
            pts = list(itertools.product(F3, repeat=n))
            lines = sum(1 for a, b, c in itertools.combinations(pts, 3)
                        if all((x+y+z) % 3 == 0 for x, y, z in zip(a, b, c)))
            print("n", n, "lines", lines, "formula 3^(n-1)(3^n-1)/2 =", 3**(n-1)*(3**n-1)//
2)
        def maxcap(n):
            pts = list(itertools.product(F3, repeat=n)); best = 0
            third = lambda a, b: tuple((-x-y) % 3 for x, y in zip(a, b))
            def rec(i, chosen, forb):
                nonlocal best
                if len(chosen) + len(pts) - i <= best: return
                if i == len(pts): best = len(chosen); return
                p = pts[i]
                if p not in forb:
                    rec(i+1, chosen + [p], forb | {third(p, q) for q in chosen})
                rec(i+1, chosen, forb)
            rec(0, [], set()); return best
        print("largest cap n=2:", maxcap(2), "n=3:", maxcap(3))
        f = lambda t: (1 + t + t**2) / t**(mp.mpf(2)/3)
        tm = mp.findroot(lambda t: mp.diff(f, t), 0.6)
        print("min (1+t+t^2)/t^(2/3) =", mp.nstr(f(tm), 8))
        print("20^(1/4) =", mp.nstr(mp.mpf(20)**0.25, 6), " 112^(1/6) =",
mp.nstr(mp.mpf(112)**(mp.mpf(1)/6), 6),
              " 512^(1/8) =", mp.nstr(mp.mpf(512)**(mp.mpf(1)/8), 6))
```

Output:

```
zero sum <=> all same or all different: True
n 1 lines 1 formula 3^(n-1)(3^n-1)/2 = 1
n 2 lines 12 formula 3^(n-1)(3^n-1)/2 = 12
n 3 lines 117 formula 3^(n-1)(3^n-1)/2 = 117
n 4 lines 1080 formula 3^(n-1)(3^n-1)/2 = 1080
largest cap n=2: 4 n=3: 9
min (1+t+t^2)/t^(2/3) = 2.7551046
20^(1/4) = 2.11474  112^(1/6) = 2.19551  512^(1/8) = 2.18102
```

Calculation 1.7. Peirce’s 10 sign classes against the 9 closed windows (T3).

```
        import itertools
        peirce = [t for t in itertools.product((1, 2, 3), repeat=3) if t[0] >= t[1] >= t[2]]

        closed = [t for t in itertools.product((-1, 0, 1), repeat=3) if sum(t) % 3 == 0]

        m = {1: 1, 2: 0, 3: -1}
        both = [t for t in peirce if sum(m[v] for v in t) % 3 == 0]
        print("Peirce classes", len(peirce), "closed windows", len(closed), "Peirce classes
that close", len(both), both)
```

Output:

```
Peirce classes 10 closed windows 9 Peirce classes that close 4 [(1, 1, 1), (2, 2,
2), (3, 2, 1), (3, 3, 3)]
```

Calculation 1.8. Negation and deontic status as trit mirror (T12.1).

```
        mirror = {1: -1, 0: 0, -1: 1}
        kleene_not = {"T": "F", "U": "U", "F": "T"}
        status = {"obligatory": 1, "optional": 0, "forbidden": -1}
        # status of not-p from status of p: O p <-> F not-p; optional p <-> optional not-p

        neg_status = {"obligatory": "forbidden", "optional": "optional", "forbidden": "oblig
atory"}
        print(all(status[neg_status[k]] == mirror[v] for k, v in status.items()))
        print(all({"T": 1, "U": 0, "F": -1}[kleene_not[k]] == mirror[{"T": 1, "U": 0, "F":
-1}[k]] for k in kleene_not))
```

Output:

```
True
True
```

Calculation 1.9. Superstable periodic windows of $x^2 + c$ by period, against the ladder (T15).

```
        from sympy import Symbol, Poly, real_roots, div, divisors, factorint
        c = Symbol('c'); G = {}; cur = Poly(0, c)
        for n in range(1, 10):
            cur = Poly(cur.as_expr()**2 + c, c); G[n] = cur        # f_c^n(0), f_c(x) = x^2
+ c
        prim = {}
        for n in range(1, 10):
            p = G[n]
            for d in divisors(n):
                if d < n:
                    p, r = div(p, prim[d]); assert r.is_zero
            prim[n] = p
        print({n: len(real_roots(prim[n])) for n in range(1, 10)})
        print("periods on the ladder 2^a 3^b:", [n for n in range(1, 10) if set(factorint(n)
) <= {2, 3}])
```

Output:

```
{1: 1, 2: 1, 3: 1, 4: 2, 5: 3, 6: 5, 7: 9, 8: 16, 9: 28}
periods on the ladder 2^a 3^b: [1, 2, 3, 4, 6, 8, 9]
```

Calculation 1.10. The ternary Golay code: perfection and the zero-sum check (T2, T3).

```
        import itertools
        g = [2, 0, 1, 2, 1, 1]          # x^5 + x^4 - x^3 + x^2 - 1 over GF(3), low to high

        gens = []
        for s in range(6):
            v = [0] * 11
            for i, a in enumerate(g): v[(i + s) % 11] = (v[(i + s) % 11] + a) % 3
            gens.append(v)
        code = set()
        for co in itertools.product(range(3), repeat=6):
```

```
            v = [0] * 11
            for a, gv in zip(co, gens):
                v = [(x + a * y) % 3 for x, y in zip(v, gv)]
            code.add(tuple(v))
        wt = lambda v: sum(1 for x in v if x)
        print("codewords", len(code), "min weight", min(wt(v) for v in code if any(v)))

        print("3^6 * (1 + 11*2 + 55*4) =", 3**6 * (1 + 22 + 220), " 3^11 =", 3**11)
        ext = [v + ((-sum(v)) % 3,) for v in code]
        print("extended: every word sums to 0 mod 3:", all(sum(v) % 3 == 0 for v in ext),

              "min weight", min(wt(v) for v in ext if any(v)))
        w6 = [v for v in ext if sum(1 for x in v if x) == 6]
        supports = {frozenset(i for i, x in enumerate(v) if x) for v in w6}
        print("weight-6 words:", len(w6), "; distinct supports:", len(supports), "; C(12,5)/
C(6,5) =", 792 // 6)
```

Output:

```
codewords 729 min weight 5
3^6 * (1 + 11*2 + 55*4) = 177147  3^11 = 177147
extended: every word sums to 0 mod 3: True min weight 6
weight-6 words: 264 ; distinct supports: 132 ; C(12,5)/C(6,5) = 132
```

Calculation 1.11. Closure probability of absmean-quantised Gaussian weights (T6).

```
        import math
        from scipy.stats import norm
        gam = math.sqrt(2 / math.pi)                 # mean |w| for w ~ N(0,1)
        p0 = norm.cdf(0.5 * gam) - norm.cdf(-0.5 * gam)   # round(w/gam) = 0
        p = (1 - p0) / 2
        w = complex(math.cos(2*math.pi/3), math.sin(2*math.pi/3))
        ph = lambda z: p0 + p*z + p*z*z
        print("p0 = %.4f, p+ = p- = %.4f, window closes with P = %.5f" % (p0, p, ((1 +
ph(w)**3 + ph(w*w)**3)/3).real))
```

Output:

```
p0 = 0.3101, p+ = p- = 0.3450, window closes with P = 0.33330
```

Calculation 1.12. Turn share of the Eisenstein digit map for growing boxes (T14, IX. 3).

```
        def bal(s):
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        def mul(x, y):                       # (a + b w)(c + d w), w^2 = -1 - w
            a, b = x; c, d = y
            return (a*c - b*d, a*d + b*c - b*d)
        def T(z):                            # T(z) = (z - d)/(1 - w) = (z - d)(2 + w)/3

            d = bal(z[0] + z[1])[0]
            a, b = mul((z[0] - d, z[1]), (2, 1))
            return (a // 3, b // 3)
        memo = {}
        def end(z):
            path = []
            while z not in memo:
```

```
                if z == (0, 0): memo[z] = 0; break
                if z in path: memo[z] = z; break
                path.append(z); z = T(z)
            r = memo[z]
            for p in path: memo[p] = r
            return r
        for B in (40, 100, 200, 400, 800):
            tot = (2*B + 1)**2
            turn = sum(1 for a in range(-B, B+1) for b in range(-B, B+1) if end((a, b)) !=
0)
            print(B, turn, tot, round(100*turn/tot, 1))
```

Output:

```
40 1788 6561 27.3
100 16546 40401 41.0
200 51816 160801 32.2
400 149198 641601 23.3
800 688602 2563201 26.9
```

Calculation 1.13. Binary words: sum divisible by 3 against even parity (T16).

```
        # Razborov-Smolensky illustration: MOD_3 of n bits as a balanced digit, and the two
ladders
        import itertools
        n = 6
        words = list(itertools.product((0, 1), repeat=n))
        mod3_closed = sum(1 for w in words if sum(w) % 3 == 0)
        parity_even = sum(1 for w in words if sum(w) % 2 == 0)
        print("6-bit words:", len(words), "; sum = 0 mod 3:", mod3_closed, "; even
parity:", parity_even)
        # ternary windows over {T,0,1}: share closing
        tw = list(itertools.product((-1, 0, 1), repeat=3))
        print("ternary windows closing:", sum(1 for w in tw if sum(w) % 3 == 0), "of", len(t
w))
```

Output:

```
6-bit words: 64 ; sum = 0 mod 3: 22 ; even parity: 32
ternary windows closing: 9 of 27
```

Calculation 1.14. Cycle lengths of the rewrite R on rings (T10, T15).

```
        import itertools
        from collections import Counter
        from sympy import factorint
        def R(t):
            n = len(t)
            return tuple((-(t[i-1] + t[(i+1) % n])) % 3 for i in range(n))
        for n in range(3, 11):
            lengths = Counter()
            for t in itertools.product((0, 1, 2), repeat=n):
                seen = {}; x = t; k = 0
                while x not in seen:
                    seen[x] = k; x = R(x); k += 1
                lengths[k - seen[x]] += 1
            cyc = sorted(lengths)
```

```
            print(n, "cycle lengths reached:", cyc, "all on ladder 2^a3^b:", all(set(factori
nt(c)) <= {2, 3} for c in cyc))
```

Output:

```
3 cycle lengths reached: [1, 3] all on ladder 2^a3^b: True
4 cycle lengths reached: [1, 2] all on ladder 2^a3^b: True
5 cycle lengths reached: [1, 8] all on ladder 2^a3^b: True
6 cycle lengths reached: [1, 2, 3, 6] all on ladder 2^a3^b: True
7 cycle lengths reached: [1, 26] all on ladder 2^a3^b: False
8 cycle lengths reached: [1, 2, 4] all on ladder 2^a3^b: True
9 cycle lengths reached: [1, 3, 9] all on ladder 2^a3^b: True
10 cycle lengths reached: [1, 2, 8] all on ladder 2^a3^b: True
```

Calculations for Chapter 2 — Particles and Fields

All scripts run with Python 3.13, sympy 1.14, mpmath and numpy 2.5 on 27 October 2026. Each block is complete and was run as shown. Input constants are those cited in the text.

Calculation 2.1. Hadron compositions: closure and carry (T1, T3, T4, T7, T7.1).

```
        def bal(s):
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        rows = [(q, a) for q in range(6) for a in range(6) if 1 <= q + a <= 5]
# q quarks, a antiquarks
        closed = [(q, a, bal(q - a)[1]) for q, a in rows if bal(q - a)[0] == 0]
        print("compositions with 1-5 constituents:", len(rows), "; triality 0:", len(closed)
)
        for q, a, carry in closed:
            print(f"  {q} q + {a} qbar: carry {carry:+d}, baryon number (q-a)/3 = {(q - a)/
3:+.0f}")
        import itertools
        words = list(itertools.product((1, -1, 0), repeat=3))       # quark, antiquark,
gluon
        closed3 = [w for w in words if sum(w) % 3 == 0]
        print("ordered three-constituent words:", len(words), "; closed:", len(closed3),

              "; all equal:", sum(len(set(w)) == 1 for w in closed3), "; all different:", su
m(len(set(w)) == 3 for w in closed3))
```

Output:

```
compositions with 1-5 constituents: 20 ; triality 0: 6
  0 q + 3 qbar: carry -1, baryon number (q-a)/3 = -1
  1 q + 1 qbar: carry +0, baryon number (q-a)/3 = +0
  1 q + 4 qbar: carry -1, baryon number (q-a)/3 = -1
  2 q + 2 qbar: carry +0, baryon number (q-a)/3 = +0
  3 q + 0 qbar: carry +1, baryon number (q-a)/3 = +1
  4 q + 1 qbar: carry +1, baryon number (q-a)/3 = +1
ordered three-constituent words: 27 ; closed: 9 ; all equal: 3 ; all different: 6
```

Calculation 2.2. Integer electric charge and zero triality.

```
        from fractions import Fraction as F
        Qu, Qd = F(2, 3), F(-1, 3)
        ok = True; n = 0
```

```
        for nu in range(4):
            for nd in range(4):
                for nub in range(4):
                    for ndb in range(4):
                        if not 1 <= nu + nd + nub + ndb <= 6: continue
                        n += 1
                        Q = (nu - nub) * Qu + (nd - ndb) * Qd
                        ok &= (Q.denominator == 1) == ((nu + nd - nub - ndb) % 3 == 0)

        print(n, "combinations of u, d, ubar, dbar; integer charge <=> triality 0:", ok)
```

Output:

```
149 combinations of u, d, ubar, dbar; integer charge <=> triality 0: True
```

Calculation 2.3. Light neutrino species from the Z width.

```
        print("N_nu = 2.9963 +- 0.0074: distance from 3 =", round((3 - 2.9963) / 0.0074, 2),
 "sigma")
```

Output:

```
N_nu = 2.9963 +- 0.0074: distance from 3 = 0.5 sigma
```

Calculation 2.4. Magic numbers, oscillator closures and periodic-table periods against the ladder (T15).

```
        from sympy import factorint
        on = lambda n: set(factorint(n)) <= {2, 3}
        magic = [2, 8, 20, 28, 50, 82, 126]
        print("magic:", [(m, on(m), dict(factorint(m))) for m in magic])
        osc = [(n + 1) * (n + 2) * (n + 3) // 3 for n in range(7)]
        print("oscillator shell closures:", [(m, on(m)) for m in osc])
        per = [2, 8, 8, 18, 18, 32, 32]
        print("periods of the table:", [(p, on(p)) for p in per], "; 2n^2 for n=1..6:",
[(2*n*n, on(2*n*n)) for n in range(1, 7)])
```

Output:

```
magic: [(2, True, {2: 1}), (8, True, {2: 3}), (20, False, {2: 2, 5: 1}), (28, False,
{2: 2, 7: 1}), (50, False, {2: 1, 5: 2}), (82, False, {2: 1, 41: 1}), (126, False,
{2: 1, 3: 2, 7: 1})]
oscillator shell closures: [(2, True), (8, True), (20, False), (40, False), (70,
False), (112, False), (168, False)]
periods of the table: [(2, True), (8, True), (8, True), (18, True), (18, True), (32,
True), (32, True)] ; 2n^2 for n=1..6: [(2, True), (8, True), (18, True), (32, True),
(50, False), (72, True)]
```

Calculation 2.5. Efimov scaling factor.

```
        import mpmath as mp
        mp.mp.dps = 30
        g = lambda s: s * mp.cosh(mp.pi * s / 2) - 8 / mp.sqrt(3) * mp.sinh(mp.pi * s / 6)

        s0 = mp.findroot(g, (0.5, 1.5), solver='anderson')
        print("s0 =", mp.nstr(s0, 8), "; length ratio e^(pi/s0) =", mp.nstr(mp.e**(mp.pi/s0)
```

```
, 6),
              "; energy ratio e^(2pi/s0) =", mp.nstr(mp.e**(2*mp.pi/s0), 6))
```

Output:

```
s0 = 1.0062378 ; length ratio e^(pi/s0) = 22.6944 ; energy ratio e^(2pi/s0) =
515.035
```

Calculation 2.6. Spin-½ rotation: return after 4π (T21).

```
        import numpy as np
        U = lambda t: np.diag([np.exp(-1j * t / 2), np.exp(1j * t / 2)])   # spin-1/2
rotation about z
        print("U(2pi) =", np.round(U(2 * np.pi).real, 12).tolist(), " U(4pi) =",
np.round(U(4 * np.pi).real, 12).tolist())
        print("Werner et al. 1975: period 62 +- 2 G for a 4pi precession; relative error", r
ound(2 / 62 * 100, 1), "%")
```

Output:

```
U(2pi) = [[-1.0, 0.0], [0.0, -1.0]]  U(4pi) = [[1.0, 0.0], [0.0, 1.0]]
Werner et al. 1975: period 62 +- 2 G for a 4pi precession; relative error 3.2 %
```

Calculation 2.7. The Morel–Fan tension in 1/α.

```
        import math
        d = 137.035999206 - 137.035999166; s = math.hypot(0.000000011, 0.000000015)
        print("Morel 2020 - Fan 2023 in 1/alpha:", round(d * 1e9, 1), "x 1e-9 =", round(d /
s, 2), "sigma")
```

Output:

```
Morel 2020 - Fan 2023 in 1/alpha: 40.0 x 1e-9 = 2.15 sigma
```

Calculation 2.8. Larson units from the hydrogen Rydberg constant (companion document, Physics T5).

```
        Rinf = 10973731.568157          # m^-1, CODATA
        mu = 1836.152673426             # m_p / m_e, CODATA 2022
        h = 6.62607015e-34; c = 299792458; e = 1.602176634e-19
        RH = Rinf / (1 + 1 / mu)
        S0 = 1 / (2 * RH); E0 = 2 * h * c * RH / e
        print("R_H =", round(RH, 4), "m^-1; S0 = 1/(2 R_H) =", f"{S0:.8e}", "m; E0 = 2 h c
R_H =", round(E0, 7), "eV")
        print("hcR_inf =", round(h * c * Rinf / e, 6), "eV; 2 hcR_inf (Hartree) =", round(2
* h * c * Rinf / e, 6), "eV")
```

Output:

```
R_H = 10967758.3403 m^-1; S0 = 1/(2 R_H) = 4.55881671e-08 m; E0 = 2 h c R_H =
27.1965745 eV
hcR_inf = 13.605693 eV; 2 hcR_inf (Hartree) = 27.211386 eV
```

Calculation 2.9. Predicted Δα/α at G191-B2B against the measurement.

```
        pred = 3.08e-8
        for v, s in [(6.36e-5, 0.33e-5 + 1.94e-5), (4.21e-5, 0.47e-5 + 2.35e-5)]:
            print(f"measured {v:.2e} +- {s:.2e}: distance from prediction {(v - pred)/s:.
2f} sigma; uncertainty/prediction = {s/pred:.0f}")
```

Output:

```
measured 6.36e-05 +- 2.27e-05: distance from prediction 2.80 sigma; uncertainty/
prediction = 737
measured 4.21e-05 +- 2.82e-05: distance from prediction 1.49 sigma; uncertainty/
prediction = 916
```

Calculation 2.10. Hypercharge, doublets and triality: $6Y \equiv 3d - 2t \pmod 6$.

```
        from fractions import Fraction as F
        # Standard Model fields: (name, triality t, SU(2) doublet d, hypercharge Y) with Q =
T3 + Y
        fields = [("Q_L", 1, 1, F(1, 6)), ("u_R", 1, 0, F(2, 3)), ("d_R", 1, 0, F(-1, 3)),

                  ("L_L", 0, 1, F(-1, 2)), ("e_R", 0, 0, F(-1)), ("H", 0, 1, F(1, 2)),

                  ("gluon", 0, 0, F(0)), ("W", 0, 0, F(0))]
        for name, t, d, Y in fields:
            print(f"{name:6s} t={t} d={d} 6Y={int(6*Y):+d}  6Y mod 6 = {int(6*Y) %
6}  (3d - 2t) mod 6 = {(3*d - 2*t) % 6}  6Y mod 3 = {int(6*Y) % 3}  t mod 3 = {t %
3}")
        print("6Y = 3d - 2t (mod 6) for all:", all(int(6*Y) % 6 == (3*d - 2*t) % 6
for _, t, d, Y in fields))
```

Output:

```
Q_L    t=1 d=1 6Y=+1  6Y mod 6 = 1  (3d - 2t) mod 6 = 1  6Y mod 3 = 1  t mod 3 = 1
u_R    t=1 d=0 6Y=+4  6Y mod 6 = 4  (3d - 2t) mod 6 = 4  6Y mod 3 = 1  t mod 3 = 1
d_R    t=1 d=0 6Y=-2  6Y mod 6 = 4  (3d - 2t) mod 6 = 4  6Y mod 3 = 1  t mod 3 = 1
L_L    t=0 d=1 6Y=-3  6Y mod 6 = 3  (3d - 2t) mod 6 = 3  6Y mod 3 = 0  t mod 3 = 0
e_R    t=0 d=0 6Y=-6  6Y mod 6 = 0  (3d - 2t) mod 6 = 0  6Y mod 3 = 0  t mod 3 = 0
H      t=0 d=1 6Y=+3  6Y mod 6 = 3  (3d - 2t) mod 6 = 3  6Y mod 3 = 0  t mod 3 = 0
gluon  t=0 d=0 6Y=+0  6Y mod 6 = 0  (3d - 2t) mod 6 = 0  6Y mod 3 = 0  t mod 3 = 0
W      t=0 d=0 6Y=+0  6Y mod 6 = 0  (3d - 2t) mod 6 = 0  6Y mod 3 = 0  t mod 3 = 0
6Y = 3d - 2t (mod 6) for all: True
```

Calculation 2.11. Dimensionless constants against the nearest ladder numbers (T15).

```
        from sympy import factorint
        import math
        on = lambda n: set(factorint(n)) <= {2, 3}
        def nearest_ladder(x):
            best = None
            for a in range(0, 40):
                for b in range(0, 30):
                    v = 2**a * 3**b
                    if best is None or abs(math.log(v / x)) < abs(math.log(best[0] / x)):

                        best = (v, a, b)
            return best
        for name, x in [("m_p/m_e", 1836.152673426), ("m_mu/m_e", 206.7682830), ("1/alpha",
137.035999177)]:
            v, a, b = nearest_ladder(x)
```

```
            print(f"{name} = {x}: nearest 2^{a}*3^{b} = {v}, off by {100*(x/v - 1):+.2f} %")

        print("1836 =", factorint(1836), " 137 =", factorint(137))
```

Output:

```
m_p/m_e = 1836.152673426: nearest 2^3*3^5 = 1944, off by -5.55 %
m_mu/m_e = 206.768283: nearest 2^3*3^3 = 216, off by -4.27 %
1/alpha = 137.035999177: nearest 2^4*3^2 = 144, off by -4.84 %
1836 = {2: 2, 3: 3, 17: 1}  137 = {137: 1}
```

Calculation 2.12. Shell sizes $2n^2$ on the ladder.

```
        from sympy import factorint
        on = lambda n: set(factorint(n)) <= {2, 3}
        print("2n^2 on the ladder iff n is 3-smooth:", all(on(2*n*n) == on(n) for n in
range(1, 200)))
        print("period lengths 2n^2, n=1..8:", [(2*n*n, on(2*n*n)) for n in range(1, 9)])
```

Output:

```
2n^2 on the ladder iff n is 3-smooth: True
period lengths 2n^2, n=1..8: [(2, True), (8, True), (18, True), (32, True), (50,
False), (72, True), (98, False), (128, True)]
```

Calculation 2.13. SU(3) weights as Eisenstein integers; triality as residue mod $1-\omega$ (T14, III.1).

```
        # SU(3) weights as Eisenstein integers a + b*w (w = e^{2 pi i/3}); triality =
balanced residue of a + b mod 3
        def bal(s):
            d = ((s + 1) % 3) - 1
            return d
        quarks = {"u": (1, 0), "d": (0, 1), "s": (-1, -1)}          # 1, w, w^2 = -1 - w

        anti = {k + "bar": (-a, -b) for k, (a, b) in quarks.items()}
        roots = {"1-w": (1, -1), "w-w2": (1, 2), "w2-1": (-2, -1)}   # differences of quark
weights (gluons)
        roots.update({"-" + k: (-a, -b) for k, (a, b) in list(roots.items())})
        for group, dct in (("quarks", quarks), ("antiquarks", anti), ("gluon roots",
roots)):
            print(group, {k: bal(a + b) for k, (a, b) in dct.items()})
        norm = lambda a, b: a*a - a*b + b*b
        print("norm of 1 - w:", norm(1, -1), "-> index of root lattice in weight lattice =
3")
        # every baryon qqq and meson q qbar weight sum has residue 0
        import itertools
        ok = all(bal(sum(q[0] for q in c) + sum(q[1] for q in c)) == 0 for c in
itertools.product(quarks.values(), repeat=3))
        ok &= all(bal(q[0] + a[0] + q[1] + a[1]) == 0 for q in quarks.values() for a in
anti.values())
        print("all qqq and q-qbar weight sums lie in the root lattice:", ok)
```

Output:

```
quarks {'u': 1, 'd': 1, 's': 1}
antiquarks {'ubar': -1, 'dbar': -1, 'sbar': -1}
gluon roots {'1-w': 0, 'w-w2': 0, 'w2-1': 0, '-1-w': 0, '-w-w2': 0, '-w2-1': 0}
```

```
norm of 1 - w: 3 -> index of root lattice in weight lattice = 3
all qqq and q-qbar weight sums lie in the root lattice: True
```

Calculations for Chapter 3 — Forces and Motion

All scripts run with Python 3.13, sympy 1.14 and numpy 2.5 on 27 October 2026. Each block is complete and was run as shown.

Calculation 3.1. Bertrand frequency ratios and Mercury's spin–orbit ratio.

```
        import math
        # Bertrand: for a power-law force F ~ r^n, near-circular orbits have radial/angular
frequency ratio beta = sqrt(n + 3)
        for n, name in [(-2, "inverse square"), (1, "Hooke"), (-1, "1/r force"), (0, "consta
nt force"), (2, "r^2 force")]:
            print(f"{name:15s} n={n:+d}  beta = {math.sqrt(n + 3):.4f}")
        print("Mercury: orbit/spin =", round(87.97 / 58.646, 5), "(3:2 spin-orbit lock)")
```

Output:

```
inverse square  n=-2  beta = 1.0000
Hooke           n=+1  beta = 2.0000
1/r force       n=-1  beta = 1.4142
constant force  n=+0  beta = 1.7321
r^2 force       n=+2  beta = 2.2361
Mercury: orbit/spin = 1.50002 (3:2 spin-orbit lock)
```

Calculation 3.2. Stability threshold of the Lagrange points L4 and L5.

```
        import math
        # Gascheau/Routh: L4, L5 linearly stable iff 27 mu (1 - mu) < 1, mu = m2/(m1 + m2)

        mu_c = (1 - math.sqrt(1 - 4/27)) / 2
        print("critical mu =", round(mu_c, 6), "; critical mass ratio m1/m2 =", round((1 -
mu_c)/mu_c, 4),
              "= (25 + 3 sqrt 69)/2 =", round((25 + 3*math.sqrt(69))/2, 4))
```

Output:

```
critical mu = 0.038521 ; critical mass ratio m1/m2 = 24.9599 = (25 + 3 sqrt 69)/2 =
24.9599
```

Calculation 3.3. Maxwell counting: the triangle as the only rigid polygon; isostatic lattices.

```
        # Maxwell counting in the plane: rigid frame of j joints needs 2j - 3 bars; a
polygon has j bars
        print([(j, 2*j - 3, "rigid polygon" if j == 2*j - 3 else "floppy polygon") for j in
range(3, 8)])
        # lattices in d = 2: isostatic when mean coordination z = 2d = 4
        for name, z in [("honeycomb", 3), ("kagome", 4), ("square", 4), ("triangular", 6)]:

            print(f"{name:10s} z = {z}: z - 2d = {z - 4:+d}")
```

Output:

```
[(3, 3, 'rigid polygon'), (4, 5, 'floppy polygon'), (5, 7, 'floppy polygon'), (6, 9,
'floppy polygon'), (7, 11, 'floppy polygon')]
honeycomb  z = 3: z - 2d = -1
kagome     z = 4: z - 2d = +0
square     z = 4: z - 2d = +0
triangular z = 6: z - 2d = +2
```

Calculation 3.4. Net current on a three-state ring and on a two-state pair (T7.1).

```
        import numpy as np, math
        def stationary(Q):
            w, v = np.linalg.eig(Q.T); p = np.real(v[:, np.argmin(abs(w))]); return p / p.su
m()
        kp, km = 2.0, 1.0
        Q3 = np.array([[-(kp+km), kp, km], [km, -(kp+km), kp], [kp, km, -(kp+km)]])
        p3 = stationary(Q3)
        print("3-state ring: stationary", np.round(p3, 4), "net current", round(p3[0]*kp -
p3[1]*km, 4),
              "cycle affinity ln((kp/km)^3) =", round(3*math.log(kp/km), 4))
        Q2 = np.array([[-kp, kp], [km, -km]]); p2 = stationary(Q2)
        print("2-state pair: net current", round(p2[0]*kp - p2[1]*km, 12))
```

Output:

```
3-state ring: stationary [0.3333 0.3333 0.3333] net current 0.3333 cycle affinity
ln((kp/km)^3) = 2.0794
2-state pair: net current 0.0
```

Calculation 3.5. T23 and the bulk modulus: B′ = 2 + γ; the free-path lever of T24.

```
        import sympy as sp
        n, g2, g3 = sp.symbols('n g2 g3', positive=True)
        e = g2/2*n**2 + g3/6*n**3                 # energy density of T23
        P = n*sp.diff(e, n) - e                   # pressure
        K = n*sp.diff(P, n)                       # bulk modulus
        Bp = sp.simplify(sp.diff(K, n) / sp.diff(P, n))
        gamma = n*g3/(g2 + g3*n)                  # gamma of T23 at reference density n

        print("K =", sp.factor(K), "; B' = dK/dP =", Bp, "; B' - 2 - gamma =",
sp.simplify(Bp - 2 - gamma))
        f = sp.Function('f')
        Pg = n*sp.diff(f(n), n) - f(n); Kg = n*sp.diff(Pg, n)
        print("general energy density: B' =", sp.simplify(sp.diff(Kg, n)/sp.diff(Pg, n)))

        print("gamma implied by measured B' = 4..8:", [b - 2 for b in (4, 5, 6, 7, 8)])

        for P3 in (0.3405373296, 0.1932016732, 0.1351786098):
            print("free-path lever -(1+P)/P for P =", P3, ":", round(-(1 + P3)/P3, 4))
```

Output:

```
K = n**2*(g2 + g3*n) ; B' = dK/dP = (2*g2 + 3*g3*n)/(g2 + g3*n) ; B' - 2 - gamma = 0
general energy density: B' = n*Derivative(f(n), (n, 3))/Derivative(f(n), (n, 2)) + 2
gamma implied by measured B' = 4..8: [2, 3, 4, 5, 6]
free-path lever -(1+P)/P for P = 0.3405373296 : -3.9365
free-path lever -(1+P)/P for P = 0.1932016732 : -6.1759
free-path lever -(1+P)/P for P = 0.1351786098 : -8.3976
```

Calculation 3.6. Pauling’s ice entropy and frustration on a triangle.

```
        import itertools, math
        R = 8.314462618
        print("Pauling: S0 = R ln(3/2) =", round(R*math.log(1.5), 3), "J/(mol K) =",
round(R*math.log(1.5)/4.184, 3), "cal/(mol K)")
        # ice rule: of the 2^4 = 16 proton arrangements around an oxygen, those with two
near (2-in 2-out)
        print("2-in-2-out arrangements:", sum(1 for s in itertools.product((0, 1), repeat=4)
 if sum(s) == 2), "of 16 ->", 6/16, "; Pauling W = 2^2 * 6/16 =", 4*6/16)
        gs = sum(1 for s in itertools.product((1, -1), repeat=3)
                 if sum(s[a] != s[b] for a, b in ((0, 1), (1, 2), (2, 0))) == 2)
        print("Ising antiferromagnet on one triangle: ground states", gs, "of 8; best is 2
of 3 bonds satisfied")
        print("square plaquette (4 AF bonds) ground states:", sum(1 for s in
itertools.product((1, -1), repeat=4)
              if all(s[i] != s[(i+1) % 4] for i in range(4))))
```

Output:

```
Pauling: S0 = R ln(3/2) = 3.371 J/(mol K) = 0.806 cal/(mol K)
2-in-2-out arrangements: 6 of 16 -> 0.375 ; Pauling W = 2^2 * 6/16 = 1.5
Ising antiferromagnet on one triangle: ground states 6 of 8; best is 2 of 3 bonds
satisfied
square plaquette (4 AF bonds) ground states: 2
```

Calculation 3.7. Landauer cost per bit and per trit.

```
        import math
        k = 1.380649e-23; q = 1.602176634e-19
        print("kT ln2 at 300 K =", round(k*300*math.log(2)/q, 5), "eV ; kT ln3 =", round(k*3
00*math.log(3)/q, 5), "eV")
        print("per bit of information both are", round(k*300*math.log(2)/q, 5), "eV, since
ln3 / log2(3) = ln2:", round(math.log(3)/math.log2(3), 6))
```

Output:

```
kT ln2 at 300 K = 0.01792 eV ; kT ln3 = 0.0284 eV
per bit of information both are 0.01792 eV, since ln3 / log2(3) = ln2: 0.693147
```

Calculation 3.8. Nanotube metallicity as closure mod 3 on the Eisenstein lattice.

```
        tubes = [(n, m) for n in range(1, 31) for m in range(0, n + 1)]
        met = [t for t in tubes if (t[0] - t[1]) % 3 == 0]
        print("chiral tubes 0 <= m <= n <= 30:", len(tubes), "; n - m = 0 mod 3:", len(met),
 "share", round(len(met)/len(tubes), 4))
        print("|C|^2 ~ n^2 + nm + m^2 = 0 mod 3 <=> n - m = 0 mod 3:",
              all(((n*n + n*m + m*m) % 3 == 0) == ((n - m) % 3 == 0) for n, m in tubes))
```

Output:

```
chiral tubes 0 <= m <= n <= 30: 495 ; n - m = 0 mod 3: 175 share 0.3535
|C|^2 ~ n^2 + nm + m^2 = 0 mod 3 <=> n - m = 0 mod 3: True
```

Calculation 3.9. The KCBS pentagon and Lovász numbers of odd cycles.

```
        import math
        theta = lambda n: n*math.cos(math.pi/n)/(1 + math.cos(math.pi/n))
        print("KCBS pentagon: classical bound alpha(C5) = 2; quantum (Lovasz) theta(C5) =",
round(theta(5), 6), "= sqrt5 =", round(math.sqrt(5), 6))
        print("odd cycles theta(C_n):", {n: round(theta(n), 4) for n in (3, 5, 7, 9)}, ";
independence numbers:", {n: n//2 for n in (3, 5, 7, 9)})
```

Output:

```
KCBS pentagon: classical bound alpha(C5) = 2; quantum (Lovasz) theta(C5) = 2.236068
= sqrt5 = 2.236068
odd cycles theta(C_n): {3: 1.0, 5: 2.2361, 7: 3.3177, 9: 4.3601} ; independence
numbers: {3: 1, 5: 2, 7: 3, 9: 4}
```

Calculation 3.10. Three charges of e/3 as a ring with carry 1 (T1, T7.1).

```
        from fractions import Fraction as F
        def bal(s):
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        # three Laughlin quasiparticles of charge e/3: in units of e/3 their sum is 3 ->
digit 0, carry 1 (one electron)
        print("3 x (e/3): digit, carry =", bal(3), "; 2 x (e/3):", bal(2))
        print("Laughlin fractions 1/m for odd m:", [str(F(1, m)) for m in (1, 3, 5, 7)])
```

Output:

```
3 x (e/3): digit, carry = (0, 1) ; 2 x (e/3): (-1, 1)
Laughlin fractions 1/m for odd m: ['1', '1/3', '1/5', '1/7']
```

Calculation 3.11. Spectral spans of f − 2f and 2f − 3f; photon bookkeeping in nonlinear optics.

```
        import math
        print("f-2f self-referencing: span ratio 2 =", round(1200*math.log2(2), 3), "cents;
2f-3f: span ratio 3/2 =", round(1200*math.log2(1.5), 3), "cents")
        # photon-number bookkeeping in nonlinear optics is conservation in the integers, not
mod 3
        for name, inp, out in [("second harmonic", 2, 1), ("third harmonic", 3, 1), ("parame
tric down-conversion", 1, 2)]:
            print(f"{name}: {inp} photons -> {out}; photon number changes by {out - inp};
energy balance exact for any count")
```

Output:

```
f-2f self-referencing: span ratio 2 = 1200.0 cents; 2f-3f: span ratio 3/2 = 701.955
cents
second harmonic: 2 photons -> 1; photon number changes by -1; energy balance exact
for any count
third harmonic: 3 photons -> 1; photon number changes by -2; energy balance exact
for any count
parametric down-conversion: 1 photons -> 2; photon number changes by 1; energy
balance exact for any count
```

Calculation 3.12. Cube roots of unity and the qutrit Fourier transform.

```
        import numpy as np, itertools
        # Sum of three unit phasors at 0, 120, 240 degrees vanishes; quantum Fourier on Z3

        w = np.exp(2j*np.pi/3)
        print("1 + w + w^2 =", np.round(1 + w + w*w, 15))
        # qutrit Fourier matrix is unitary
        F3 = np.array([[w**(j*k) for k in range(3)] for j in range(3)])/np.sqrt(3)
        print("F3 unitary:", np.allclose(F3 @ F3.conj().T, np.eye(3)))
```

Output:

```
1 + w + w^2 = (-0+0j)
F3 unitary: True
```

Calculations for Chapter 4 — The Cosmos

All scripts run with Python 3.13 and sympy 1.14 on 27 October 2026. Each block is complete and was run as shown. Input values are those cited in the text.

Calculation 4.1. Published resonance chains and Kirkwood gaps against the ladder (T15).

```
        from sympy import factorint
        on = lambda n: set(factorint(n)) <= {2, 3}
        chains = {"Io:Europa:Ganymede periods": (1, 2, 4), "Pluto:Neptune periods": (3, 2),
"Titan:Hyperion": (3, 4),
                  "Enceladus:Dione": (1, 2), "Kepler-223 periods": (3, 4, 6, 8), "HD 110067
ratio": (54, 36, 24, 16, 12, 9),
                  "TRAPPIST-1 orbit ratio": (24, 15, 9, 6, 4, 3, 2), "Mercury spin:orbit":
(3, 2),
                  "Kirkwood 4:1": (4, 1), "Kirkwood 3:1": (3, 1), "Kirkwood 5:2": (5, 2), "K
irkwood 7:3": (7, 3), "Kirkwood 2:1": (2, 1)}
        for k, v in chains.items():
            print(f"{k:28s} {v}  on ladder: {all(on(x) for x in v)}  off: {[x for x in v if
not on(x)]}")
```

Output:

```
Io:Europa:Ganymede periods   (1, 2, 4)  on ladder: True  off: []
Pluto:Neptune periods        (3, 2)  on ladder: True  off: []
Titan:Hyperion               (3, 4)  on ladder: True  off: []
Enceladus:Dione              (1, 2)  on ladder: True  off: []
Kepler-223 periods           (3, 4, 6, 8)  on ladder: True  off: []
HD 110067 ratio              (54, 36, 24, 16, 12, 9)  on ladder: True  off: []
TRAPPIST-1 orbit ratio       (24, 15, 9, 6, 4, 3, 2)  on ladder: False  off: [15]
Mercury spin:orbit           (3, 2)  on ladder: True  off: []
Kirkwood 4:1                 (4, 1)  on ladder: True  off: []
Kirkwood 3:1                 (3, 1)  on ladder: True  off: []
Kirkwood 5:2                 (5, 2)  on ladder: False  off: [5]
Kirkwood 7:3                 (7, 3)  on ladder: False  off: [7]
Kirkwood 2:1                 (2, 1)  on ladder: True  off: []
```

Calculation 4.2. Measured period ratios of resonant moons; the Laplace combination.

```
        moons = {"Io": 1.762732, "Europa": 3.525463, "Ganymede": 7.155588, "Mimas":
0.942422, "Enceladus": 1.370218,
                 "Tethys": 1.887802, "Dione": 2.736916, "Titan": 15.945448, "Hyperion": 21.2
```

```
76658}
        for a, b in [("Europa", "Io"), ("Ganymede", "Europa"), ("Tethys", "Mimas"),
("Dione", "Enceladus"), ("Hyperion", "Titan")]:
            print(f"{a}/{b} = {moons[a]/moons[b]:.5f}")
        nI, nE, nG = (1/moons[k] for k in ("Io", "Europa", "Ganymede"))
        print("Laplace combination n_Io - 3 n_Europa + 2 n_Ganymede =", round(nI - 3*nE +
2*nG, 5), "per day (mean periods; the true resonance is in longitudes)")
```

Output:

```
Europa/Io = 2.00000
Ganymede/Europa = 2.02969
Tethys/Mimas = 2.00314
Dione/Enceladus = 1.99743
Hyperion/Titan = 1.33434
Laplace combination n_Io - 3 n_Europa + 2 n_Ganymede = -0.00415 per day (mean
periods; the true resonance is in longitudes)
```

Calculation 4.3. Adjacent planetary period ratios; baseline share of ladder fractions.

```
        from sympy import factorint
        from fractions import Fraction as F
        on = lambda n: set(factorint(n)) <= {2, 3}
        pl = {"Mercury": 0.2408467, "Venus": 0.61519726, "Earth": 1.0000174, "Mars": 1.88084
76, "Jupiter": 11.862615,
              "Saturn": 29.447498, "Uranus": 84.016846, "Neptune": 164.79132, "Pluto": 247.9
2065}
        names = list(pl)
        for a, b in zip(names[1:], names[:-1]):
            r = pl[a] / pl[b]
            best = min(((round(r*q), q) for q in range(1, 7)), key=lambda pq: abs(r - pq[0]/
pq[1]) / r)
            p, q = best
            print(f"{a}/{b} = {r:.5f}  nearest p/q with q <= 6: {p}/{q} ({100*abs(r - p/q)/
r:.2f} %)  on ladder: {on(p) and on(q)}")
        fr = {F(p, q) for p in range(1, 10) for q in range(1, 10)}
        print("baseline: reduced fractions p/q with p, q <= 9:", len(fr), "; with both on
the ladder:",
              sum(1 for f in fr if on(f.numerator) and on(f.denominator)))
```

Output:

```
Venus/Mercury = 2.55431  nearest p/q with q <= 6: 13/5 (1.79 %)  on ladder: False
Earth/Venus = 1.62552  nearest p/q with q <= 6: 8/5 (1.57 %)  on ladder: False
Mars/Earth = 1.88081  nearest p/q with q <= 6: 11/6 (2.52 %)  on ladder: False
Jupiter/Mars = 6.30706  nearest p/q with q <= 6: 19/3 (0.42 %)  on ladder: False
Saturn/Jupiter = 2.48238  nearest p/q with q <= 6: 5/2 (0.71 %)  on ladder: False
Uranus/Saturn = 2.85311  nearest p/q with q <= 6: 17/6 (0.69 %)  on ladder: False
Neptune/Uranus = 1.96141  nearest p/q with q <= 6: 2/1 (1.97 %)  on ladder: True
Pluto/Neptune = 1.50445  nearest p/q with q <= 6: 3/2 (0.30 %)  on ladder: True
baseline: reduced fractions p/q with p, q <= 9: 55 ; with both on the ladder: 25
```

Calculation 4.4. $a_0 = cH_0/2\pi$ and the solar radius $\sqrt{(GM_\odot/a_0)}$ (companion document).

```
        import math
        c = 299792458.0; Mpc = 3.0856775814913673e22
        for H0 in (67.4, 70.0, 70.39, 73.04, 73.5):
            print(f"H0 = {H0}: cH0/2pi = {c*H0*1e3/Mpc/(2*math.pi):.4e} m/s^2")
        GM = 1.32712440042e20; AU = 149597870700.0
```

```
        for a0 in (1.2e-10, 1.0824e-10, 1.0422e-10, 1.1294e-10):
            print(f"a0 = {a0:.4e}: sqrt(GM_sun/a0) = {math.sqrt(GM/a0)/AU:.0f} AU")
```

Output:

```
H0 = 67.4: cH0/2pi = 1.0422e-10 m/s^2
H0 = 70.0: cH0/2pi = 1.0824e-10 m/s^2
H0 = 70.39: cH0/2pi = 1.0884e-10 m/s^2
H0 = 73.04: cH0/2pi = 1.1294e-10 m/s^2
H0 = 73.5: cH0/2pi = 1.1365e-10 m/s^2
a0 = 1.2000e-10: sqrt(GM_sun/a0) = 7030 AU
a0 = 1.0824e-10: sqrt(GM_sun/a0) = 7402 AU
a0 = 1.0422e-10: sqrt(GM_sun/a0) = 7543 AU
a0 = 1.1294e-10: sqrt(GM_sun/a0) = 7246 AU
```

Calculation 4.5. Lunisolar convergents: Metonic cycle and saros (T17 method).

```
        import math
        Y, M = 365.24219, 29.530589
        x = Y / M; cf = []; y = x
        for _ in range(8):
            a = int(math.floor(y)); cf.append(a); y = 1/(y - a)
        p0, q0, p1, q1 = 1, 0, cf[0], 1; conv = [(p1, q1)]
        for a in cf[1:]:
            p0, q0, p1, q1 = p1, q1, a*p1 + p0, a*q1 + q0; conv.append((p1, q1))
        print("year/month =", round(x, 6), "; continued fraction", cf, "; convergents
(months, years):", conv)
        print("235 months =", round(235*M, 3), "d; 19 years =", round(19*Y, 3), "d;
difference", round((235*M - 19*Y)*24, 2), "h")
        syn, dra, ano = 29.53059, 27.212220, 27.554551
        print("Saros: 223 synodic =", round(223*syn, 3), "d; 242 draconic =", round(242*dra,
 3), "d; 239 anomalistic =", round(239*ano, 3), "d")
```

Output:

```
year/month = 12.368266 ; continued fraction [12, 2, 1, 2, 1, 1, 17, 3] ; convergents
(months, years): [(12, 1), (25, 2), (37, 3), (99, 8), (136, 11), (235, 19), (4131,
334), (12628, 1021)]
235 months = 6939.688 d; 19 years = 6939.602 d; difference 2.08 h
Saros: 223 synodic = 6585.322 d; 242 draconic = 6585.357 d; 239 anomalistic =
6585.538 d
```

Calculation 4.6. The Hubble tension in standard deviations.

```
        import math
        def sig(a, sa, b, sb): return (a - b) / math.hypot(sa, sb)
        print("SH0ES 73.04 +- 1.04 vs Planck 67.4 +- 0.5:", round(sig(73.04, 1.04, 67.4,
0.5), 2), "sigma")
        print("H0DN 73.50 +- 0.81 vs Planck:", round(sig(73.50, 0.81, 67.4, 0.5), 2), "sigma
")
        print("CCHP 70.39 +- 1.94 vs Planck:", round(sig(70.39, 1.94, 67.4, 0.5), 2), "sigma
; vs H0DN:", round(sig(73.50, 0.81, 70.39, 1.94), 2), "sigma")
```

Output:

```
SH0ES 73.04 +- 1.04 vs Planck 67.4 +- 0.5: 4.89 sigma
H0DN 73.50 +- 0.81 vs Planck: 6.41 sigma
CCHP 70.39 +- 1.94 vs Planck: 1.49 sigma; vs H0DN: 1.48 sigma
```

Calculation 4.7. The 7,400 AU edge against the heliopause, Oort cloud and Hill sphere.

```
        print("Sun's Hill radius 230000 AU / 7400 AU =", round(230000/7400, 1))
        print("7400 AU / Voyager 1 heliopause 122 AU =", round(7400/122, 1))
        print("inner Oort cloud 2000-20000 AU contains 7400 AU:", 2000 <= 7400 <= 20000)

        print("Planet Nine estimate 290 +- 30 AU (Siraj et al. 2025) / 7400 =", round(290/74
00, 3))
```

Output:

```
Sun's Hill radius 230000 AU / 7400 AU = 31.1
7400 AU / Voyager 1 heliopause 122 AU = 60.7
inner Oort cloud 2000-20000 AU contains 7400 AU: True
Planet Nine estimate 290 +- 30 AU (Siraj et al. 2025) / 7400 = 0.039
```

Calculation 4.8. Solar cycles, Kepler’s third law and calendar numbers against the ladder.

```
        from sympy import factorint
        on = lambda n: set(factorint(n)) <= {2, 3}
        print("solar cycle 11 yr:", on(11), "; Hale cycle 22 yr:", on(22))
        print("Kepler III: T^2 ~ a^3; doubling a multiplies T by 2^(3/2) =", round(2**1.5,
4))
        print("Saros 223:", dict(factorint(223)), " 242:", dict(factorint(242)), " 239:", di
ct(factorint(239)), " Metonic 235:", dict(factorint(235)), " 19 years")
```

Output:

```
solar cycle 11 yr: False ; Hale cycle 22 yr: False
Kepler III: T^2 ~ a^3; doubling a multiplies T by 2^(3/2) = 2.8284
Saros 223: {223: 1}  242: {2: 1, 11: 2}  239: {239: 1}  Metonic 235: {5: 1, 47: 1}
19 years
```

Calculation 4.9. The white-dwarf prediction against present precision.

```
        import math
        pred = 3.08e-8
        print("G191-B2B: predicted Delta alpha/alpha", pred,
"; present uncertainty ~2e-5 -> factor", round(2e-5/pred))
        print("clock limit on drift of alpha 1.0e-18 per year (Lange et al. 2021)")
```

Output:

```
G191-B2B: predicted Delta alpha/alpha 3.08e-08 ; present uncertainty ~2e-5 -> factor
649
clock limit on drift of alpha 1.0e-18 per year (Lange et al. 2021)
```

Calculation 4.10. First-order resonances on the ladder; adjacent ratios of HD 110067 and TRAPPIST-1.

```
        from sympy import factorint
        on = lambda n: set(factorint(n)) <= {2, 3}
        first = [(k + 1, k) for k in range(1, 12)]
        print("first-order resonances (k+1):k on the ladder:", [f"{a}:{b}" for a, b in
first if on(a) and on(b)])
        print("off the ladder:", [f"{a}:{b}" for a, b in first if not (on(a) and on(b))])

        # HD 110067 adjacent period ratios from 54:36:24:16:12:9 (orbit counts) -> period
ratios
        seq = [54, 36, 24, 16, 12, 9]
        from fractions import Fraction as F
        print("HD 110067 adjacent period ratios:", [str(F(seq[i], seq[i+1])) for i in
range(5)])
        seqT = [24, 15, 9, 6, 4, 3, 2]
        print("TRAPPIST-1 adjacent period ratios:", [str(F(seqT[i], seqT[i+1])) for i in ran
ge(6)])
```

Output:

```
first-order resonances (k+1):k on the ladder: ['2:1', '3:2', '4:3', '9:8']
off the ladder: ['5:4', '6:5', '7:6', '8:7', '10:9', '11:10', '12:11']
HD 110067 adjacent period ratios: ['3/2', '3/2', '3/2', '4/3', '4/3']
TRAPPIST-1 adjacent period ratios: ['8/5', '5/3', '3/2', '3/2', '4/3', '3/2']
```

Calculations for Chapter 5 — Engineering of Energy and Signal

All scripts run with Python 3.13 and numpy 2.5 on 27 October 2026. Each block is complete and was run as shown. Input values are those cited in the text.

Calculation 5.1. Sequence of harmonic h as the balanced digit of h (T1; Fortescue).

```
        import cmath, math
        def bal(s):
            d = ((s + 1) % 3) - 1
            return d
        # balanced set of harmonic h: phases 0, -h*120, -h*240 degrees
        for h in range(1, 14):
            s = sum(cmath.exp(-1j*h*k*2*math.pi/3) for k in range(3))
            seq = {1: "positive", -1: "negative", 0: "zero"}[bal(h)]
            print(f"h = {h:2d}: balanced digit {bal(h):+d} -> {seq:
8s} sequence; |sum of the three phases| = {abs(s):.0f}")
```

Output:

```
h =  1: balanced digit +1 -> positive sequence; |sum of the three phases| = 0
h =  2: balanced digit -1 -> negative sequence; |sum of the three phases| = 0
h =  3: balanced digit +0 -> zero     sequence; |sum of the three phases| = 3
h =  4: balanced digit +1 -> positive sequence; |sum of the three phases| = 0
h =  5: balanced digit -1 -> negative sequence; |sum of the three phases| = 0
h =  6: balanced digit +0 -> zero     sequence; |sum of the three phases| = 3
h =  7: balanced digit +1 -> positive sequence; |sum of the three phases| = 0
h =  8: balanced digit -1 -> negative sequence; |sum of the three phases| = 0
h =  9: balanced digit +0 -> zero     sequence; |sum of the three phases| = 3
h = 10: balanced digit +1 -> positive sequence; |sum of the three phases| = 0
h = 11: balanced digit -1 -> negative sequence; |sum of the three phases| = 0
h = 12: balanced digit +0 -> zero     sequence; |sum of the three phases| = 3
h = 13: balanced digit +1 -> positive sequence; |sum of the three phases| = 0
```

Calculation 5.2. Symmetrical components of a balanced set and of an earth fault.

```
        import numpy as np, cmath, math
        a = cmath.exp(2j*math.pi/3)
        A = np.array([[1, 1, 1], [1, a, a*a], [1, a*a, a]]) / 3
        print("1 + a + a^2 =", abs(1 + a + a*a))
        for name, V in [("balanced", [1, a*a, a]), ("phase b shorted to earth", [1, 0, a])]:

            V0, V1, V2 = A @ np.array(V)
            print(f"{name:26s}: |V0| = {abs(V0):.4f}, |V1| = {abs(V1):.4f}, |V2| = {abs(V2):
.4f}")
```

Output:

```
1 + a + a^2 = 3.510833468576701e-16
balanced                   : |V0| = 0.0000, |V1| = 1.0000, |V2| = 0.0000
phase b shorted to earth   : |V0| = 0.3333, |V1| = 0.6667, |V2| = 0.3333
```

Calculation 5.3. Constant power of a balanced three-phase load.

```
        import numpy as np
        t = np.linspace(0, 1, 1001)
        p3 = sum(np.cos(2*np.pi*t - k*2*np.pi/3)**2 for k in range(3))
        p1 = np.cos(2*np.pi*t)**2
        print("three-phase sum of cos^2: min", round(p3.min(), 12), "max", round(p3.max(), 1
2))
        print("single phase cos^2: min", round(p1.min(), 6), "max", round(p1.max(), 6))
```

Output:

```
three-phase sum of cos^2: min 1.5 max 1.5
single phase cos^2: min 0.0 max 1.0
```

Calculation 5.4. Harmonics of a six-pulse converter: only 6k ± 1.

```
        import numpy as np
        N = 3600; th = (np.arange(N) + 0.5) * 2*np.pi/N
        x = np.zeros(N)
        x[(th > np.pi/6) & (th < 5*np.pi/6)] = 1; x[(th > 7*np.pi/6) & (th < 11*np.pi/6)] =
-1
        F = np.abs(np.fft.rfft(x)) / N * 2
        present = [h for h in range(1, 26) if F[h] > 1e-3 * F[1]]
        print("six-pulse line current, harmonics present up to 25:", present)
        print("relative amplitudes:", {h: round(F[h]/F[1], 4) for h in present[1:5]})

        print("all present harmonics are 6k +- 1:", all(h % 6 in (1, 5) for h in present))
```

Output:

```
six-pulse line current, harmonics present up to 25: [1, 5, 7, 11, 13, 17, 19, 23,
25]
relative amplitudes: {5: np.float64(0.2), 7: np.float64(0.1429), 11:
np.float64(0.0909), 13: np.float64(0.0769)}
all present harmonics are 6k +- 1: True
```

Calculation 5.5. Stable states of rings of inverters (T7).

```
        import itertools
        for n in range(1, 9):
            latch = [s for s in itertools.product((0, 1), repeat=n) if all(s[i] == 1 -
s[i-1] for i in range(n))]
            print(f"ring of {n} inverters: {len(latch)} stable states -> {'latch' if latch e
lse 'oscillates'}")
```

Output:

```
ring of 1 inverters: 0 stable states -> oscillates
ring of 2 inverters: 2 stable states -> latch
ring of 3 inverters: 0 stable states -> oscillates
ring of 4 inverters: 2 stable states -> latch
ring of 5 inverters: 0 stable states -> oscillates
ring of 6 inverters: 2 stable states -> latch
ring of 7 inverters: 0 stable states -> oscillates
ring of 8 inverters: 2 stable states -> latch
```

Calculation 5.6. Surface-code sizes and error suppression.

```
        import math
        for d in (3, 5, 7):
            print(f"surface code distance {d}: data + measure qubits 2d^2 - 1 = {2*d*d -
1}")
        L, e7 = 2.14, 0.143
        print("logical error per cycle extrapolated back: d=5", round(e7*L, 3), "%, d=3", ro
und(e7*L*L, 3), "%")
        print("cycles to halve: each +2 distance divides error by", L)
```

Output:

```
surface code distance 3: data + measure qubits 2d^2 - 1 = 17
surface code distance 5: data + measure qubits 2d^2 - 1 = 49
surface code distance 7: data + measure qubits 2d^2 - 1 = 97
logical error per cycle extrapolated back: d=5 0.306 %, d=3 0.655 %
cycles to halve: each +2 distance divides error by 2.14
```

Calculation 5.7. Target gains of NIF ignition shots.

```
        for date, Ein, Eout in [("5 Dec 2022", 2.05, 3.15), ("23 Feb 2025", 2.05, 5.0), ("7
Apr 2025", 2.08, 8.6), ("1 Oct 2025", 2.065, 3.5)]:
            print(f"NIF {date}: {Eout} MJ / {Ein} MJ = gain {Eout/Ein:.3f}")
```

Output:

```
NIF 5 Dec 2022: 3.15 MJ / 2.05 MJ = gain 1.537
NIF 23 Feb 2025: 5.0 MJ / 2.05 MJ = gain 2.439
NIF 7 Apr 2025: 8.6 MJ / 2.08 MJ = gain 4.135
NIF 1 Oct 2025: 3.5 MJ / 2.065 MJ = gain 1.695
```

Calculation 5.8. Digits for 10^6 states in radix 2 and 3 (III.1, T12.3).

```
        import math
        for r in (2, 3):
            d = math.ceil(math.log(10**6 + 1, r))
            print(f"radix {r}: {d} digits for 10^6 states; radix economy r*d = {r*d}")
```

```
        print("signed range of 13 balanced trits: +-", (3**13 - 1)//2)
```

Output:

```
radix 2: 20 digits for 10^6 states; radix economy r*d = 40
radix 3: 13 digits for 10^6 states; radix economy r*d = 39
signed range of 13 balanced trits: +- 797161
```

Calculation 5.9. HVDC and AC losses over 3,300 km.

```
        for name, per1000 in (("HVDC", 0.035), ("AC", 0.067)):
            print(f"{name}: loss over 3300 km = {1 - (1 - per1000)**3.3:.4f}")
```

Output:

```
HVDC: loss over 3300 km = 0.1109
AC: loss over 3300 km = 0.2046
```

Calculation 5.10. Nanotube metallicity counts.

```
        tubes = [(n, m) for n in range(1, 31) for m in range(0, n + 1)]
        met = [t for t in tubes if (t[0] - t[1]) % 3 == 0]
        arm = [t for t in tubes if t[0] == t[1]]
        print("tubes:", len(tubes), "n - m = 0 mod 3:", len(met), "of which armchair:", len(
arm), "quasi-metallic (n != m):", len(met) - len(arm))
```

Output:

```
tubes: 495 n - m = 0 mod 3: 175 of which armchair: 30 quasi-metallic (n != m): 145
```

Calculation 5.11. Ternary-fission rates.

```
        print("ternary fission 2-4 per 1000; alpha third particle ~90% -> alpha-accompanied
per 1000:", 0.9*2, "to", 0.9*4)
        print("true ternary 1e-8 relative to ternary 2e-3..4e-3:", 1e-8/4e-3, "to", 1e-8/2e-
3)
```

Output:

```
ternary fission 2-4 per 1000; alpha third particle ~90% -> alpha-accompanied per
1000: 1.8 to 3.6
true ternary 1e-8 relative to ternary 2e-3..4e-3: 2.5e-06 to 5e-06
```

Calculations for Chapter 6 — The Elements, Molecules and Materials

All scripts were run on 27 October 2026 with Python 3 and sympy 1.14. Calculations 6.1 to 6.5 import this helper, saved as `lad.py`:

```
        def on_ladder(n):
            """True if n = 2^a * 3^b (Foundations T15)."""
            if n <= 0: return False
            while n % 2 == 0: n //= 2
```

```
            while n % 3 == 0: n //= 3
            return n == 1
```

6.1 Period lengths, blocks and the ladder.

```
        from lad import on_ladder
        L = 'spdfghi'
        orbs = sorted((n + l, n, l) for n in range(1, 10) for l in range(n))
        periods, cur = [], []
        for _, n, l in orbs:
            if l == 0 and cur: periods.append(cur); cur = []
            cur.append((n, l))
        for i, p in enumerate(periods[:8], 1):
            size = sum(2 * (2 * l + 1) for n, l in p)
            print(i, ' '.join(f'{n}{L[l]}' for n, l in p), size, on_ladder(size))
        print('2n^2:', [(n, 2*n*n, on_ladder(2*n*n)) for n in range(1, 8)])
        print('blocks 2(2l+1):', [(L[l], 2*(2*l+1), on_ladder(2*(2*l+1))) for l in range(6)]
)
        print('n<=4 all 3-smooth:', all(on_ladder(n) for n in range(1, 5)), '; first n not
3-smooth:', 5)
        print('Fricke period 8 = 119..164:', 164-119+1, on_ladder(46), '| Kulsha period 8:',
 54, on_ladder(54))
        ev = list(range(4, 34, 2)); share = sum(on_ladder(x) for x in ev) / len(ev)
        print('share of even 4..32 on ladder:', round(share, 4), '; three distinct lengths
by chance:', round(share**3, 4))
        print('nuclear magic:', [(m, on_ladder(m)) for m in (2, 8, 20, 28, 50, 82, 126)])
```

Output:

```
1 1s 2 True
2 2s 2p 8 True
3 3s 3p 8 True
4 4s 3d 4p 18 True
5 5s 4d 5p 18 True
6 6s 4f 5d 6p 32 True
7 7s 5f 6d 7p 32 True
8 8s 5g 6f 7d 8p 50 False
2n^2: [(1, 2, True), (2, 8, True), (3, 18, True), (4, 32, True), (5, 50, False), (6,
72, True), (7, 98, False)]
blocks 2(2l+1): [('s', 2, True), ('p', 6, True), ('d', 10, False), ('f', 14, False),
('g', 18, True), ('h', 22, False)]
n<=4 all 3-smooth: True ; first n not 3-smooth: 5
Fricke period 8 = 119..164: 46 False | Kulsha period 8: 54 True
share of even 4..32 on ladder: 0.5333 ; three distinct lengths by chance: 0.1517
nuclear magic: [(2, True), (8, True), (20, False), (28, False), (50, False), (82,
False), (126, False)]
```

6.2 Döbereiner’s triads in atomic number and in weight. Weights as tabulated on the Wikipedia page cited in 6.1.

```
        groups = {'1': [3, 11, 19, 37, 55, 87], '2': [4, 12, 20, 38, 56, 88],
                  '16': [8, 16, 34, 52, 84, 116], '17': [9, 17, 35, 53, 85, 117]}
        for g, zs in groups.items():
            print('group', g, [(zs[i], zs[i+1], zs[i+2], zs[i+1]-zs[i], zs[i+2]-zs[i+1],

                                (zs[i]+zs[i+2]) == 2*zs[i+1]) for i in range(len(zs)-2)])

        W = {'Li': 6.941, 'Na': 22.989769, 'K': 39.0983, 'Ca': 40.078, 'Sr': 87.62, 'Ba': 13
7.327,
```

```
             'Cl': 35.470, 'Br': 79.904, 'I': 126.904, 'S': 32.065, 'Se': 78.971, 'Te': 127.
60}
        for a, b, c in [('Li','Na','K'), ('Ca','Sr','Ba'), ('Cl','Br','I'), ('S','Se','Te')]
:
            m = (W[a] + W[c]) / 2
            print(a, b, c, 'mean', round(m, 3), 'actual', W[b], 'dev %', round(100*(W[b]-m)/
W[b], 2))
```

Output:

```
group 1 [(3, 11, 19, 8, 8, True), (11, 19, 37, 8, 18, False), (19, 37, 55, 18, 18,
True), (37, 55, 87, 18, 32, False)]
group 2 [(4, 12, 20, 8, 8, True), (12, 20, 38, 8, 18, False), (20, 38, 56, 18, 18,
True), (38, 56, 88, 18, 32, False)]
group 16 [(8, 16, 34, 8, 18, False), (16, 34, 52, 18, 18, True), (34, 52, 84, 18,
32, False), (52, 84, 116, 32, 32, True)]
group 17 [(9, 17, 35, 8, 18, False), (17, 35, 53, 18, 18, True), (35, 53, 85, 18,
32, False), (53, 85, 117, 32, 32, True)]
Li Na K mean 23.02 actual 22.989769 dev % -0.13
Ca Sr Ba mean 88.703 actual 87.62 dev % -1.24
Cl Br I mean 81.187 actual 79.904 dev % -1.61
S Se Te mean 79.832 actual 78.971 dev % -1.09
```

6.3 The ice rule as meetings; Pauling's count.

```
        from itertools import product
        from fractions import Fraction
        from math import log
        closed = [w for w in product((1, -1), repeat=4) if sum(w) == 0]
        print('vertices with two in, two out:', len(closed), 'of', 2**4)
        W = 4 * Fraction(len(closed), 16)      # 2^(2N) bond states times (6/16)^N, per
molecule
        print('configurations per molecule:', W)
        print('S = R ln W =', round(8.314462618 * log(W), 3), 'J/(mol K)')
        print('odd number of +-1 arrows can sum to 0:', any(sum(w) == 0 for w in product((1,
 -1), repeat=3)))
```

Output:

```
vertices with two in, two out: 6 of 16
configurations per molecule: 3/2
S = R ln W = 3.371 J/(mol K)
odd number of +-1 arrows can sum to 0: False
```

6.4 Knot determinants and three-colourability. Determinants of non-torus knots and the Jones polynomial of 8_{19} are from the Knot Atlas.

```
        import sympy as sp, cmath
        from lad import on_ladder
        t = sp.symbols('t')
        def torus_alex(p, q):
            return sp.cancel((t**(p*q) - 1)*(t - 1)/((t**p - 1)*(t**q - 1)))
        for name, (p, q) in [('3_1', (2, 3)), ('5_1', (2, 5)), ('7_1', (2, 7)), ('8_19', (3,
 4))]:
            d = abs(torus_alex(p, q).subs(t, -1))
            print(f'{name:5s} T({p},{q}) det {d}  three-colourable {d % 3 == 0}  p,q on
ladder {on_ladder(p) and on_ladder(q)}')
        for name, d in [('4_1', 5), ('5_2', 7), ('7_4', 15), ('8_18', 45)]:
```

```
            print(f'{name:5s} det {d}  three-colourable {d % 3 == 0}')
        q = cmath.exp(1j*cmath.pi/3); V = -q**8 + q**5 + q**3
        print('8_19: V(e^{i pi/3}) =', complex(round(V.real, 12), round(V.imag, 12)), '; 3|
V|^2 =', round(3*abs(V)**2, 10))
```

Output:

```
3_1   T(2,3) det 3  three-colourable True  p,q on ladder True
5_1   T(2,5) det 5  three-colourable False  p,q on ladder False
7_1   T(2,7) det 7  three-colourable False  p,q on ladder False
8_19  T(3,4) det 3  three-colourable True  p,q on ladder True
4_1   det 5  three-colourable False
5_2   det 7  three-colourable False
7_4   det 15  three-colourable True
8_18  det 45  three-colourable True
8_19: V(e^{i pi/3}) = (-0-1.732050807569j) ; 3|V|^2 = 9.0
```

6.5 Isomer counts, aromatic numbers and the ladder.

```
        from itertools import product
        from lad import on_ladder
        h = [4*k + 2 for k in range(8)]
        print('Huckel 4k+2:', [(x, on_ladder(x), x % 3) for x in h])
        print('Hirsch 2(k+1)^2:', [(2*(k+1)**2, on_ladder(2*(k+1)**2)) for k in range(7)])

        def classes(dihedral):
            seen, counts = set(), [0]*7
            for w in product((0, 1), repeat=6):
                rots = [w[i:] + w[:i] for i in range(6)]
                if dihedral: rots += [r[::-1] for r in rots]
                c = min(rots)
                if c not in seen: seen.add(c); counts[sum(w)] += 1
            return counts
        print('benzene C6H(6-k)X(k), dihedral classes by k:', classes(True), 'total', sum(cl
asses(True)))
        print('two-colour necklaces of 6 (rotation only):', classes(False), 'total', sum(cla
sses(False)))
        alk = [1, 1, 1, 2, 3, 5, 9, 18, 35, 75, 159, 355, 802, 1858]   # OEIS A000602, n =
1..14
        print('alkane isomer counts on ladder:', [(i+1, a) for i, a in enumerate(alk) if
on_ladder(a)])
        print('share of integers 1..1858 on ladder:', round(sum(on_ladder(k) for k in
range(1, 1859))/1858, 4))
```

Output:

```
Huckel 4k+2: [(2, True, 2), (6, True, 0), (10, False, 1), (14, False, 2), (18, True,
0), (22, False, 1), (26, False, 2), (30, False, 0)]
Hirsch 2(k+1)^2: [(2, True), (8, True), (18, True), (32, True), (50, False), (72,
True), (98, False)]
benzene C6H(6-k)X(k), dihedral classes by k: [1, 1, 3, 3, 3, 1, 1] total 13
two-colour necklaces of 6 (rotation only): [1, 1, 3, 4, 3, 1, 1] total 14
alkane isomer counts on ladder: [(1, 1), (2, 1), (3, 1), (4, 2), (5, 3), (7, 9), (8,
18)]
share of integers 1..1858 on ladder: 0.0248
```

6.6 Polytype stacking: residue of the Hägg word and identity period. Positions A, B, C are 0, 1, 2 mod 3.

```
        from itertools import product
        def info(h):
            s = sum(1 if c == '+' else -1 for c in h)
            N = len(h); period = N if s % 3 == 0 else 3 * N
            pos = [0]
            for c in (h * 3)[:period - 1]: pos.append((pos[-1] + (1 if c == '+' else -1)) %
3)
            return s, s % 3, period, ''.join('ABC'[p] for p in pos)
        for name, h in [('2H', '+-'), ('4H', '++--'), ('6H', '+++---'), ('3C', '+'),
                        ('9R (Zhdanov 21)', '++-'), ('15R (Zhdanov 32)', '+++--'), ('21R
(Zhdanov 34)', '+++----')]:
            s, r, p, st = info(h)
            print(f'{name:17s} Hagg {h:8s} sum {s:+d} residue {r}  identity period {p:2d}
layers  {st}')
        print('share of Hagg words of length N with sum = 0 mod 3:')
        for N in range(2, 13):
            c = sum(1 for w in product((1, -1), repeat=N) if sum(w) % 3 == 0)
            print(N, c, 2**N, round(c / 2**N, 4))
```

Output:

```
2H                Hagg +-       sum +0 residue 0  identity period  2 layers  AB
4H                Hagg ++--     sum +0 residue 0  identity period  4 layers  ABCB
6H                Hagg +++---   sum +0 residue 0  identity period  6 layers  ABCACB
3C                Hagg +        sum +1 residue 1  identity period  3 layers  ABC
9R (Zhdanov 21)   Hagg ++-      sum +1 residue 1  identity period  9 layers
ABCBCACAB
15R (Zhdanov 32)  Hagg +++--    sum +1 residue 1  identity period 15 layers
ABCACBCABACABCB
21R (Zhdanov 34)  Hagg +++----  sum -1 residue 2  identity period 21 layers
ABCACBACABCBACBCABACB
share of Hagg words of length N with sum = 0 mod 3:
2 2 4 0.5
3 2 8 0.25
4 6 16 0.375
5 10 32 0.3125
6 22 64 0.3438
7 42 128 0.3281
8 86 256 0.3359
9 170 512 0.332
10 342 1024 0.334
11 682 2048 0.333
12 1366 4096 0.3335
```

6.7 Crystallographic restriction and the ladder; counts of crystallography.

```
        import sympy as sp
        from math import cos, pi
        from lad import on_ladder
        orders = [n for n in range(1, 61) if (1 if n <= 2 else sp.totient(n)) <= 2]
        print('orders with phi(n) <= 2 (2D/3D lattices):', orders, all(on_ladder(n) for n
in orders))
        orders4 = [n for n in range(1, 61) if (1 if n <= 2 else sp.totient(n)) <= 4]
        print('orders with phi(n) <= 4 (4D/5D):', [(n, on_ladder(n)) for n in orders4])

        print('2cos(2pi/n):', [(n, round(2*cos(2*pi/n), 4)) for n in range(1, 9)])
        for name, c in [('crystal systems', 7), ('Bravais lattices 3D', 14), ('point groups
3D', 32),
                        ('space groups', 230), ('Bravais lattices 2D', 5), ('plane point
groups', 10),
```

```
                        ('wallpaper groups', 17)]:
            print(f'{name:20s} {c:4d} on ladder: {on_ladder(c)}')
```

Output:

```
orders with phi(n) <= 2 (2D/3D lattices): [1, 2, 3, 4, 6] True
orders with phi(n) <= 4 (4D/5D): [(1, True), (2, True), (3, True), (4, True), (5,
False), (6, True), (8, True), (10, False), (12, True)]
2cos(2pi/n): [(1, 2.0), (2, -2.0), (3, -1.0), (4, 0.0), (5, 0.618), (6, 1.0), (7,
1.247), (8, 1.4142)]
crystal systems         7 on ladder: False
Bravais lattices 3D    14 on ladder: False
point groups 3D        32 on ladder: True
space groups          230 on ladder: False
Bravais lattices 2D     5 on ladder: False
plane point groups     10 on ladder: False
wallpaper groups       17 on ladder: False
```

6.8 T23 and the bulk modulus: K′ = 2 + γ. K₀′ values as tabulated in the Murnaghan source cited in 6.5.

```
        import sympy as sp
        n, g2, g3 = sp.symbols('n g2 g3', positive=True)
        e = g2/2*n**2 + g3/6*n**3                 # energy density of Foundations T23

        P = sp.simplify(n*sp.diff(e, n) - e)       # pressure from energy density
        K = sp.simplify(n*sp.diff(P, n))           # bulk modulus K = n dP/dn
        geff = sp.diff(e, n, 2)
        print('K =', sp.factor(K), '; K/n^2 - g_eff =', sp.simplify(K/n**2 - geff))
        Kp = sp.simplify(sp.diff(K, n)/sp.diff(P, n))   # K' = dK/dP
        gamma = g3*n/(g2 + g3*n)                   # gamma of T23 at reference density n

        print("K' - (2 + gamma) =", sp.simplify(Kp - (2 + gamma)))
        data = {'NaF': 5.28, 'NaCl': 5.39, 'NaBr': 5.46, 'NaI': 5.59, 'MgO': 4.7,
                'calcite': 4.63, 'magnesite': 3.08, '3C-SiC': 4.0}
        for k, v in data.items(): print(f"{k:10s} K0'={v:5.2f}  gamma=K0'-2={v-2:5.2f}")

        print('free-path lever d=3 (T24):', round(-(1 + 0.3405373)/0.3405373, 4))
        print("Thomas-Fermi limit K'=5/3 -> gamma =", sp.Rational(5, 3) - 2)
```

Output:

```
K = n**2*(g2 + g3*n) ; K/n^2 - g_eff = 0
K' - (2 + gamma) = 0
NaF        K0'= 5.28  gamma=K0'-2= 3.28
NaCl       K0'= 5.39  gamma=K0'-2= 3.39
NaBr       K0'= 5.46  gamma=K0'-2= 3.46
NaI        K0'= 5.59  gamma=K0'-2= 3.59
MgO        K0'= 4.70  gamma=K0'-2= 2.70
calcite    K0'= 4.63  gamma=K0'-2= 2.63
magnesite  K0'= 3.08  gamma=K0'-2= 1.08
3C-SiC     K0'= 4.00  gamma=K0'-2= 2.00
free-path lever d=3 (T24): -3.9365
Thomas-Fermi limit K'=5/3 -> gamma = -1/3
```

6.9 Coincidence-site index of combined twins.

```
        import sympy as sp
        def rot(axis, ang):
            a = sp.Matrix(axis) / sp.sqrt(sum(x**2 for x in axis)); c, s = sp.cos(ang),
sp.sin(ang)
            K = sp.Matrix([[0, -a[2], a[1]], [a[2], 0, -a[0]], [-a[1], a[0], 0]])
            return sp.simplify(sp.eye(3)*c + (1 - c)*a*a.T + s*K)
        def sigma(R):
            d = 1
            for e in R: d = sp.ilcm(d, sp.fraction(sp.nsimplify(e))[1])
            while d % 2 == 0: d //= 2
            return d
        A, B, C = rot([1, 1, 1], sp.pi/3), rot([1, 1, -1], sp.pi/3), rot([1, -1, 1], sp.pi/
3)
        print('Sigma(60 deg about <111>) =', sigma(A))
        print('Sigma3 * Sigma3 (different twin axes) =', {sigma(A*B), sigma(A*B.T)})
        print('Sigma3 * Sigma3 * Sigma3 =', sorted({sigma(X*Y*Z) for X in (A, A.T) for Y in
(B, B.T) for Z in (C, C.T)}))
        print('Sigma3 * Sigma3 (same axis, A*A.T) =', sigma(A*A.T))
```

Output:

```
Sigma(60 deg about <111>) = 3
Sigma3 * Sigma3 (different twin axes) = {9}
Sigma3 * Sigma3 * Sigma3 = [27]
Sigma3 * Sigma3 (same axis, A*A.T) = 1
```

6.10–6.12 Nanotubes, ribbons, cages and the index-3 sublattice. In the code a point x + yω of Z[ω] is the pair (x, y).

```
        from fractions import Fraction
        met = tot = 0
        for n in range(31):
            for m in range(n + 1):
                if (n, m) == (0, 0): continue
                tot += 1; met += (n - m) % 3 == 0
        print('tubes with 0<=m<=n<=30:', met, 'metallic of', tot, '=', round(met/tot, 4),
'; T6 uniform baseline 1/3')
        ok = all((((n + m) + m) % 3 == 0) == ((n - m) % 3 == 0) for n in range(100) for m
in range(100))
        print('n - m = 0 mod 3  <=>  C = 0 mod (1 - w):', ok)
        print('armchair ribbon width families N mod 3:', [(N, N % 3) for N in range(3, 15)])

        norms = sorted({h*h + h*k + k*k for h in range(12) for k in range(12) if h + k > 0})
[:12]
        print('Eisenstein norms T:', norms); print('icosahedral cages C_20T:', [20*t for t i
n norms])
        print('C60: T = 3, hexagons 10(T-1) =', 10*2, ', pentagons 12')
        def in_lat(v, B):
            (a, _), (b, d) = B; x, y = v
            return y % d == 0 and (x - b*(y//d)) % a == 0
        turn = lambda v: (v[0] - v[1], v[0])           # (x + y w)(1 + w)
        subs = [((a, 0), (b, d)) for a, d in ((1, 3), (3, 1)) for b in range(a)]
        inv = [B for B in subs if all(in_lat(turn(g), B) for g in [(B[0][0], 0), (B[1][0],
B[1][1])])]
        print('index-3 sublattices:', len(subs), '; invariant under 60 deg:', inv)
        print('it equals (1 - w)Z[w]: contains 1 - w:', in_lat((1, -1), inv[0]), ', all
points have a + b = 0 mod 3:',
              all((x + y) % 3 == 0 for x in range(-9, 10) for y in range(-9, 10) if
```

```
in_lat((x, y), inv[0])))
        print('LiC6: carbon atoms per hexagon 2, so hexagons per Li =', Fraction(6, 2))
```

Output:

```
tubes with 0<=m<=n<=30: 175 metallic of 495 = 0.3535 ; T6 uniform baseline 1/3
n - m = 0 mod 3  <=>  C = 0 mod (1 - w): True
armchair ribbon width families N mod 3: [(3, 0), (4, 1), (5, 2), (6, 0), (7, 1), (8,
2), (9, 0), (10, 1), (11, 2), (12, 0), (13, 1), (14, 2)]
Eisenstein norms T: [1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27]
icosahedral cages C_20T: [20, 60, 80, 140, 180, 240, 260, 320, 380, 420, 500, 540]
C60: T = 3, hexagons 10(T-1) = 20 , pentagons 12
index-3 sublattices: 4 ; invariant under 60 deg: [((3, 0), (2, 1))]
it equals (1 - w)Z[w]: contains 1 - w: True , all points have a + b = 0 mod 3: True
LiC6: carbon atoms per hexagon 2, so hexagons per Li = 3
```

Calculation 6.10 is the nanotube count and the residue identity, 6.11 the cage series, 6.12 the sublattice.

6.13 Coordination numbers and ice counts on the ladder.

```
        from lad import on_ladder
        cn = {2: 'linear', 3: 'trigonal', 4: 'tetrahedral/square', 5: 'trigonal bipyramid',
6: 'octahedral',
              7: 'pentagonal bipyramid', 8: 'cubic/square antiprism', 9: 'tricapped prism',
12: 'icosahedral/cuboctahedral'}
        print('coordination numbers on ladder:', [(k, on_ladder(k)) for k in cn])
        print('share on ladder:', sum(on_ladder(k) for k in cn), 'of', len(cn))
        print('ice phases 22 on ladder:', on_ladder(22), '; ordered/disordered pairs 7:',
on_ladder(7))
```

Output:

```
coordination numbers on ladder: [(2, True), (3, True), (4, True), (5, False), (6,
True), (7, False), (8, True), (9, True), (12, True)]
share on ladder: 7 of 9
ice phases 22 on ladder: False ; ordered/disordered pairs 7: False
```

6.14 Hexagonality and residue of SiC polytypes. A layer is h-type when the Hägg sign changes across it.

```
        from fractions import Fraction
        def hexagonality(h):
            n = len(h); hs = sum(1 for i in range(n) if h[i] != h[(i + 1) % n])
            return Fraction(hs, n)
        for name, h in [('2H', '+-'), ('4H', '++--'), ('6H', '+++---'), ('8H', '++++----'),

                        ('9R', '++-'), ('15R', '+++--'), ('21R', '+++----'), ('3C', '+')]:

            s = sum(1 if c == '+' else -1 for c in h)
            print(f'{name:4s} hexagonality {str(hexagonality(h)):5s} = {float(hexagonality(h
))*100:5.1f} %  residue {s % 3}')
```

Output:

```
2H   hexagonality 1     = 100.0 %  residue 0
4H   hexagonality 1/2   =  50.0 %  residue 0
6H   hexagonality 1/3   =  33.3 %  residue 0
```

```
8H   hexagonality 1/4   =  25.0 %  residue 0
9R   hexagonality 2/3   =  66.7 %  residue 1
15R  hexagonality 2/5   =  40.0 %  residue 1
21R  hexagonality 2/7   =  28.6 %  residue 2
3C   hexagonality 0     =   0.0 %  residue 1
```

6.15 Shockley partials: a ring that carries a lattice vector. Vectors in units of a/6 unless stated.

```
        import numpy as np
        b1, b2, b3 = np.array([-1, 2, -1]), np.array([2, -1, -1]), np.array([-1, -1, 2])

        print('b1+b2+b3 =', b1 + b2 + b3, '; each lies in (111):', [int(v @ np.array([1, 1,
1])) for v in (b1, b2, b3)])
        print('b1 - b2 =', b1 - b2, '= a/2[-1 1 0] in units of a/6 (perfect dislocation)')

        cos = (b1 @ b2) / np.sqrt((b1 @ b1) * (b2 @ b2)); print('angle b1,b2 =', round(np.de
grees(np.arccos(cos)), 3))
        t = 3*b1 // 3
# 3*b1 in units of a/6 is 3*b1; in units of a/2 it is b1
        print('3*b1 in units of a/2 =', t, '; fcc lattice vector (integer, even sum):', int(
t.sum()) % 2 == 0)
```

Output:

```
b1+b2+b3 = [0 0 0] ; each lies in (111): [0, 0, 0]
b1 - b2 = [-3  3  0] = a/2[-1 1 0] in units of a/6 (perfect dislocation)
angle b1,b2 = 120.0
3*b1 in units of a/2 = [-1  2 -1] ; fcc lattice vector (integer, even sum): True
```

6.16 Armchair tubes: the quotient by β.

```
        import cmath
        w = cmath.exp(2j*cmath.pi/3); beta = 1 - w
        for n in range(1, 6):
            C = (2*n) + n*w
            print(n, 'C/beta =', complex(round((C/beta).real, 12), round((C/beta).imag, 12))
, '; n(1+w) =', n*(1 + w))
        for n, m in [(3, 0), (6, 0), (4, 1), (5, 2), (7, 1)]:
            C = (n + m) + m*w; z = C/beta
            print((n, m), 'C/beta =', complex(round(z.real, 12), round(z.imag, 12)), 'norm',
 round(abs(z)**2, 10))
```

Output:

```
1 C/beta = (0.5+0.866025403784j) ; n(1+w) = (0.5000000000000002+0.8660254037844387j)
2 C/beta = (1+1.732050807569j) ; n(1+w) = (1.0000000000000004+1.7320508075688774j)
3 C/beta = (1.5+2.598076211353j) ; n(1+w) = (1.5000000000000007+2.598076211353316j)
4 C/beta = (2+3.464101615138j) ; n(1+w) = (2.000000000000001+3.464101615137755j)
5 C/beta = (2.5+4.330127018922j) ; n(1+w) = (2.500000000000001+4.330127018922194j)
(3, 0) C/beta = (1.5+0.866025403784j) norm 3.0
(6, 0) C/beta = (3+1.732050807569j) norm 12.0
(4, 1) C/beta = (2+1.732050807569j) norm 7.0
(5, 2) C/beta = (2.5+2.598076211353j) norm 13.0
(7, 1) C/beta = (3.5+2.598076211353j) norm 19.0
```

6.17 Werner’s isomer counts by skeleton. Configurations of two or three B ligands up to the rotations of the skeleton (Burnside, as in T13).

```
        import itertools, numpy as np
        def orbits(points, group, k):
            seen = set(); count = 0
            for S in itertools.combinations(range(len(points)), k):
                key = frozenset(S)
                if key in seen: continue
                count += 1
                for g in group: seen.add(frozenset(g[i] for i in S))
            return count
        def closure(gens, n):
            G = {tuple(range(n))}; frontier = list(G)
            while frontier:
                new = []
                for g in frontier:
                    for h in gens:
                        c = tuple(h[g[i]] for i in range(n))
                        if c not in G: G.add(c); new.append(c)
                frontier = new
            return G
        def perm_from(R, pts):
            out = []
            for p in pts:
                q = R @ p; out.append(int(np.argmin([np.linalg.norm(q - r) for r in pts])))

            return tuple(out)
        octa = [np.array(v) for v in [(1,0,0),(-1,0,0),(0,1,0),(0,-1,0),(0,0,1),(0,0,-1)]]

        Rz = np.array([[0,-1,0],[1,0,0],[0,0,1]]); Rx = np.array([[1,0,0],[0,0,-1],[0,1,0]])

        G_oct = closure([perm_from(Rz, octa), perm_from(Rx, octa)], 6)
        hexpl = [np.array([np.cos(k*np.pi/3), np.sin(k*np.pi/3), 0]) for k in range(6)]

        R6 = np.array([[np.cos(np.pi/3), -np.sin(np.pi/3), 0], [np.sin(np.pi/3),
np.cos(np.pi/3), 0], [0, 0, 1]])
        F = np.array([[1,0,0],[0,-1,0],[0,0,-1]])
        G_hex = closure([perm_from(R6, hexpl), perm_from(F, hexpl)], 6)
        tri = [np.array([np.cos(k*2*np.pi/3), np.sin(k*2*np.pi/3), s]) for s in (1, -1) for
k in range(3)]
        R3 = np.array([[np.cos(2*np.pi/3), -np.sin(2*np.pi/3), 0], [np.sin(2*np.pi/3),
np.cos(2*np.pi/3), 0], [0, 0, 1]])
        G_tri = closure([perm_from(R3, tri), perm_from(F, tri)], 6)
        for name, pts, G in [('octahedron', octa, G_oct), ('hexagon (planar)', hexpl,
G_hex), ('trigonal prism', tri, G_tri)]:
            print(f'{name:17s} |G| = {len(G):2d}  MA4B2 isomers = {orbits(pts, G,
2)}  MA3B3 isomers = {orbits(pts, G, 3)}')
```

Output:

```
octahedron        |G| = 24  MA4B2 isomers = 2  MA3B3 isomers = 2
hexagon (planar)  |G| = 12  MA4B2 isomers = 3  MA3B3 isomers = 3
trigonal prism    |G| =  6  MA4B2 isomers = 4  MA3B3 isomers = 4
```

Calculations for Chapter 7 — Reactions and Heat

All scripts were run on 27 October 2026 with Python 3, numpy 2.5, scipy and sympy 1.14. `lad.py` is the ladder helper of Chapter 6.

7.1 Invariant points of the phase rule.

```
        from lad import on_ladder
        for C in (1, 2, 3, 4):
            print(f'C = {C}: F = {C} - P + 2; invariant point at P = {C + 2} phases; on
ladder: {on_ladder(C + 2)}')
```

Output:

```
C = 1: F = 1 - P + 2; invariant point at P = 3 phases; on ladder: True
C = 2: F = 2 - P + 2; invariant point at P = 4 phases; on ladder: True
C = 3: F = 3 - P + 2; invariant point at P = 5 phases; on ladder: False
C = 4: F = 4 - P + 2; invariant point at P = 6 phases; on ladder: True
```

7.2 Debye–Hückel limiting law from fundamental constants. Input: relative permittivity of water 78.54 and density 997 kg/m³ at 25 °C.

```
        from math import pi, sqrt, log
        e, eps0, kB, NA = 1.602176634e-19, 8.8541878128e-12, 1.380649e-23, 6.02214076e23

        T, epsr, rho = 298.15, 78.54, 997.0
        kappa_per_sqrtI = sqrt(2*NA*e**2*rho/(eps0*epsr*kB*T))
        lB = e**2/(4*pi*eps0*epsr*kB*T)
        A = lB*kappa_per_sqrtI/(2*log(10))
        print('Bjerrum length (nm):', round(lB*1e9, 4), '; Debye-Hueckel A (kg^1/2
mol^-1/2):', round(A, 4))
        for I in (1e-4, 1e-3, 1e-2, 0.1):
            print(f'I = {I:g} mol/kg: log10 gamma(1:1) = {-A*sqrt(I):.4f}, gamma = {10**(-
A*sqrt(I)):.4f}')
```

Output:

```
Bjerrum length (nm): 0.7136 ; Debye-Hueckel A (kg^1/2 mol^-1/2): 0.5085
I = 0.0001 mol/kg: log10 gamma(1:1) = -0.0051, gamma = 0.9884
I = 0.001 mol/kg: log10 gamma(1:1) = -0.0161, gamma = 0.9637
I = 0.01 mol/kg: log10 gamma(1:1) = -0.0508, gamma = 0.8895
I = 0.1 mol/kg: log10 gamma(1:1) = -0.1608, gamma = 0.6906
```

7.3 Geminate re-encounters: lever of the encounter count. P_3 from Foundations T24.

```
        import numpy as np
        P3 = 0.3405373
        print('T24 lever d=3:', round(-(1 + P3)/P3, 4), '; mean number of encounters 1/(1-
P3):', round(1/(1 - P3), 4))
        rng = np.random.default_rng(1)
        moves = np.array([[1,0,0],[-1,0,0],[0,1,0],[0,-1,0],[0,0,1],[0,0,-1]])
        def encounters(L):
            pos = np.cumsum(moves[rng.integers(0, 6, size=L)], axis=0)
            return 1 + int(np.sum(np.all(pos == 0, axis=1)))
        V = np.array([encounters(2000) for _ in range(20000)])
        m, k2 = V.mean(), V.var(); k3 = ((V - m)**3).mean()
        print('Monte Carlo, walks of 2000 steps, 20000 pairs: mean', round(m, 4), ' return
share', round(1 - 1/m, 4),
              ' lever', round(-m*k3/k2**2, 3))
```

Output:

```
T24 lever d=3: -3.9365 ; mean number of encounters 1/(1-P3): 1.5164
Monte Carlo, walks of 2000 steps, 20000 pairs: mean 1.5012  return share 0.3338
lever -3.933
```

7.4 Closure and carry of ligand words (T7.1); molecularity and the ladder.

```
        from itertools import product
        from lad import on_ladder
        print('elementary closure lengths (T7): standing 1, meeting 2, ring 3;
molecularities observed: 1, 2, 3')
        print('powers of two are on the ladder:', all(on_ladder(2**k) for k in range(10)), '
; 3*2^k on ladder:', all(on_ladder(3*2**k) for k in range(10)))
        def bt(n):
            d = []
            while n:
                r = n % 3
                if r == 2: r = -1
                d.append(r); n = (n - r)//3
            return ''.join({1:'1', 0:'0', -1:'T'}[x] for x in reversed(d)) or '0'
        for s in (0, 2, 3, -3, -2):
            print(f'sum {s:+d} -> balanced ternary {bt(s)}')
```

Output:

```
elementary closure lengths (T7): standing 1, meeting 2, ring 3; molecularities
observed: 1, 2, 3
powers of two are on the ladder: True ; 3*2^k on ladder: True
sum +0 -> balanced ternary 0
sum +2 -> balanced ternary 1T
sum +3 -> balanced ternary 10
sum -3 -> balanced ternary T0
sum -2 -> balanced ternary T1
```

The sums 0, ±2 and ±3 are those of (R, S), (R, R) or (S, S), and (R, R, R) or (S, S, S).

7.5 Kagan's ML_2 model. Random pairing (K = 4); g is the relative reactivity of the heterochiral complex.

```
        from scipy.optimize import brentq
        def ml2(ee, K=4.0, g=0.0):
            R, S = (1 + ee)/2, (1 - ee)/2
            z = brentq(lambda z: z*z - K*((R - z)/2)*((S - z)/2), 0, min(R, S))
            x, y = (R - z)/2, (S - z)/2
            beta = z/(x + y)
            return ee*(1 + beta)/(1 + g*beta), beta
        for ee in (0.1, 0.2, 0.5, 0.8):
            r0, b = ml2(ee, g=0.0); r1, _ = ml2(ee, g=1.0); r2, _ = ml2(ee, g=2.0)
            print(f'ee_aux {ee:.1f}  beta {b:.4f}  ee_prod/ee_max: g=0 {r0:.4f} | g=1 {r1:.
4f} | g=2 {r2:.4f}')
```

Output:

```
ee_aux 0.1  beta 0.9802  ee_prod/ee_max: g=0 0.1980 | g=1 0.1000 | g=2 0.0669
ee_aux 0.2  beta 0.9231  ee_prod/ee_max: g=0 0.3846 | g=1 0.2000 | g=2 0.1351
ee_aux 0.5  beta 0.6000  ee_prod/ee_max: g=0 0.8000 | g=1 0.5000 | g=2 0.3636
ee_aux 0.8  beta 0.2195  ee_prod/ee_max: g=0 0.9756 | g=1 0.8000 | g=2 0.6780
```

7.6 Haber–Bosch recycle at 15 % per pass.

```
        for n in (1, 5, 10, 20, 22, 25):
            print(f'passes {n:2d}: cumulative conversion {1 - 0.85**n:.4f}')
```

Output:

```
passes  1: cumulative conversion 0.1500
passes  5: cumulative conversion 0.5563
passes 10: cumulative conversion 0.8031
passes 20: cumulative conversion 0.9612
passes 22: cumulative conversion 0.9720
passes 25: cumulative conversion 0.9828
```

7.7 Marcus parabola. λ = 1 eV, kT = 0.0257 eV.

```
        import numpy as np
        lam, kT = 1.0, 0.0257
        for dG in (-0.25, -0.5, -0.75, -1.0, -1.25, -1.5, -2.0):
            print(f'dG = {dG:5.2f} eV  relative rate = {np.exp(-(lam + dG)**2/(4*lam*kT)):.
6f}')
```

Output:

```
dG = -0.25 eV  relative rate = 0.004204
dG = -0.50 eV  relative rate = 0.087869
dG = -0.75 eV  relative rate = 0.544451
dG = -1.00 eV  relative rate = 1.000000
dG = -1.25 eV  relative rate = 0.544451
dG = -1.50 eV  relative rate = 0.087869
dG = -2.00 eV  relative rate = 0.000060
```

7.8 Nernst slope, electrolysis charge and the LiC_6 capacity.

```
        from math import log
        R, F, T = 8.314462618, 96485.33212, 298.15
        print('Nernst slope RT ln10 / F =', round(1000*R*T*log(10)/F, 3), 'mV per decade (n
= 1)')
        for n in (1, 2, 3, 4): print(f'  n = {n}: {1000*R*T*log(10)/(n*F):.3f} mV per
decade')
        print('charge to make 1 mol H2 (2 e-):', round(2*F, 1), 'C ; 1 mol O2 (4 e-):', roun
d(4*F, 1), 'C')
        print('LiC6 theoretical capacity: F/(6*12.011 g) =', round(F/(6*12.011)/3.6, 1), 'mA
h/g')
```

Output:

```
Nernst slope RT ln10 / F = 59.159 mV per decade (n = 1)
  n = 1: 59.159 mV per decade
  n = 2: 29.580 mV per decade
  n = 3: 19.720 mV per decade
  n = 4: 14.790 mV per decade
charge to make 1 mol H2 (2 e-): 192970.7 C ; 1 mol O2 (4 e-): 385941.3 C
LiC6 theoretical capacity: F/(6*12.011 g) = 371.9 mAh/g
```

7.9 Three-state excitable automaton: winding defects. States 0 resting, 1 excited, 2 refractory; a resting cell fires if a neighbour is excited.

```
        import numpy as np
        def step(S):
            exc = (S == 1)
            nb = np.roll(exc, 1, 0) | np.roll(exc, -1, 0) | np.roll(exc, 1, 1) |
np.roll(exc, -1, 1)
            N = S.copy(); N[S == 1] = 2; N[S == 2] = 0; N[(S == 0) & nb] = 1
            return N
        def bal(d):
            d = d % 3; return np.where(d == 2, -1, d)
        def winding(S):
            a, b = S, np.roll(S, -1, 1); c = np.roll(b, -1, 0); d = np.roll(S, -1, 0)

            return (bal(b - a) + bal(c - b) + bal(d - c) + bal(a - d)) // 3
        for seed in (0, 1, 2):
            rng = np.random.default_rng(seed); S = rng.integers(0, 3, size=(120, 120))

            w0 = winding(S)
            for t in range(300): S = step(S)
            w = winding(S)
            print(f'seed {seed}: t=0 defects {np.count_nonzero(w0):4d} (net {w0.sum():
+d}) | t=300 defects {np.count_nonzero(w):3d} '
                  f'(net {w.sum():+d}), active cells {np.count_nonzero(S):5d}')
        S = np.zeros((120, 120), int); S[60, 60] = 1
        for t in range(300): S = step(S)
        print('single excitation, no defect: active cells after 300 steps =',
np.count_nonzero(S))
        S = np.zeros((120, 120), int); S[60, 40:80] = 1; S[61, 40:80] = 2
        w = winding(S); print('broken wave seed: defects', np.count_nonzero(w), 'net',
w.sum())
        for t in range(300): S = step(S)
        w = winding(S); print('after 300 steps: defects', np.count_nonzero(w), 'net', w.sum(
), 'active', np.count_nonzero(S))
        from itertools import product
        cnt = {-1: 0, 0: 0, 1: 0}
        for a, b, c, d in product(range(3), repeat=4):
            w = (int(bal(b - a)) + int(bal(c - b)) + int(bal(d - c)) + int(bal(a - d))) // 3

            cnt[w] += 1
        print('random plaquette windings over 81 states:', cnt, '; share nonzero', round((cn
t[1] + cnt[-1])/81, 4))
```

Output:

```
seed 0: t=0 defects 4268 (net +0) | t=300 defects 4268 (net +0), active cells  9612
seed 1: t=0 defects 4330 (net +0) | t=300 defects 4330 (net +0), active cells  9601
seed 2: t=0 defects 4246 (net +0) | t=300 defects 4246 (net +0), active cells  9543
single excitation, no defect: active cells after 300 steps = 0
broken wave seed: defects 2 net 0
after 300 steps: defects 2 net 0 active 7200
random plaquette windings over 81 states: {-1: 12, 0: 57, 1: 12} ; share nonzero
0.2963
```

7.10 Logistic map: period-3 window and Feigenbaum ratios.

```
        from math import sqrt
        from scipy.optimize import brentq
        print('period-3 window opens at r = 1 + sqrt(8) =', round(1 + sqrt(8), 6))
        x = 0.5
        for _ in range(2000): x = 3.83*x*(1 - x)
        orb = []
```

```
        for _ in range(6): x = 3.83*x*(1 - x); orb.append(round(x, 6))
        print('r = 3.83, orbit:', orb)
        def f(r, n):
            x = 0.5
            for _ in range(2**n): x = r*x*(1 - x)
            return x - 0.5
        Rs = [2.0, 1 + sqrt(5)]
        for g in [3.4985616, 3.5546408, 3.5666673, 3.5692435, 3.5697952]:
            n = len(Rs); Rs.append(brentq(lambda r: f(r, n), g - 1e-4, g + 1e-4))
        print('superstable r (periods 1, 2, 4, ..., 64):', [round(r, 7) for r in Rs])

        print('Feigenbaum delta estimates:', [round((Rs[i] - Rs[i-1])/(Rs[i+1] - Rs[i]), 4)
for i in range(1, len(Rs) - 1)])
```

Output:

```
period-3 window opens at r = 1 + sqrt(8) = 3.828427
r = 3.83, orbit: [0.504666, 0.957417, 0.156149, 0.504666, 0.957417, 0.156149]
superstable r (periods 1, 2, 4, ..., 64): [2.0, 3.236068, 3.4985617, 3.5546409,
3.5666674, 3.5692435, 3.5697953]
Feigenbaum delta estimates: [4.7089, 4.6808, 4.663, 4.6684, 4.669]
```

7.11 Brusselator: onset of oscillation at B = 1 + A². Reactions A → X, B + X → Y + D, 2X + Y → 3X, X → E, with unit rate constants.

```
        from scipy.integrate import solve_ivp
        def bru(t, u, A, B):
            X, Y = u
            return [A - (B + 1)*X + X*X*Y, B*X - X*X*Y]
        for A, B in ((1, 1.8), (1, 2.2), (1, 3.0)):
            s = solve_ivp(bru, (0, 400), [1.1, B/A], args=(A, B), rtol=1e-9, max_step=0.1)

            X = s.y[0][s.t > 300]
            print(f'A = {A}, B = {B} (Hopf threshold 1 + A^2 = {1 +
A*A}): late amplitude of X = {X.max() - X.min():.4f}')
```

Output:

```
A = 1, B = 1.8 (Hopf threshold 1 + A^2 = 2): late amplitude of X = 0.0000
A = 1, B = 2.2 (Hopf threshold 1 + A^2 = 2): late amplitude of X = 1.1284
A = 1, B = 3.0 (Hopf threshold 1 + A^2 = 2): late amplitude of X = 3.3812
```

7.12 Amplification of a small enantiomeric excess. A minimal model: each hand catalyses its own formation from a constant supply, the two hands inhibit each other by forming an inactive mirror pair, and both flow out.

```
        from scipy.integrate import solve_ivp
        def model(t, u, k=1.0, mu=1.0, A=1.0, d=0.5):
            R, S = u
            return [k*R*A - mu*R*S - d*R, k*S*A - mu*R*S - d*S]
        sol = solve_ivp(model, (0, 200), [0.500001, 0.499999], method='LSODA', rtol=1e-10,
atol=1e-14,
                        t_eval=[0, 10, 20, 40, 80, 200])
        for t, R, S in zip(sol.t, *sol.y):
            print(f't = {t:5.0f}  ee = {(R - S)/(R + S):.6f}')
```

Output:

```
t =     0  ee = 0.000002
t =    10  ee = 0.000297
t =    20  ee = 0.044028
t =    40  ee = 1.000000
t =    80  ee = 1.000000
t =   200  ee = 1.000000
```

7.13 Resonant triads and hexagons.

```
        import numpy as np
        k = [np.array([np.cos(a), np.sin(a)]) for a in (0, 2*np.pi/3, 4*np.pi/3)]
        print('k1 + k2 + k3 =', np.round(sum(k), 15))
        def u(x): return sum(np.cos(kk @ x) for kk in k)
        rng = np.random.default_rng(0); pts = rng.normal(size=(1000, 2))*5
        R60 = np.array([[np.cos(np.pi/3), -np.sin(np.pi/3)], [np.sin(np.pi/3), np.cos(np.pi/
3)]])
        print('max |u(R60 x) - u(x)| over 1000 points:', round(max(abs(u(R60 @ p) - u(p)) fo
r p in pts), 12))
        # quadratic coupling: products of two modes of a triad land on the third (k1 + k2 =
-k3)
        print('k1 + k2 + k3 = 0 means k1 + k2 = -k3:', np.allclose(k[0] + k[1], -k[2]))

        # four modes at 90 degrees (squares): no three of them sum to zero
        sq = [np.array([np.cos(a), np.sin(a)]) for a in (0, np.pi/2, np.pi, 3*np.pi/2)]

        from itertools import combinations
        print('triads summing to zero among square modes:', sum(np.allclose(a + b + c, 0) fo
r a, b, c in combinations(sq, 3)))
```

Output:

```
k1 + k2 + k3 = [-0.  0.]
max |u(R60 x) - u(x)| over 1000 points: 0.0
k1 + k2 + k3 = 0 means k1 + k2 = -k3: True
triads summing to zero among square modes: 0
```

Calculations for Chapter 8 — The Earth

All scripts were run on 27 October 2026 with Python 3 and numpy 2.5. `lad.py` is the ladder helper of Chapter 6.

8.1 Closure at a triple junction.

```
        import numpy as np
        from itertools import combinations_with_replacement
        rng = np.random.default_rng(0)
        vA, vB, vC = rng.normal(size=(3, 2))
        print('AvB + BvC + CvA =', np.round((vB - vA) + (vC - vB) + (vA - vC), 15))
        w = np.exp(2j*np.pi/3); print('equal spreading at 120 deg, 1 + w + w^2 =',
np.round(1 + w + w*w, 15))
        print('unordered triples of boundary types {R, T, F}:', len(list(combinations_with_r
eplacement('RTF', 3))))
```

Output:

```
AvB + BvC + CvA = [0. 0.]
equal spreading at 120 deg, 1 + w + w^2 = (-0+0j)
unordered triples of boundary types {R, T, F}: 10
```

8.2 Gutenberg-Richter factors.

```
        from lad import on_ladder
        print('Gutenberg-Richter b = 1: count ratio per magnitude unit = 10, on ladder:',
on_ladder(10))
        print('energy ratio per magnitude unit 10^1.5 =', round(10**1.5, 2))
        print('USGS 2000-2021 yearly ranges: M6-6.9 104-185, M7-7.9 6-23 -> ratios of range
ends', round(104/6, 1), round(185/23, 1))
```

Output:

```
Gutenberg-Richter b = 1: count ratio per magnitude unit = 10, on ladder: False
energy ratio per magnitude unit 10^1.5 = 31.62
USGS 2000-2021 yearly ranges: M6-6.9 104-185, M7-7.9 6-23 -> ratios of range ends
17.3 8.0
```

8.3 Spring-neap beat; Doodson numbers in balanced ternary.

```
        M2, S2 = 12.4206012, 12.0
        beat = 1/(1/S2 - 1/M2)
        print('spring-neap period:', round(beat, 2), 'h =', round(beat/24, 3), 'days')

        def decode(dn):
            s = dn.replace('.', ''); return [int(s[0])] + [int(c) - 5 for c in s[1:]]

        def bt(n):
            d = []
            while n:
                r = n % 3
                if r == 2: r = -1
                d.append(r); n = (n - r)//3
            return ''.join({1:'1', 0:'0', -1:'T'}[x] for x in reversed(d)) or '0'
        for name, dn in [('M2', '255.555'), ('S2', '273.555'), ('K1', '165.555'), ('O1', '14
5.555')]:
            k = decode(dn); print(name, dn, '->', k, '-> balanced ternary', [bt(x) for x in
k])
```

Output:

```
spring-neap period: 354.37 h = 14.765 days
M2 255.555 -> [2, 0, 0, 0, 0, 0] -> balanced ternary ['1T', '0', '0', '0', '0', '0']
S2 273.555 -> [2, 2, -2, 0, 0, 0] -> balanced ternary ['1T', '1T', 'T1', '0', '0',
'0']
K1 165.555 -> [1, 1, 0, 0, 0, 0] -> balanced ternary ['1', '1', '0', '0', '0', '0']
O1 145.555 -> [1, -1, 0, 0, 0, 0] -> balanced ternary ['1', 'T', '0', '0', '0', '0']
```

8.4 Cells, QBO, ENSO and Chern number against the ladder.

```
        from lad import on_ladder
        print('Hadley-Ferrel-polar cells per hemisphere 3, on ladder:', on_ladder(3), ';
both hemispheres 6:', on_ladder(6))
        for m in (28, 29): print(f'QBO {m} months / 12 = {m/12:.3f}')
        print('ENSO periods 2..7 years on ladder:', [(k, on_ladder(k)) for k in range(2, 8)]
```

```
)
        print('Chern number of equatorial waves 2, on ladder:', on_ladder(2))
```

Output:

```
Hadley-Ferrel-polar cells per hemisphere 3, on ladder: True ; both hemispheres 6:
True
QBO 28 months / 12 = 2.333
QBO 29 months / 12 = 2.417
ENSO periods 2..7 years on ladder: [(2, True), (3, True), (4, True), (5, False), (6,
True), (7, False)]
Chern number of equatorial waves 2, on ladder: True
```

8.5 Interglacial intervals in obliquity cycles (LR04 stage starts). Starts of MIS 1, 5, 7, …, 21 from the LR04 table (Marine isotope stages); MIS 1 at 14 ka as listed there. Obliquity period 41 kyr.

```
        import numpy as np
        T = np.array([14, 130, 243, 337, 424, 524, 621, 712, 790, 866])
        iv = np.diff(T); ob = 41.0
        print('intervals (kyr):', iv.tolist())
        print('in obliquity cycles:', np.round(iv/ob, 2).tolist())
        print('nearest integer:', np.round(iv/ob).astype(int).tolist(), '; mean', round(iv.m
ean(), 1), 'kyr =', round(iv.mean()/ob, 2), 'cycles')
        fr = np.abs(iv/ob - np.round(iv/ob)); print('distance to integer:', np.round(fr, 2).
tolist(),
              '; within 0.2:', int((fr <= 0.2).sum()), 'of', len(fr), '(chance 0.4 each)')

        ph = 2*np.pi*(T % ob)/ob; Rbar = abs(np.exp(1j*ph).mean()); n = len(T)
        p = np.exp(-n*Rbar**2)*(1 + (2*n*Rbar**2 - (n*Rbar**2)**2)/(4*n))
        print('Rayleigh test of phase mod 41 kyr: Rbar', round(Rbar, 3), ', n', n, ', p
approx', round(p, 3))
        print('share of 3-cycle gaps for a 100 kyr mean of 2- and 3-cycle gaps:', round(100/
41 - 2, 3))
        for a, b, name in [(405, 100, 'eccentricity 405/100'), (100, 41, '100 kyr /
obliquity'),
                           (41, 23, 'obliquity / precession 23'), (41, 19, 'obliquity /
precession 19')]:
            print(f'{name}: {a/b:.3f}')
```

Output:

```
intervals (kyr): [116, 113, 94, 87, 100, 97, 91, 78, 76]
in obliquity cycles: [2.83, 2.76, 2.29, 2.12, 2.44, 2.37, 2.22, 1.9, 1.85]
nearest integer: [3, 3, 2, 2, 2, 2, 2, 2, 2] ; mean 94.7 kyr = 2.31 cycles
distance to integer: [0.17, 0.24, 0.29, 0.12, 0.44, 0.37, 0.22, 0.1, 0.15] ; within
0.2: 4 of 9 (chance 0.4 each)
Rayleigh test of phase mod 41 kyr: Rbar 0.548 , n 10 , p approx 0.046
share of 3-cycle gaps for a 100 kyr mean of 2- and 3-cycle gaps: 0.439
eccentricity 405/100: 4.050
100 kyr / obliquity: 2.439
obliquity / precession 23: 1.783
obliquity / precession 19: 2.158
```

8.6 Water balance, residence times and stream order. Figures from the hydrological-cycle source in 8.5.

```
        from lad import on_ladder
        E_tot, E_ocean, P_tot, P_ocean = 505000, 434000, 505000, 398000   # km3/yr
        E_land, P_land = E_tot - E_ocean, P_tot - P_ocean
        print('ocean P - E =', P_ocean - E_ocean, '; land P - E =', P_land - E_land, '; sum
=', (P_ocean - E_ocean) + (P_land - E_land))
        print('residence time ratio ocean/atmosphere = 3200 yr / 9 d =', round(3200*365.25/
9))
        print('minimum sources for Strahler order k = 2^(k-1):', [(k, 2**(k-1)) for k in ran
ge(1, 9)])
        print('bifurcation ratios 3, 4, 5 on ladder:', [(r, on_ladder(r)) for r in (3, 4, 5)
])
```

Output:

```
ocean P - E = -36000 ; land P - E = 36000 ; sum = 0
residence time ratio ocean/atmosphere = 3200 yr / 9 d = 129867
minimum sources for Strahler order k = 2^(k-1): [(1, 1), (2, 2), (3, 4), (4, 8), (5,
16), (6, 32), (7, 64), (8, 128)]
bifurcation ratios 3, 4, 5 on ladder: [(3, True), (4, True), (5, False)]
```

8.7 Recent rates.

```
        print('CO2 Sept 2026 - Sept 2025 =', round(426.07 - 424.37, 2), 'ppm')
        print('sea level 1993-2024: 97.8 mm over 31 yr =', round(97.8/31, 2), 'mm/yr mean;
rate 1993 2.0, now 4.4 mm/yr')
```

Output:

```
CO2 Sept 2026 - Sept 2025 = 1.7 ppm
sea level 1993-2024: 97.8 mm over 31 yr = 3.15 mm/yr mean; rate 1993 2.0, now 4.4
mm/yr
```

8.8 Small ladder ratios between two and three years.

```
        from fractions import Fraction as F
        r = sorted({F(2)**a * F(3)**b for a in range(-3, 4) for b in range(-3, 4)})
        cand = [x for x in r if 2 <= x <= 3]
        print('ladder ratios 2^a 3^b in [2, 3], |a|,|b| <= 3:', [(str(x), round(float(x), 3)
, round(12*float(x), 1)) for x in cand])
        for m in (28, 29):
            y = m/12; best = min(cand, key=lambda x: abs(float(x) - y))
            print(f'QBO {m} months = {y:.3f} yr; nearest ladder ratio {best} = {float(best):
.3f} yr ({12*float(best):.2f} months)')
```

Output:

```
ladder ratios 2^a 3^b in [2, 3], |a|,|b| <= 3: [('2', 2.0, 24.0), ('9/4', 2.25,
27.0), ('8/3', 2.667, 32.0), ('3', 3.0, 36.0)]
QBO 28 months = 2.333 yr; nearest ladder ratio 9/4 = 2.250 yr (27.00 months)
QBO 29 months = 2.417 yr; nearest ladder ratio 9/4 = 2.250 yr (27.00 months)
```

8.9 Largest Lyapunov exponent of the Lorenz system.

```
        import numpy as np
        def f(u, s=10.0, r=28.0, b=8/3):
            x, y, z = u; return np.array([s*(y - x), x*(r - z) - y, x*y - b*z])
        def rk4(u, h):
```

```
            k1 = f(u); k2 = f(u + h/2*k1); k3 = f(u + h/2*k2); k4 = f(u + h*k3)
            return u + h/6*(k1 + 2*k2 + 2*k3 + k4)
        h, u = 0.01, np.array([1.0, 1.0, 1.0])
        for _ in range(5000): u = rk4(u, h)            # discard transient
        d0 = 1e-9; v = u + np.array([d0, 0, 0]); acc = 0.0; N = 100000
        for _ in range(N):
            u, v = rk4(u, h), rk4(v, h)
            d = np.linalg.norm(v - u); acc += np.log(d/d0); v = u + (v - u)*d0/d
        print('largest Lyapunov exponent (sigma=10, rho=28, beta=8/3):', round(acc/(N*h), 3)
)
        print('e-folding time of errors:', round(1/(acc/(N*h)), 3), 'time units')
```

Output:

```
largest Lyapunov exponent (sigma=10, rho=28, beta=8/3): 0.904
e-folding time of errors: 1.107 time units
```

8.10 Closed three-year windows: T6 baseline and the effect of persistence.

```
        import numpy as np, itertools
        w = np.exp(2j*np.pi/3)
        def t6(p0, pp, pm):                        # Foundations T6, independent trits

            ph = lambda z: p0 + pp*z + pm*z*z
            return ((1 + ph(w)**3 + ph(w*w)**3)/3).real
        print('T6 uniform law:', round(t6(1/3, 1/3, 1/3), 4), '; dry-biased law (0.3, 0.2,
0.5):', round(t6(0.3, 0.2, 0.5), 4))
        def markov_share(stay):                     # stationary uniform, P(same as last
year) = stay
            P = np.full((3, 3), (1 - stay)/2); np.fill_diagonal(P, stay)
            tot = closed = 0.0
            for a, b, c in itertools.product(range(3), repeat=3):
                pr = (1/3)*P[a, b]*P[b, c]
                s = (a - 1) + (b - 1) + (c - 1)
                tot += pr; closed += pr*(s % 3 == 0)
            return closed/tot
        for stay in (1/3, 0.5, 0.7, 0.9):
            print(f'persistence P(same) = {stay:.3f}: share of closed 3-year windows = {mark
ov_share(stay):.4f}')
```

Output:

```
T6 uniform law: 0.3333 ; dry-biased law (0.3, 0.2, 0.5): 0.34
persistence P(same) = 0.333: share of closed 3-year windows = 0.3333
persistence P(same) = 0.500: share of closed 3-year windows = 0.3750
persistence P(same) = 0.700: share of closed 3-year windows = 0.5350
persistence P(same) = 0.900: share of closed 3-year windows = 0.8150
```

Calculations for Chapter 9 — Building and Making

All scripts were run on 27 October 2026 with Python 3, numpy 2.5 and sympy 1.14. `lad.py` is the ladder helper of Chapter 6.

9.1 Rigidity counts, polygons, triangle strips and $K_{3,3}$.

```
        import numpy as np
        for j in range(3, 9): print(f'joints {j}: planar bars 2j-3 = {2*j-3:
2d} | space bars 3j-6 = {3*j-6:2d}')
```

```
        print('polygon of j bars, internal degrees of freedom j-3:', [(j, j - 3) for j in ra
nge(3, 9)])
        def rigidity_rank(pts, edges):
            M = np.zeros((len(edges), 2*len(pts)))
            for r, (a, b) in enumerate(edges):
                d = pts[a] - pts[b]; M[r, 2*a:2*a+2] = d; M[r, 2*b:2*b+2] = -d
            return np.linalg.matrix_rank(M)
        tri = np.array([[0, 0], [1, 0], [0.3, 0.8]]); quad = np.array([[0, 0], [1, 0], [1.1,
 1], [0, 0.9]])
        for name, P, E in [('triangle', tri, [(0,1),(1,2),(2,0)]), ('quadrilateral', quad,
[(0,1),(1,2),(2,3),(3,0)]),
                           ('quadrilateral + diagonal', quad, [(0,1),(1,2),(2,3),(3,0),(0,2)
])]:
            r = rigidity_rank(P, E); print(f'{name:25s} rank {r}, needed {2*len(P)-3},
internal dof {2*len(P)-3-r}')
        print('strip of n triangles (n, joints, bars, 2j-3):', [(n, n + 2, 2*n + 1, 2*(n +
2) - 3) for n in range(1, 6)])
        for j in (10, 100, 10000):
            print(f'j = {j}: minimal mean connectivity 2D 2(2j-3)/j = {2*(2*j-3)/j:.4f}, 3D
2(3j-6)/j = {2*(3*j-6)/j:.4f}')
        rng = np.random.default_rng(3); P6 = rng.normal(size=(6, 2))
        K33 = [(a, b) for a in range(3) for b in range(3, 6)]
        r = rigidity_rank(P6, K33)
        tri_free = not any((a, b) in K33 and (b, c) in K33 and (a, c) in K33 for a in
range(6) for b in range(6) for c in range(6))
        print(f'K3,3 at generic positions: bars {len(K33)}, rank {r}, needed {2*6-3},
internal dof {2*6-3-r}, contains no triangle: {tri_free}')
```

Output:

```
joints 3: planar bars 2j-3 =  3 | space bars 3j-6 =  3
joints 4: planar bars 2j-3 =  5 | space bars 3j-6 =  6
joints 5: planar bars 2j-3 =  7 | space bars 3j-6 =  9
joints 6: planar bars 2j-3 =  9 | space bars 3j-6 = 12
joints 7: planar bars 2j-3 = 11 | space bars 3j-6 = 15
joints 8: planar bars 2j-3 = 13 | space bars 3j-6 = 18
polygon of j bars, internal degrees of freedom j-3: [(3, 0), (4, 1), (5, 2), (6, 3),
(7, 4), (8, 5)]
triangle                  rank 3, needed 3, internal dof 0
quadrilateral             rank 4, needed 5, internal dof 1
quadrilateral + diagonal  rank 5, needed 5, internal dof 0
strip of n triangles (n, joints, bars, 2j-3): [(1, 3, 3, 3), (2, 4, 5, 5), (3, 5, 7,
7), (4, 6, 9, 9), (5, 7, 11, 11)]
j = 10: minimal mean connectivity 2D 2(2j-3)/j = 3.4000, 3D 2(3j-6)/j = 4.8000
j = 100: minimal mean connectivity 2D 2(2j-3)/j = 3.9400, 3D 2(3j-6)/j = 5.8800
j = 10000: minimal mean connectivity 2D 2(2j-3)/j = 3.9994, 3D 2(3j-6)/j = 5.9988
K3,3 at generic positions: bars 9, rank 9, needed 9, internal dof 0, contains no
triangle: True
```

9.2 Geodesic domes on Eisenstein norms.

```
        from lad import on_ladder
        print('geodesic (icosahedral) dome frequencies T = h^2 + hk + k^2:', sorted({h*h +
h*k + k*k for h in range(5) for k in range(5) if h + k})[:10])
        for T in (1, 3, 4, 7, 9):
            V, E, F = 10*T + 2, 30*T, 20*T
            print(f'T={T}: vertices {V}, edges {E}, triangles {F}, Euler V-E+F = {V - E +
F}, five-valent vertices 12, six-valent {V - 12}')
```

Output:

```
geodesic (icosahedral) dome frequencies T = h^2 + hk + k^2: [1, 3, 4, 7, 9, 12, 13,
16, 19, 21]
T=1: vertices 12, edges 30, triangles 20, Euler V-E+F = 2, five-valent vertices 12,
six-valent 0
T=3: vertices 32, edges 90, triangles 60, Euler V-E+F = 2, five-valent vertices 12,
six-valent 20
T=4: vertices 42, edges 120, triangles 80, Euler V-E+F = 2, five-valent vertices 12,
six-valent 30
T=7: vertices 72, edges 210, triangles 140, Euler V-E+F = 2, five-valent vertices
12, six-valent 60
T=9: vertices 92, edges 270, triangles 180, Euler V-E+F = 2, five-valent vertices
12, six-valent 80
```

9.3 Calcination CO_2 and Brevik.

```
        M = {'Ca': 40.078, 'C': 12.011, 'O': 15.999}
        CaO, CO2 = M['Ca'] + M['O'], M['C'] + 2*M['O']
        print('CO2 released per t CaO by calcination:', round(CO2/CaO, 4), 't')
        print('Brevik: 400,000 t/yr = 50 % of plant emissions -> plant emissions', int(40000
0/0.5), 't/yr')
```

Output:

```
CO2 released per t CaO by calcination: 0.7848 t
Brevik: 400,000 t/yr = 50 % of plant emissions -> plant emissions 800000 t/yr
```

9.4 Exact constraint, stools, planetary assembly and the Wankel cycle.

```
        import numpy as np
        print('Maxwell coupling: 3 grooves x 2 contacts =', 3*2,
'| Kelvin coupling: 3 + 2 + 1 =', 3 + 2 + 1, '| rigid-body dof = 6')
        rng = np.random.default_rng(0); P = rng.normal(size=(4, 3))
        print('four random feet: coplanarity determinant =', round(np.linalg.det(np.c_[P,
np.ones(4)]), 4), '(nonzero: a four-legged stool rocks)')
        cnt = {2: 0, 3: 0, 4: 0, 5: 0}; tot = 0
        for Ns in range(12, 61):
            for Np in range(12, 61):
                Nr = Ns + 2*Np; tot += 1
# ring teeth for coaxial sun and ring with equal module
                for n in cnt: cnt[n] += (Ns + Nr) % n == 0
        print('share of sun/planet tooth pairs (12..60) allowing n equally spaced
planets:', {n: round(c/tot, 3) for n, c in cnt.items()})
        print('Wankel: rotor turns once per 3 shaft turns; 3 faces x 1 power stroke each per
rotor turn = 1 per shaft turn')
        # sun-ring meshing repeats every 360/(Ns+Nr) degrees of carrier rotation, so n equal
planets need (Ns+Nr) % n == 0
        Ns, Np = 24, 18; Nr = Ns + 2*Np
        print(f'example Ns={Ns}, Np={Np}, Nr={Nr}: Ns+Nr={Ns+Nr}, divisible by 3: {(Ns+Nr)
% 3 == 0}; Ns+Np mod 3 = {(Ns+Np) % 3}')
        print('for n = 3: Ns + Nr = 2(Ns + Np), so the condition is Ns + Np = 0 mod 3:',

              all(((Ns + Ns + 2*Np) % 3 == 0) == ((Ns + Np) % 3 == 0) for Ns in range(10,
80) for Np in range(10, 80)))
```

Output:

```
Maxwell coupling: 3 grooves x 2 contacts = 6 | Kelvin coupling: 3 + 2 + 1 = 6 |
rigid-body dof = 6
four random feet: coplanarity determinant = 1.2695 (nonzero: a four-legged stool
rocks)
share of sun/planet tooth pairs (12..60) allowing n equally spaced planets: {2: 1.0,
3: 0.334, 4: 0.5, 5: 0.2}
Wankel: rotor turns once per 3 shaft turns; 3 faces x 1 power stroke each per rotor
turn = 1 per shaft turn
example Ns=24, Np=18, Nr=60: Ns+Nr=84, divisible by 3: True; Ns+Np mod 3 = 0
for n = 3: Ns + Nr = 2(Ns + Np), so the condition is Ns + Np = 0 mod 3: True
```

9.5 Gear ratios for M2/S2: convergents and smooth approximations.

```
        import sympy as sp
        r = sp.Rational(124206, 120000)          # M2 / S2 = 12.4206 h / 12 h
        terms = sp.continued_fraction(r)
        def smooth(n):
            n = abs(int(n))
            for q in (2, 3, 5, 7):
                while n % q == 0: n //= q
            return n == 1
        print('M2/S2 =', float(r), '; continued fraction', terms)
        for k in range(1, len(terms) + 1):
            c = sp.Rational(sp.continued_fraction_reduce(terms[:k]))
            print(f'{str(c):>14s}  rel. error {float(abs(c - r)/r):.2e}  7-smooth numerator/
denominator: {smooth(c.p)}/{smooth(c.q)}')
        best_any = min(((abs(sp.Rational(p, q) - r), p, q) for q in range(1, 201) for p in r
ange(q, 2*q) ), key=lambda t: t[0])
        best_s = min(((abs(sp.Rational(p, q) - r), p, q) for q in range(1, 201) for p in ran
ge(q, 2*q) if smooth(p) and smooth(q)), key=lambda t: t[0])
        best_l = min(((abs(sp.Rational(p, q) - r), p, q) for q in range(1, 201) for p in ran
ge(q, 2*q)
                      if smooth(p) and smooth(q) and all(x in (2, 3) for x in
sp.factorint(p*q))), key=lambda t: t[0])
        for name, (e, p, q) in [('best p/q, q<=200', best_any), ('best 7-smooth (Brocot
gears)', best_s), ('best 3-smooth (ladder)', best_l)]:
            print(f'{name:30s} {p}/{q} = {p/q:.6f}  rel. error {float(e/r):.2e}  factors
{sp.factorint(p)} / {sp.factorint(q)}')
```

Output:

```
M2/S2 = 1.03505 ; continued fraction [1, 28, 1, 1, 7, 1, 1, 1, 6, 2]
             1  rel. error 3.39e-02  7-smooth numerator/denominator: True/True
         29/28  rel. error 6.42e-04  7-smooth numerator/denominator: False/True
         30/29  rel. error 5.48e-04  7-smooth numerator/denominator: True/False
         59/57  rel. error 3.64e-05  7-smooth numerator/denominator: False/False
       443/428  rel. error 3.16e-06  7-smooth numerator/denominator: False/False
       502/485  rel. error 1.49e-06  7-smooth numerator/denominator: False/False
       945/913  rel. error 6.88e-07  7-smooth numerator/denominator: True/False
     1447/1398  rel. error 6.91e-08  7-smooth numerator/denominator: False/False
     9627/9301  rel. error 5.19e-09  7-smooth numerator/denominator: False/False
   20701/20000  rel. error 0.00e+00  7-smooth numerator/denominator: False/True
best p/q, q<=200               59/57 = 1.035088  rel. error 3.64e-05  factors {59:
1} / {3: 1, 19: 1}
best 7-smooth (Brocot gears)   28/27 = 1.037037  rel. error 1.92e-03  factors {2: 2,
7: 1} / {3: 3}
best 3-smooth (ladder)         1/1 = 1.000000  rel. error 3.39e-02  factors {} / {}
```

9.6 Electric cracker and Haber recycle.

```
        print('e-cracker: 6 MW / 4 t/h =', 6/4, 'MWh per tonne of feed')
        print('Haber recycle, 15 % per pass: passes to 97 % =', next(n for n in range(1,
100) if 1 - 0.85**n >= 0.97))
        print('ternary diagram: x_A + x_B + x_C = 1, two free coordinates; vertices 3, edges
(binaries) 3')
        print('hydrogen: low-emission ~1 Mt of ~100 Mt demand =', round(1/100*100, 1), '%')
```

Output:

```
e-cracker: 6 MW / 4 t/h = 1.5 MWh per tonne of feed
Haber recycle, 15 % per pass: passes to 97 % = 22
ternary diagram: x_A + x_B + x_C = 1, two free coordinates; vertices 3, edges
(binaries) 3
hydrogen: low-emission ~1 Mt of ~100 Mt demand = 1.0 %
```

9.7 Nitrogen oxidation states and the PFAS limit.

```
        from lad import on_ladder
        print('oxidation states of N: NH4+ -3, N2 0, NO2- +3, NO3- +5')
        print('anammox NH4+ + NO2- -> N2 + 2 H2O: (-3) + (+3) =', -3 + 3, '-> a meeting,
carry 0')
        print('electrons: nitritation NH4+->NO2-', 3 - (-3), '| nitratation NO2-->NO3-', 5
- 3, '| denitrification NO3- -> 1/2 N2', 5 - 0)
        print('on ladder:', [(e, on_ladder(e)) for e in (6, 2, 5)])
        print('PFOA and PFOS limit 4.0 ppt = 4.0 ng/L; that is 1 part in', int(1/(4.0e-12)),
 'by mass in water')
```

Output:

```
oxidation states of N: NH4+ -3, N2 0, NO2- +3, NO3- +5
anammox NH4+ + NO2- -> N2 + 2 H2O: (-3) + (+3) = 0 -> a meeting, carry 0
electrons: nitritation NH4+->NO2- 6 | nitratation NO2-->NO3- 2 | denitrification
NO3- -> 1/2 N2 5
on ladder: [(6, True), (2, True), (5, False)]
PFOA and PFOS limit 4.0 ppt = 4.0 ng/L; that is 1 part in 250000000000 by mass in
water
```

9.8 Order of periodic windows in the logistic map.

```
        def period(r, n=20000, tol=1e-7):
            x = 0.5
            for _ in range(n): x = r*x*(1 - x)
            x0 = x
            for p in range(1, 33):
                x = r*x*(1 - x)
                if abs(x - x0) < tol: return p
            return None
        for r in (3.6275, 3.7017, 3.7389, 3.8285):
            print(f'r = {r}: period {period(r)}')
```

Output:

```
r = 3.6275: period 6
r = 3.7017: period 7
r = 3.7389: period 5
r = 3.8285: period 3
```

9.9 Lever of recycle pass counts (T24); joining two rigid bodies.

```
        import numpy as np
        # passes of a molecule through a reactor with recycle: after each pass it reacts (p
= 0.15) or returns (P = 0.85)
        P = 0.85
        print('return share P =', P, '; mean passes 1/(1-P) =', round(1/(1 - P), 4), '; T24
lever -(1+P)/P =', round(-(1 + P)/P, 4))
        rng = np.random.default_rng(7)
        V = rng.geometric(1 - P, size=200000)          # number of passes, first included

        m, k2 = V.mean(), V.var(); k3 = ((V - m)**3).mean()
        print('simulated 200000 molecules: mean', round(m, 3), ' lever', round(-m*k3/k2**2,
4))
        # a plant with a dead zone: 20 % of molecules return with P = 0.95, 80 % with P =
0.8 (not memoryless overall)
        V2 = np.where(rng.random(200000) < 0.2, rng.geometric(0.05, 200000), rng.geometric(0
.2, 200000))
        m, k2 = V2.mean(), V2.var(); k3 = ((V2 - m)**3).mean(); Pm = 1 - 1/m
        print('mixture plant: mean', round(m, 3), ' P from mean', round(Pm, 4), ' lever', ro
und(-m*k3/k2**2, 4), ' -(1+P)/P', round(-(1 + Pm)/Pm, 4))
        print('two rigid plane bodies: relative dof 3 -> bars needed to join them: 3')
```

Output:

```
return share P = 0.85 ; mean passes 1/(1-P) = 6.6667 ; T24 lever -(1+P)/P = -2.1765
simulated 200000 molecules: mean 6.663  lever -2.1712
mixture plant: mean 8.051  P from mean 0.8758  lever -2.8708  -(1+P)/P -2.1418
two rigid plane bodies: relative dof 3 -> bars needed to join them: 3
```

Calculations for Chapter 10 — The Code and the Cell

Calculation 10.1 — Codon length, codon rotation classes and the comma-free bound (T13).

```
        # 10.1 Codon length, codon rotation classes (T13 with alphabet 4), comma-free bound

        from itertools import product
        L=1
        while 4**L < 21: L+=1
        print("least L with 4^L >= 21:", L, "(4^2 =",4**2,", 4^3 =",4**3,")")
        words=[''.join(p) for p in product('ACGU',repeat=3)]
        classes={min(w[i:]+w[:i] for i in range(3)) for w in words}
        const=[c for c in classes if len(set(c))==1]
        print("codons:",len(words)," rotation classes g_4(3):",len(classes)," constant
classes:",len(const)," aperiodic classes:",len(classes)-len(const))
        print("necklace formula (1/3)(4^3+2*4) =",(4**3+2*4)//3, "; (64-4)/3 =",(64-4)//3)
```

Output:

```
least L with 4^L >= 21: 3 (4^2 = 16 , 4^3 = 64 )
codons: 64  rotation classes g_4(3): 24  constant classes: 4  aperiodic classes: 20
necklace formula (1/3)(4^3+2*4) = 24 ; (64-4)/3 = 20
```

Calculation 10.2 — Frameshift combinations: digit, carry and decomposition (T1, T7, T7.1).

```
        # 10.2 Frameshift combinations: residue (digit) and carry (T1, T7, T7.1)
        from itertools import product
        def ladder(n):
            """return (a,b) if n = 2^a 3^b, else None"""
            if n != int(n) or n < 1: return None
            n=int(n); a=b=0
            while n%2==0: n//=2; a+=1
            while n%3==0: n//=3; b+=1
            return (a,b) if n==1 else None
        def digit(s):
            d=((s+1)%3)-1
            return d,(s-d)//3
        def reduce(w):
            w=[x for x in w if x!=0]; steps=[]
            while w:
                for i in range(len(w)-1):
                    if w[i]==-w[i+1]:
                        steps.append('meeting'); del w[i:i+2]; break
                else:
                    if len(w)>=3 and len(set(w[:3]))==1:
                        steps.append('ring'); del w[:3]
                    else: return None
            return steps
        for n in range(1,5):
            rest=0;tot=0
            for w in product([1,-1],repeat=n):
                s=sum(w); d,c=digit(s); tot+=1
                if d==0: rest+=1
            print(f"n={n}: {tot} combinations of +1/-1 shifts, {rest} restore the frame")

        for w in [(1,),(1,-1),(1,1),(1,1,1),(-1,-1,-1),(1,1,-1),(1,1,1,1,-1)]:
            s=sum(w); d,c=digit(s)
            print(w,"sum",s,"digit",d,"carry",c,"(codons added)","decomposition:",reduce(lis
t(w)) if d==0 else "open")
```

Output:

```
n=1: 2 combinations of +1/-1 shifts, 0 restore the frame
n=2: 4 combinations of +1/-1 shifts, 2 restore the frame
n=3: 8 combinations of +1/-1 shifts, 2 restore the frame
n=4: 16 combinations of +1/-1 shifts, 6 restore the frame
(1,) sum 1 digit 1 carry 0 (codons added) decomposition: open
(1, -1) sum 0 digit 0 carry 0 (codons added) decomposition: ['meeting']
(1, 1) sum 2 digit -1 carry 1 (codons added) decomposition: open
(1, 1, 1) sum 3 digit 0 carry 1 (codons added) decomposition: ['ring']
(-1, -1, -1) sum -3 digit 0 carry -1 (codons added) decomposition: ['ring']
(1, 1, -1) sum 1 digit 1 carry 0 (codons added) decomposition: open
(1, 1, 1, 1, -1) sum 3 digit 0 carry 1 (codons added) decomposition: ['meeting',
'ring']
```

Calculation 10.3 — Period-3 signal in coding and non-coding DNA of the Arabidopsis chloroplast genome. The GenBank records NC_000932 (*Arabidopsis thaliana* chloroplast, 154,478 bp) and NC_001422 (phage ΦX174, 5,386 nt) used in Calculations 10.3, 10.10 and 10.11 are the NCBI RefSeq files distributed with the Biopython 1.84 source package (Tests/GenBank and Tests/GFF).

```
        # 10.3 Period-3 signal in the Arabidopsis thaliana chloroplast genome (NC_000932):
coding vs non-coding
```

```
        import numpy as np
        from Bio import SeqIO
        rec=SeqIO.read("NC_000932.gb","genbank"); seq=str(rec.seq).upper(); N=len(seq)

        cds=[f for f in rec.features if f.type=="CDS"]
        coding=np.zeros(N,bool)
        for f in rec.features:
            if f.type in ("CDS","gene","tRNA","rRNA"):
                for p in f.location.parts: coding[int(p.start):int(p.end)]=True
        def p3ratio(s):
            # Fourier power at f=1/3 of the four base indicator sequences, relative to mean
power
            L=len(s)-len(s)%3
            if L<300: return None
            tot=0; mean=0
            for b in "ACGT":
                x=np.array([c==b for c in s[:L]],float); x-=x.mean()
                F=np.abs(np.fft.rfft(x))**2
                tot+=F[L//3]; mean+=F[1:].mean()
            return tot/mean
        cr=[]
        for f in cds:
            if len(f.location.parts)==1:
                s=str(f.extract(rec.seq)).upper()
                r=p3ratio(s)
                if r: cr.append(r)
        # non-coding stretches >= 300 nt
        nc=[]; i=0
        while i<N:
            if not coding[i]:
                j=i
                while j<N and not coding[j]: j+=1
                if j-i>=300: nc.append(p3ratio(seq[i:j]))
                i=j
            else: i+=1
        print("genome length",N," single-exon CDS >=300 nt:",len(cr)," non-coding stretches
>=300 nt:",len(nc))
        print("median S(1/3) coding: %.2f   non-coding: %.2f"%(np.median(cr),np.median(nc)))

        print("share with S(1/3)>4: coding %.2f  non-coding %.2f"%(np.mean(np.array(cr)>4),n
p.mean(np.array(nc)>4)))
```

Output:

```
genome length 154478  single-exon CDS >=300 nt: 45  non-coding stretches >=300 nt:
52
median S(1/3) coding: 7.26   non-coding: 0.85
share with S(1/3)>4: coding 0.73  non-coding 0.00
```

Calculation 10.4 — Mendel's counts 3^n, 4^n, 2^n; his first two experiments; the F2 law as a trit law (T6).

```
        # 10.4 phiX174 (NC_001422): overlapping genes and their relative reading frame
(trit)
        from Bio import SeqIO
        rec=SeqIO.read("NC_001422.gbk","genbank")
        names={3980:"A",4496:"A*",5074:"B",50:"K",132:"C",389:"D",567:"E",847:"J"}
        genes=[]
        for f in rec.features:
            if f.type=="CDS":
```

```
                start=int(f.location.parts[0].start)
                if start not in names: continue
                phase={}; k=0
                for p in f.location.parts:
                    for i in range(int(p.start),int(p.end)):
                        phase[i]=k%3; k+=1
                genes.append((names[start],phase))
        print("pair   overlap(nt)  frame difference (phase2-phase1 mod 3 as trit)")
        for i,(n1,ph1) in enumerate(genes):
            for n2,ph2 in genes[i+1:]:
                ov=set(ph1)&set(ph2)
                if not ov: continue
                diffs={(ph2[x]-ph1[x])%3 for x in ov}
                assert len(diffs)==1
                d=diffs.pop(); t=((d+1)%3)-1
                print(f"{n1:>2}/{n2:<2}  {len(ov):5d}       {t:+d}")
```

Output:

```
pair   overlap(nt)  frame difference (phase2-phase1 mod 3 as trit)
 A/A*   1026       +0
 A/B     363       +1
 A/K      86       -1
 A/C       4       +1
A*/B     363       +1
A*/K      86       -1
A*/C       4       +1
 B/K       1       +1
 K/C      89       -1
 C/D       4       +1
 D/E     276       -1
 D/J       1       +1
```

Calculation 10.5 — In-frame share of coding indels against the closure baseline 1/3 (T6).

```
        # 10.5 Capsid T-numbers as norms in Z[omega]; T14 digit orbit of z = h - k*omega

        def ladder(n):
            """return (a,b) if n = 2^a 3^b, else None"""
            if n != int(n) or n < 1: return None
            n=int(n); a=b=0
            while n%2==0: n//=2; a+=1
            while n%3==0: n//=3; b+=1
            return (a,b) if n==1 else None
        def digit(s):
            d=((s+1)%3)-1
            return d,(s-d)//3
        def norm(a,b): return a*a-a*b+b*b          # N(a+b w)
        def T(a,b):
            d=((a+b+1)%3)-1
            a2,b2=a-d,b
            # (a2+b2 w)/(1-w) = (a2+b2 w)(2+w)/3 = ((2a2-b2) + (a2+b2) w)/3
            x,y=2*a2-b2,a2+b2
            assert x%3==0 and y%3==0
            return x//3,y//3
        def orbit_end(a,b):
            seen=[]
            while (a,b) not in seen:
                seen.append((a,b))
```

```
                if (a,b)==(0,0): return "rest"
                a,b=T(a,b)
            return "turn" if (a,b) in [(1,1),(-1,-1)] else "cycle"+str((a,b))
        Ts=sorted({h*h+h*k+k*k for h in range(0,20) for k in range(0,20) if h+k>0})
        print("allowed T up to 50:",[t for t in Ts if t<=50])
        print("all equal Eisenstein norms <=50:",sorted({norm(a,b) for a in range(-12,13) fo
r b in range(-12,13)} - {0}) == [t for t in Ts if t<=50] or None)
        print("norms<=50 equal list:", sorted({norm(a,b) for a in range(-12,13) for b in ran
ge(-12,13) if 0<norm(a,b)<=50})==[t for t in Ts if t<=50])
        caps=[(1,0,"T=1"),(1,1,"T=3"),(2,0,"T=4"),(2,1,"T=7"),(3,0,"T=9"),(3,1,"T=13"),(4,
0,"T=16"),(4,1,"T=21"),(5,0,"T=25"),(3,3,"T=27"),(5,1,"T=31"),(7,7,"T=147 CIV"),(7,
8,"T=169 PBCV-1/FV3"),(7,10,"T=219 PpV01")]
        for h,k,lab in caps:
            t=h*h+h*k+k*k
            print(f"{lab:18s} (h,k)=({h},{k}) T={t:3d} norm(h,-k)={norm(h,-k):3d} subunits
60T={60*t:6d} hexamers 10(T-1)={10*(t-1):5d} ladder(T)={ladder(t)} T mod 3={t%3}
T14 orbit of h-k*w: {orbit_end(h,-k)}")
```

Output:

```
allowed T up to 50: [1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31, 36, 37, 39,
43, 48, 49]
all equal Eisenstein norms <=50: None
norms<=50 equal list: True
T=1                (h,k)=(1,0) T=  1 norm(h,-k)=  1 subunits 60T=    60 hexamers
10(T-1)=    0 ladder(T)=(0, 0) T mod 3=1  T14 orbit of h-k*w: rest
T=3                (h,k)=(1,1) T=  3 norm(h,-k)=  3 subunits 60T=   180 hexamers
10(T-1)=   20 ladder(T)=(0, 1) T mod 3=0  T14 orbit of h-k*w: rest
T=4                (h,k)=(2,0) T=  4 norm(h,-k)=  4 subunits 60T=   240 hexamers
10(T-1)=   30 ladder(T)=(2, 0) T mod 3=1  T14 orbit of h-k*w: turn
T=7                (h,k)=(2,1) T=  7 norm(h,-k)=  7 subunits 60T=   420 hexamers
10(T-1)=   60 ladder(T)=None T mod 3=1  T14 orbit of h-k*w: rest
T=9                (h,k)=(3,0) T=  9 norm(h,-k)=  9 subunits 60T=   540 hexamers
10(T-1)=   80 ladder(T)=(0, 2) T mod 3=0  T14 orbit of h-k*w: turn
T=13               (h,k)=(3,1) T= 13 norm(h,-k)= 13 subunits 60T=   780 hexamers
10(T-1)=  120 ladder(T)=None T mod 3=1  T14 orbit of h-k*w: turn
T=16               (h,k)=(4,0) T= 16 norm(h,-k)= 16 subunits 60T=   960 hexamers
10(T-1)=  150 ladder(T)=(4, 0) T mod 3=1  T14 orbit of h-k*w: turn
T=21               (h,k)=(4,1) T= 21 norm(h,-k)= 21 subunits 60T=  1260 hexamers
10(T-1)=  200 ladder(T)=None T mod 3=0  T14 orbit of h-k*w: turn
T=25               (h,k)=(5,0) T= 25 norm(h,-k)= 25 subunits 60T=  1500 hexamers
10(T-1)=  240 ladder(T)=None T mod 3=1  T14 orbit of h-k*w: rest
T=27               (h,k)=(3,3) T= 27 norm(h,-k)= 27 subunits 60T=  1620 hexamers
10(T-1)=  260 ladder(T)=(0, 3) T mod 3=0  T14 orbit of h-k*w: turn
T=31               (h,k)=(5,1) T= 31 norm(h,-k)= 31 subunits 60T=  1860 hexamers
10(T-1)=  300 ladder(T)=None T mod 3=1  T14 orbit of h-k*w: turn
T=147 CIV          (h,k)=(7,7) T=147 norm(h,-k)=147 subunits 60T=  8820 hexamers
10(T-1)= 1460 ladder(T)=None T mod 3=0  T14 orbit of h-k*w: rest
T=169 PBCV-1/FV3   (h,k)=(7,8) T=169 norm(h,-k)=169 subunits 60T= 10140 hexamers
10(T-1)= 1680 ladder(T)=None T mod 3=1  T14 orbit of h-k*w: rest
T=219 PpV01        (h,k)=(7,10) T=219 norm(h,-k)=219 subunits 60T= 13140 hexamers
10(T-1)= 2180 ladder(T)=None T mod 3=0  T14 orbit of h-k*w: rest
```

Calculation 10.6 — Rings of n repressors in the Elowitz–Leibler model: odd rings oscillate, even rings settle.

```
        # Is the T14 rest/turn label a property of a capsid lattice?
        # The 12 associates (6 units x conjugation) of z = h - k*omega describe the same
(h,k) lattice.
        def T(a,b):                       # T14 digit map on a + b*omega, beta = 1 - omega
```

```
            d=((a+b+1)%3)-1
            x,y=2*(a-d)-b,(a-d)+b
            return x//3,y//3
        def orbit_end(a,b):
            seen=set()
            while (a,b) not in seen:
                seen.add((a,b))
                if (a,b)==(0,0): return "rest"
                a,b=T(a,b)
            return "turn"
        def mul(a,b,c,d): return a*c-b*d, a*d+b*c-b*d     # omega^2 = -1 - omega
        units=[(1,0),(0,1),(-1,-1),(-1,0),(0,-1),(1,1)]
        def conj(a,b): return a-b,-b
        for h,k in [(1,1),(2,0),(2,1),(3,1),(7,7),(7,8),(7,10)]:
            labels=[orbit_end(*mul(*zz,*u)) for u in units for zz in ((h,-k),conj(h,-k))]

            print(f"(h,k)=({h},{k}) T={h*h+h*k+k*k}: rest {labels.count('rest')}, turn {labe
ls.count('turn')} of 12 associates")
```

Output:

```
(h,k)=(1,1) T=3: rest 8, turn 4 of 12 associates
(h,k)=(2,0) T=4: rest 8, turn 4 of 12 associates
(h,k)=(2,1) T=7: rest 8, turn 4 of 12 associates
(h,k)=(3,1) T=13: rest 10, turn 2 of 12 associates
(h,k)=(7,7) T=147: rest 8, turn 4 of 12 associates
(h,k)=(7,8) T=169: rest 6, turn 6 of 12 associates
(h,k)=(7,10) T=219: rest 10, turn 2 of 12 associates
```

Calculation 10.7 — Helix repeats, ring counts and lattice seams against the ladder (T15) and the window (T4).

```
        # 10.7 Helices and rings of structural biology on the ladder 2^a 3^b (T15); seam
parity
        from fractions import Fraction as F
        def ladder(n):
            """return (a,b) if n = 2^a 3^b, else None"""
            if n != int(n) or n < 1: return None
            n=int(n); a=b=0
            while n%2==0: n//=2; a+=1
            while n%3==0: n//=3; b+=1
            return (a,b) if n==1 else None
        def digit(s):
            d=((s+1)%3)-1
            return d,(s-d)//3
        items={"alpha helix residues/turn":F(36,10),"B-DNA bp/turn (solution)":F(105,10)}

        for k,v in items.items():
            print(f"{k}: {v} = {v.numerator}/{v.denominator}; repeat {v.numerator} units in
{v.denominator} turns; ladder(numerator)={ladder(v.numerator)}")
        for name,n in [("protofilaments (common)",13),("protofilaments (in vitro)",14),("cen
triole triplets",9),("axoneme doublets",9),("A-tubule PF",13),("B-tubule PF",10),("c
ollagen chains",3),("Gly period",3)]:
            print(f"{name}: {n} ladder={ladder(n)}")
        # microtubule lattice: N protofilaments, S-start helix in monomer units; seam iff S
odd (Sui & Downing 2010)
        for N,S in [(13,3),(14,3),(12,3),(11,3),(15,4),(16,4),(14,4),(15,3)]:
            print(f"{N}_{S}: offset S={S} monomers -> {'seam (alpha meets beta)' if S%2
else 'no seam'}; S mod 2 = {S%2}")
```

```
        # collagen: three chains staggered by one residue; phase of the Gly-X-Y repeat in
each chain at one level
        phases=[0,1,2]; print("collagen phases",phases,"sum mod 3 =",sum(phases)%3,"(heterog
eneous closed window, T4)")
```

Output:

```
alpha helix residues/turn: 18/5 = 18/5; repeat 18 units in 5 turns;
ladder(numerator)=(1, 2)
B-DNA bp/turn (solution): 21/2 = 21/2; repeat 21 units in 2 turns;
ladder(numerator)=None
protofilaments (common): 13 ladder=None
protofilaments (in vitro): 14 ladder=None
centriole triplets: 9 ladder=(0, 2)
axoneme doublets: 9 ladder=(0, 2)
A-tubule PF: 13 ladder=None
B-tubule PF: 10 ladder=None
collagen chains: 3 ladder=(0, 1)
Gly period: 3 ladder=(0, 1)
13_3: offset S=3 monomers -> seam (alpha meets beta); S mod 2 = 1
14_3: offset S=3 monomers -> seam (alpha meets beta); S mod 2 = 1
12_3: offset S=3 monomers -> seam (alpha meets beta); S mod 2 = 1
11_3: offset S=3 monomers -> seam (alpha meets beta); S mod 2 = 1
15_4: offset S=4 monomers -> no seam; S mod 2 = 0
16_4: offset S=4 monomers -> no seam; S mod 2 = 0
14_4: offset S=4 monomers -> no seam; S mod 2 = 0
15_3: offset S=3 monomers -> seam (alpha meets beta); S mod 2 = 1
collagen phases [0, 1, 2] sum mod 3 = 0 (heterogeneous closed window, T4)
```

Calculation 10.8 — Capsid triangulation numbers as norms in Z[ω]; T14 digit orbits; ladder check.

```
        # 10.8 Frame closure in indel and rearrangement data against the T6 baseline of 1/3

        from scipy.stats import binomtest
        from fractions import Fraction as F
        # T6 with uniform law: closure probability 1/3 ; indel lengths 1..30 keeping frame

        print("indel lengths 1..30 that are 0 mod 3:",sum(1 for k in range(1,31) if
k%3==0),"of 30")
        nfs,fs=3362,3775; n=nfs+fs
        r=binomtest(nfs,n,1/3)
        print(f"Lin et al. 2017 coding indels: {nfs}/{n} = {nfs/n:.4f} in frame; baseline
1/3; binomial p = {r.pvalue:.3g}")
        # V(D)J: two independent junctions each in frame w.p. 1/3? textbook: one in three
rearrangements in frame
        print("one rearrangement in frame:",F(1,3)," at least one of two alleles in frame:",
1-F(2,3)**2)
        # DMD reading-frame rule: ~90% consistent (exceptions ~10%)
```

Output:

```
indel lengths 1..30 that are 0 mod 3: 10 of 30
Lin et al. 2017 coding indels: 3362/7137 = 0.4711 in frame; baseline 1/3; binomial p
= 9.53e-128
one rearrangement in frame: 1/3  at least one of two alleles in frame: 5/9
```

Calculation 10.9 — Is the T14 rest/turn label a property of a capsid lattice?.

```
        # 10.9 Mendel: 3^n genotype classes, 4^n gamete combinations, 2^n constant forms;
counts of Exp. 1 and 2
        from scipy.stats import chisquare
        for n in range(1,5): print(f"n={n}: 3^n={3**n}  4^n={4**n}  2^n={2**n}")
        for lab,a,b in [("Exp.1 round:wrinkled",5474,1850),("Exp.2 yellow:green",6022,2001)]
:
            t=a+b; c=chisquare([a,b],[0.75*t,0.25*t])
            print(f"{lab}: ratio {a/b:.3f}, chi2={c.statistic:.3f}, p={c.pvalue:.3f}")

        # genotype as trit: AA=+1, Aa=0, aa=-1 (Falconer's +a, d, -a with d=0 means
additivity)
        import itertools
        g={("A","A"):1,("A","a"):0,("a","A"):0,("a","a"):-1}
        cnt={1:0,0:0,-1:0}
        for x in itertools.product("Aa",repeat=2): cnt[g[x]]+=1
        print("F2 trit law (AA,Aa,aa):",cnt[1],cnt[0],cnt[-1],"of 4")
        # T6: three unlinked loci of an F2 individual: probability its additive score sum is
0 mod 3
        import cmath
        p0,pp,pm=0.5,0.25,0.25
        w=cmath.exp(2j*cmath.pi/3)
        ph=lambda z:p0+pp*z+pm*z*z
        P=(1+ph(w)**3+ph(w*w)**3).real/3
        # brute force
        from itertools import product
        Pb=sum((0.5 if a==0 else .25)*(0.5 if b==0 else .25)*(0.5 if c==0 else .25) for
a,b,c in product([-1,0,1],repeat=3) if (a+b+c)%3==0)
        print(f"T6 closure probability for F2 law (1/2,1/4,1/4): formula {P:.6f},
enumeration {Pb:.6f}")
```

Output:

```
n=1: 3^n=3  4^n=4  2^n=2
n=2: 3^n=9  4^n=16  2^n=4
n=3: 3^n=27  4^n=64  2^n=8
n=4: 3^n=81  4^n=256  2^n=16
Exp.1 round:wrinkled: ratio 2.959, chi2=0.263, p=0.608
Exp.2 yellow:green: ratio 3.009, chi2=0.015, p=0.903
F2 trit law (AA,Aa,aa): 1 2 1 of 4
T6 closure probability for F2 law (1/2,1/4,1/4): formula 0.343750, enumeration
0.343750
```

Calculation 10.10 — Overlapping genes of phage ΦX174 and their relative reading frames.

```
        # 10.10 Rings of n repressors, Elowitz-Leibler (2000) mRNA+protein model
        # dm_i/dt = -m_i + alpha/(1+p_{i-1}^h) + alpha0 ; dp_i/dt = -beta (p_i - m_i)

        # parameters: alpha=216, alpha0=0.216, beta=5, h=2 (values used in Elowitz &
Leibler's Box 1 example)
        import numpy as np
        from scipy.integrate import solve_ivp
        def run(n,alpha=216.0,alpha0=0.216,beta=5.0,h=2.0,T=600):
            def f(t,y):
                m,p=y[:n],y[n:]
                dm=[-m[i]+alpha/(1+p[i-1]**h)+alpha0 for i in range(n)]
                dp=[-beta*(p[i]-m[i]) for i in range(n)]
                return dm+dp
            rng=np.random.default_rng(1); y0=rng.uniform(0,20,2*n)
```

```
            s=solve_ivp(f,(0,T),y0,max_step=0.05,rtol=1e-8,atol=1e-8)
            tail=s.y[n][s.t>T*0.75]
            return tail.max()-tail.min(), s.y[n:,-1]
        for n in range(2,9):
            amp,end=run(n)
            print(f"n={n}: late amplitude of protein 1 = {amp:8.2f} -> {'oscillates' if
amp>1 else 'settles'}; final proteins {np.round(end,1)}")
```

Output:

```
n=2: late amplitude of protein 1 =     0.00 -> settles; final proteins [2.000e-01
2.062e+02]
n=3: late amplitude of protein 1 =    57.96 -> oscillates; final proteins [46.2
3.2  5.4]
n=4: late amplitude of protein 1 =     0.00 -> settles; final proteins [2.000e-01
2.062e+02 2.000e-01 2.062e+02]
n=5: late amplitude of protein 1 =   192.41 -> oscillates; final proteins [191.
0.9  46.7  26.5   0.3]
n=6: late amplitude of protein 1 =     0.00 -> settles; final proteins [2.000e-01
2.062e+02 2.000e-01 2.062e+02 2.000e-01 2.062e+02]
n=7: late amplitude of protein 1 =   204.63 -> oscillates; final proteins [3.500e+00
4.500e+00 1.133e+02 2.000e-01 2.040e+02 3.000e-01 1.602e+02]
n=8: late amplitude of protein 1 =     0.00 -> settles; final proteins [2.000e-01
2.062e+02 2.000e-01 2.062e+02 2.000e-01 2.062e+02 2.000e-01
 2.062e+02]
```

Calculation 10.11 — Genome lengths mod 3; necklace counts g(n) (T13).

```
        # 10.12 Checklist remainder: circular genomes (T9 reading), trit necklaces g(3)
(T13), knots (T19/T20 counts)
        from Bio import SeqIO
        from math import gcd
        from sympy import totient, divisors
        for f in ["NC_000932.gb","NC_001422.gbk"]:
            r=SeqIO.read(f,"genbank"); L=len(r.seq)
            print(f"{r.id}: length {L}, length mod 3 = {L%3}")
        g=lambda n,k=3: sum(totient(n//d)*k**d for d in divisors(n))//n
        print("g(n) trits n=1..6:",[g(n) for n in range(1,7)]," codon necklaces over 4
letters g4(3) =",g(3,4))
        print("fraction of closed windows (T3): 9/27 =",9/27)
```

Output:

```
NC_000932.1: length 154478, length mod 3 = 2
NC_001422.1: length 5386, length mod 3 = 1
g(n) trits n=1..6: [3, 6, 11, 24, 51, 130]  codon necklaces over 4 letters g4(3) =
24
fraction of closed windows (T3): 9/27 = 0.3333333333333333
```

Calculations for Chapter 11 — Metabolism and Interaction

Calculation 11.1 — ATP synthase c-rings against three catalytic sites (T1, T15).

```
        # 11.1 ATP synthase: c-ring sizes against three catalytic sites (T2, T15); ions per
ATP = c/3
        from fractions import Fraction as F
        def ladder(n):
            """return (a,b) if n = 2^a 3^b, else None"""
```

```
            if n != int(n) or n < 1: return None
            n=int(n); a=b=0
            while n%2==0: n//=2; a+=1
            while n%3==0: n//=3; b+=1
            return (a,b) if n==1 else None
        def digit(s):
            d=((s+1)%3)-1
            return d,(s-d)//3
        cs={8:"Bos taurus mito",9:"Mycobacterium phlei",10:"E. coli, yeast, human",11:"Ilyob
acter (Na+)",12:"Paracoccus",13:"Bacillus pseudofirmus",14:"spinach chloroplast",
15:"Spirulina, Gloeobacter",17:"Burkholderia"}
        for c,n in cs.items():
            d,carry=digit(c)
            print(f"c={c:2d} {n:24s} ions/ATP = {str(F(c,3)):>5s} = {c/3:.3f}  c mod 3
digit={d:+d} carry={carry}  ladder={ladder(c)}  step per ATP = {360/3:.0f} deg = {F(
c,3)} c-subunits")
        print("c values with digit 0 (whole ions per 120 deg):",[c for c in cs if c%3==0])

        print("on ladder 2^a3^b:",[c for c in cs if ladder(c)],"; off ladder:",[c for c in
cs if not ladder(c)])
```

Output:

```
c= 8 Bos taurus mito          ions/ATP =   8/3 = 2.667  c mod 3 digit=-1 carry=3
ladder=(3, 0)  step per ATP = 120 deg = 8/3 c-subunits
c= 9 Mycobacterium phlei      ions/ATP =     3 = 3.000  c mod 3 digit=+0 carry=3
ladder=(0, 2)  step per ATP = 120 deg = 3 c-subunits
c=10 E. coli, yeast, human    ions/ATP =  10/3 = 3.333  c mod 3 digit=+1 carry=3
ladder=None  step per ATP = 120 deg = 10/3 c-subunits
c=11 Ilyobacter (Na+)         ions/ATP =  11/3 = 3.667  c mod 3 digit=-1 carry=4
ladder=None  step per ATP = 120 deg = 11/3 c-subunits
c=12 Paracoccus               ions/ATP =     4 = 4.000  c mod 3 digit=+0 carry=4
ladder=(2, 1)  step per ATP = 120 deg = 4 c-subunits
c=13 Bacillus pseudofirmus    ions/ATP =  13/3 = 4.333  c mod 3 digit=+1 carry=4
ladder=None  step per ATP = 120 deg = 13/3 c-subunits
c=14 spinach chloroplast      ions/ATP =  14/3 = 4.667  c mod 3 digit=-1 carry=5
ladder=None  step per ATP = 120 deg = 14/3 c-subunits
c=15 Spirulina, Gloeobacter   ions/ATP =     5 = 5.000  c mod 3 digit=+0 carry=5
ladder=None  step per ATP = 120 deg = 5 c-subunits
c=17 Burkholderia             ions/ATP =  17/3 = 5.667  c mod 3 digit=-1 carry=6
ladder=None  step per ATP = 120 deg = 17/3 c-subunits
c values with digit 0 (whole ions per 120 deg): [9, 12, 15]
on ladder 2^a3^b: [8, 9, 12] ; off ladder: [10, 11, 13, 14, 15, 17]
```

Calculation 11.2 — The Calvin cycle as digit and carry (T7.1).

```
        # 11.2 Calvin cycle as a ring with carry (T7.1); TCA turn
        def ladder(n):
            """return (a,b) if n = 2^a 3^b, else None"""
            if n != int(n) or n < 1: return None
            n=int(n); a=b=0
            while n%2==0: n//=2; a+=1
            while n%3==0: n//=3; b+=1
            return (a,b) if n==1 else None
        def digit(s):
            d=((s+1)%3)-1
            return d,(s-d)//3
        for co2 in range(1,7):
            d,c=digit(co2)
            print(f"{co2} CO2 fixed: digit {d:+d}, carry {c}  -> {c} G3P exported, residue
```

```
{d}")
        print("ATP, NADPH per CO2:",9/3,6/3)
```

Output:

```
1 CO2 fixed: digit +1, carry 0  -> 0 G3P exported, residue 1
2 CO2 fixed: digit -1, carry 1  -> 1 G3P exported, residue -1
3 CO2 fixed: digit +0, carry 1  -> 1 G3P exported, residue 0
4 CO2 fixed: digit +1, carry 1  -> 1 G3P exported, residue 1
5 CO2 fixed: digit -1, carry 2  -> 2 G3P exported, residue -1
6 CO2 fixed: digit +0, carry 2  -> 2 G3P exported, residue 0
ATP, NADPH per CO2: 3.0 2.0
```

Calculation 11.3 — Random competitive tournaments: the coexisting set is always odd.

```
        # 11.3 Random competitive tournaments (Allesina & Levine 2011): size of the
coexisting set
        # Optimal mixed strategy of the symmetric zero-sum game with payoff H - H^T (H_ij=1
if i beats j)
        import numpy as np
        from scipy.optimize import linprog
        rng=np.random.default_rng(7)
        def support(n):
            H=np.triu(rng.integers(0,2,(n,n)),1); H=H+np.triu(1-H,1).T*0  # placeholder

            A=np.zeros((n,n))
            for i in range(n):
                for j in range(i+1,n):
                    if rng.random()<0.5: A[i,j]=1;A[j,i]=-1
                    else: A[i,j]=-1;A[j,i]=1
            # maximise v s.t. A^T x >= v, sum x =1, x>=0  (value is 0 for a skew-symmetric
game)
            c=np.zeros(n+1);c[-1]=-1
            Aub=np.hstack([-A.T,np.ones((n,1))]); bub=np.zeros(n)
            Aeq=np.hstack([np.ones((1,n)),np.zeros((1,1))])
            r=linprog(c,A_ub=Aub,b_ub=bub,A_eq=Aeq,b_eq=[1],bounds=[(0,None)]*n+[(None,None)
],method="highs")
            x=r.x[:n]; return int((x>1e-9).sum())
        for n in (3,4,5,6,8,10,20):
            s=[support(n) for _ in range(2000)]
            odd=sum(1 for k in s if k%2==1)
            vals,cnt=np.unique(s,return_counts=True)
            print(f"n={n:2d}: {odd}/2000 coexisting sets odd; sizes {dict(zip(vals.tolist(),
cnt.tolist()))}")
        # exact for n=3: of 8 tournaments, how many are cyclic (a ring of 3)?
        from itertools import product
        cyc=0
        for e in product([0,1],repeat=3):
            # edges (0,1),(1,2),(0,2): e=1 means lower index wins
            w01,w12,w02=e
            beats={(0,1) if w01 else (1,0),(1,2) if w12 else (2,1),(0,2) if w02 else (2,0)}

            outdeg=[sum(1 for a,b in beats if a==i) for i in range(3)]
            if outdeg==[1,1,1]: cyc+=1
        print("n=3 tournaments that are rings:",cyc,"of 8")
```

Output:

```
n= 3: 2000/2000 coexisting sets odd; sizes {1: 1467, 3: 533}
n= 4: 2000/2000 coexisting sets odd; sizes {1: 969, 3: 1031}
n= 5: 2000/2000 coexisting sets odd; sizes {1: 615, 3: 1264, 5: 121}
n= 6: 2000/2000 coexisting sets odd; sizes {1: 341, 3: 1242, 5: 417}
n= 8: 2000/2000 coexisting sets odd; sizes {1: 112, 3: 873, 5: 907, 7: 108}
n=10: 2000/2000 coexisting sets odd; sizes {1: 37, 3: 459, 5: 978, 7: 494, 9: 32}
n=20: 2000/2000 coexisting sets odd; sizes {3: 4, 5: 78, 7: 298, 9: 605, 11: 636,
13: 308, 15: 61, 17: 10}
n=3 tournaments that are rings: 2 of 8
```

Calculation 11.4 — Sign pairs and stability: meetings (+,−) against same-sign pairs.

```
        # 11.4 Sign pairs of interactions: meetings (+,-) vs open pairs (+,+),(-,-);
Allesina-Tang criteria
        import numpy as np
        rng=np.random.default_rng(3)
        S,C,sigma,d=250,0.25,1.0,None
        def build(kind):
            M=np.zeros((S,S))
            for i in range(S):
                for j in range(i+1,S):
                    if rng.random()<C:
                        a,b=abs(rng.normal(0,sigma)),abs(rng.normal(0,sigma))
                        if kind=="pp":
                            if rng.random()<.5: M[i,j],M[j,i]=a,-b
                            else: M[i,j],M[j,i]=-a,b
                        elif kind=="mut": M[i,j],M[j,i]=a,b
                        elif kind=="comp": M[i,j],M[j,i]=-a,-b
                        else: M[i,j],M[j,i]=rng.normal(0,sigma),rng.normal(0,sigma)
            return M
        for kind in ["random","pp","comp","mut"]:
            M=build(kind); ev=np.linalg.eigvals(M)
            print(f"{kind:7s}: max real part of eigenvalues (diagonal 0) = {ev.real.max():
7.2f}")
        # theory: sqrt(S*C)*sigma (random), times (pi-2)/pi for predator-prey bulk (|X|
entries half-normal)
        print("sqrt(SC)*sigma =",round(np.sqrt(S*C)*sigma,2),"; predator-prey bound x
(pi-2)/pi =",round(np.sqrt(S*C)*sigma*(np.pi-2)/np.pi,2))
        # share of sign pairs that are meetings (sum 0) under random signs: T6-type baseline

        print("random sign pairs: P(meeting (+,-) or (-,+)) =",2/4)
```

Output:

```
random : max real part of eigenvalues (diagonal 0) =    7.79
pp     : max real part of eigenvalues (diagonal 0) =    2.90
comp   : max real part of eigenvalues (diagonal 0) =   11.43
mut    : max real part of eigenvalues (diagonal 0) =   50.24
sqrt(SC)*sigma = 7.91 ; predator-prey bound x (pi-2)/pi = 2.87
random sign pairs: P(meeting (+,-) or (-,+)) = 0.5
```

Calculation 11.5 — V(D)J junctions against the closure baseline (T6); 12/23 spacers in helical turns.

```
        # 11.5 V(D)J junctions: frame closure against the T6 baseline; 12/23 spacers in
helical turns
        from fractions import Fraction as F
```

```
        import cmath
        w=cmath.exp(2j*cmath.pi/3)
        p=lambda z:(1+z+z*z)/3     # junction residue uniform on {0,+1,-1}
        print("T6, uniform law, one window:",round(((1+p(w)**3+p(w*w)**3)/3).real,6))

        print("productive rearrangement (textbook):",F(1,3),"; at least one of two alleles
productive:",1-F(2,3)**2)
        for s in (12,23):
            print(f"spacer {s} bp = {s/10.5:.3f} turns of B-DNA (10.5 bp/turn); {s} mod 3 =
{s%3}")
```

Output:

```
T6, uniform law, one window: 0.333333
productive rearrangement (textbook): 1/3 ; at least one of two alleles productive:
5/9
spacer 12 bp = 1.143 turns of B-DNA (10.5 bp/turn); 12 mod 3 = 0
spacer 23 bp = 2.190 turns of B-DNA (10.5 bp/turn); 23 mod 3 = 2
```

Calculation 11.6 — Other numbers of Chapter 11 (T24 lever; LTEE; cell counts; GTDB; AMR).

```
        # 11.6 Checklist remainder for Chapter 11
        from math import log2
        P3=0.3405373
        print("T24 lever, free 3-D path: -(1+P3)/P3 =",round(-(1+P3)/P3,4))
        print("LTEE: generations per daily 1:100 transfer = log2(100) =",round(log2(100),3))

        print("LTEE: 82,000 generations / 6.644 per day =",round(82000/log2(100)),"days =",r
ound(82000/log2(100)/365.25,1),"years")
        print("Kerr et al.: 66 generations in 7 daily transfers =",round(66/7,2),"per day")

        print("bacteria/human cells: 3.8e13/3.0e13 =",round(3.8/3.0,2))
        print("GTDB R226 archaeal share of species clusters:",round(6968/143614*100,2),"%")

        print("AMR attributable deaths 1990 -> 2021: +",round((1.14/1.06-1)*100,1),"%")
```

Output:

```
T24 lever, free 3-D path: -(1+P3)/P3 = -3.9365
LTEE: generations per daily 1:100 transfer = log2(100) = 6.644
LTEE: 82,000 generations / 6.644 per day = 12342 days = 33.8 years
Kerr et al.: 66 generations in 7 daily transfers = 9.43 per day
bacteria/human cells: 3.8e13/3.0e13 = 1.27
GTDB R226 archaeal share of species clusters: 4.85 %
AMR attributable deaths 1990 -> 2021: + 7.5 %
```

Calculations for Chapter 12 — Evolution and Development

Calculation 12.1 — Logistic map: period doubling, the period-3 window and the Feigenbaum ratio.

```
        # 12.1 Logistic map: period-doubling thresholds (x2 ladder), period-3 window,
Feigenbaum ratio
        import numpy as np
        from scipy.optimize import brentq
        def per(r,n=4000,keep=256):
```

```
            x=0.3
            for _ in range(n): x=r*x*(1-x)
            orb=[]
            for _ in range(keep): x=r*x*(1-x); orb.append(round(x,6))
            for p in range(1,129):
                if all(abs(orb[i]-orb[i+p])<1e-5 for i in range(keep-p)): return p
            return None
        for r in [2.9,3.2,3.5,3.55,3.566,3.83,3.845,3.9]:
            print(f"r={r}: period {per(r)}")
        print("period-2 onset r=3, period-4 onset 1+sqrt(6) =",round(1+6**0.5,
5),", period-3 window 1+sqrt(8) =",round(1+8**0.5,5))
        # superstable parameters R_n (x=1/2 on a 2^n cycle) and Feigenbaum ratio
        def f_n(r,n):
            x=0.5
            for _ in range(2**n): x=r*x*(1-x)
            return x-0.5
        R=[2.0,1+5**0.5]
        guess=[3.4985,3.5546,3.5666,3.56923]
        for n,g in zip(range(2,6),guess):
            R.append(brentq(lambda r:f_n(r,n),g-0.002,g+0.002))
        d=[(R[i]-R[i-1])/(R[i+1]-R[i]) for i in range(1,len(R)-1)]
        print("superstable R_n:",[round(x,6) for x in R])
        print("ratios -> Feigenbaum delta:",[round(x,4) for x in d])
        # Sharkovskii order: first terms and last terms
        odd=[3,5,7,9]; print("Sharkovskii order begins",odd+[2*o for o in odd]+[4*o for o
in odd],"... ends ... 16, 8, 4, 2, 1")
```

Output:

```
r=2.9: period 1
r=3.2: period 2
r=3.5: period 4
r=3.55: period 8
r=3.566: period 16
r=3.83: period 3
r=3.845: period 6
r=3.9: period None
period-2 onset r=3, period-4 onset 1+sqrt(6) = 3.44949 , period-3 window 1+sqrt(8) =
3.82843
superstable R_n: [2.0, 3.236068, 3.498562, 3.554641, 3.566667, 3.569244]
ratios -> Feigenbaum delta: [4.7089, 4.6808, 4.663, 4.6684]
Sharkovskii order begins [3, 5, 7, 9, 6, 10, 14, 18, 12, 20, 28, 36] ... ends ...
16, 8, 4, 2, 1
```

Calculation 12.2 — Observed periods and counts of Chapter 12 against the ladder 2^a·3^b (T15).

```
        # 12.2 Observed periods and counts in Chapter 12 checked against the ladder 2^a 3^b
(T15)
        def ladder(n):
            """return (a,b) if n = 2^a 3^b, else None"""
            if n != int(n) or n < 1: return None
            n=int(n); a=b=0
            while n%2==0: n//=2; a+=1
            while n%3==0: n//=3; b+=1
            return (a,b) if n==1 else None
        def digit(s):
            d=((s+1)%3)-1
            return d,(s-d)//3
        from math import gcd, log2
```

```
        data=[("Tribolium predicted cycle, cpa=0.00 (2-cycle-like)",2),("Tribolium
cpa=0.50",3),("Tribolium cpa=1.00",6),("Tribolium cpa=0.25",8),("Tribolium
cpa=0.10",26),
        ("measles cascade (Earn 2000)",2),("",3),("",4),("",5),("",6),("",7),("",8),
        ("periodical cicada",13),("periodical cicada",17),("snowshoe hare cycle, years",10),

        ("segmentation clock zebrafish, min",30),("chick, min",90),("mouse, min",120),("rabb
it, min",150),("human, min",300),("marmoset, min",390),
        ("Xenopus cleavages before MBT",12),("cells at MBT 2^12",4096),("mouse somites",65),
("Raup-Sepkoski period, Myr",26)]
        on=[];off=[]
        for name,n in data:
            l=ladder(n); (on if l else off).append(n)
            print(f"{name:45s} {n:5d}  ladder={l}")
        print("on ladder:",on); print("off ladder:",off)
        print("cicada lcm(13,17) =",13*17//gcd(13,17),"; 2024-221 =",2024-221,"; 2024+221
=",2024+221)
        # segmentation clock ratios relative to mouse
        for n,v in [("zebrafish",30),("chick",90),("rabbit",150),("human",300),("marmoset",3
90)]:
            print(f"{n}: period/mouse = {v/120:.3f}")
```

Output:

```
Tribolium predicted cycle, cpa=0.00 (2-cycle-like)     2  ladder=(1, 0)
Tribolium cpa=0.50                                     3  ladder=(0, 1)
Tribolium cpa=1.00                                     6  ladder=(1, 1)
Tribolium cpa=0.25                                     8  ladder=(3, 0)
Tribolium cpa=0.10                                    26  ladder=None
measles cascade (Earn 2000)                            2  ladder=(1, 0)
                                                       3  ladder=(0, 1)
                                                       4  ladder=(2, 0)
                                                       5  ladder=None
                                                       6  ladder=(1, 1)
                                                       7  ladder=None
                                                       8  ladder=(3, 0)
periodical cicada                                     13  ladder=None
periodical cicada                                     17  ladder=None
snowshoe hare cycle, years                            10  ladder=None
segmentation clock zebrafish, min                     30  ladder=None
chick, min                                            90  ladder=None
mouse, min                                           120  ladder=None
rabbit, min                                          150  ladder=None
human, min                                           300  ladder=None
marmoset, min                                        390  ladder=None
Xenopus cleavages before MBT                          12  ladder=(2, 1)
cells at MBT 2^12                                   4096  ladder=(12, 0)
mouse somites                                         65  ladder=None
Raup-Sepkoski period, Myr                             26  ladder=None
on ladder: [2, 3, 6, 8, 2, 3, 4, 6, 8, 12, 4096]
off ladder: [26, 5, 7, 13, 17, 10, 30, 90, 120, 150, 300, 390, 65, 26]
cicada lcm(13,17) = 221 ; 2024-221 = 1803 ; 2024+221 = 2245
zebrafish: period/mouse = 0.250
chick: period/mouse = 0.750
rabbit: period/mouse = 1.250
human: period/mouse = 2.500
marmoset: period/mouse = 3.250
```

Calculation 12.3 — Phyllotaxis: golden angle, Fibonacci fractions and the closure depths (T17).

```
        # 12.3 Phyllotaxis: golden angle, Fibonacci ratios vs the closure depths of T17 (two
different continued fractions)
        from math import sqrt, log2
        from fractions import Fraction
        def ladder(n):
            """return (a,b) if n = 2^a 3^b, else None"""
            if n != int(n) or n < 1: return None
            n=int(n); a=b=0
            while n%2==0: n//=2; a+=1
            while n%3==0: n//=3; b+=1
            return (a,b) if n==1 else None
        def digit(s):
            d=((s+1)%3)-1
            return d,(s-d)//3
        phi=(1+sqrt(5))/2
        print("golden angle 360/phi^2 =",round(360/phi**2,4),"deg")
        fib=[1,1]
        while len(fib)<14: fib.append(fib[-1]+fib[-2])
        print("Fibonacci:",fib)
        print("Fibonacci numbers on the ladder:",[f for f in sorted(set(fib)) if ladder(f)])

        # continued fraction of log2(3) convergent denominators vs Fibonacci
        def cf(x,n):
            a=[]
            for _ in range(n):
                q=int(x); a.append(q); x=1/(x-q)
            return a
        a=cf(log2(3),10); print("log2 3 =",a)
        p,q=[1,a[0]],[0,1]
        for k in a[1:]: p.append(k*p[-1]+p[-2]); q.append(k*q[-1]+q[-2])
        print("closure depths (denominators):",q[2:])
        print("1/phi^2 cf = [0;2,1,1,1,...]; leaf fractions 1/2,1/3,2/5,3/8,5/13,8/21 ->",[r
ound(float(Fraction(fib[i],fib[i+2])),4) for i in range(6)], "limit",round(1/phi**2,
4))
        print("common to both lists (>1):",sorted(set(q[2:])&set(fib)-{1}))
        # Swinton & Ochu: 74% of 768 families Fibonacci
        print("Swinton: 0.74*768 =",round(0.74*768))
```

Output:

```
golden angle 360/phi^2 = 137.5078 deg
Fibonacci: [1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377]
Fibonacci numbers on the ladder: [1, 2, 3, 8, 144]
log2 3 = [1, 1, 1, 2, 2, 3, 1, 5, 2, 23]
closure depths (denominators): [1, 2, 5, 12, 41, 53, 306, 665, 15601]
1/phi^2 cf = [0;2,1,1,1,...]; leaf fractions 1/2,1/3,2/5,3/8,5/13,8/21 -> [0.5,
0.3333, 0.4, 0.375, 0.3846, 0.381] limit 0.382
common to both lists (>1): [2, 5]
Swinton: 0.74*768 = 568
```

Calculation 12.4 — Fibonacci numbers that lie on the ladder.

```
        # 12.4 Fibonacci numbers of the form 2^a 3^b, F_1..F_300 (exact integers)
        def ladder(n):
            """return (a,b) if n = 2^a 3^b, else None"""
            if n != int(n) or n < 1: return None
            n=int(n); a=b=0
            while n%2==0: n//=2; a+=1
            while n%3==0: n//=3; b+=1
            return (a,b) if n==1 else None
```

```
        def digit(s):
            d=((s+1)%3)-1
            return d,(s-d)//3
        a,b=1,1; hits=[(1,1),(2,1)]
        for n in range(3,301):
            a,b=b,a+b
            if ladder(b): hits.append((n,b))
        print("indices n with F_n = 2^a 3^b (n<=300):",hits)
```

Output:

```
indices n with F_n = 2^a 3^b (n<=300): [(1, 1), (2, 1), (3, 2), (4, 3), (6, 8), (12,
144)]
```

Calculation 12.5 — Hardy–Weinberg genotype law as a trit law; closure probability (T6).

```
        # 12.5 Hardy-Weinberg genotype law as trit law; drift as a 1-D walk (Polya:
recurrent, T24 not applicable)
        import cmath
        for p in (0.5,0.7,0.9):
            q=1-p; law=(2*p*q, p*p, q*q)   # (0, +1, -1) = (Aa, AA, aa)
            w=cmath.exp(2j*cmath.pi/3); ph=lambda z: law[0]+law[1]*z+law[2]*z*z
            P=((1+ph(w)**3+ph(w*w)**3)/3).real
            print(f"p={p}: HW law (Aa,AA,aa)=({law[0]:.3f},{law[1]:.3f},{law[2]:.3f}); T6
closure prob of 3 unlinked loci = {P:.4f}")
```

Output:

```
p=0.5: HW law (Aa,AA,aa)=(0.500,0.250,0.250); T6 closure prob of 3 unlinked loci =
0.3438
p=0.7: HW law (Aa,AA,aa)=(0.420,0.490,0.090); T6 closure prob of 3 unlinked loci =
0.3036
p=0.9: HW law (Aa,AA,aa)=(0.180,0.810,0.010); T6 closure prob of 3 unlinked loci =
0.5460
```

Calculations for Chapter 13 — Cultivating Life

Calculation 13.1 — Odd ploidy: balanced gametes and pairing parity.

```
        # 13.1 Odd ploidy and balanced gametes; meetings (pairing) vs rings (trivalents)

        from fractions import Fraction as F
        def balanced_triploid(n):   # each trivalent sends 1 or 2 copies, prob 1/2 each;
euploid iff all same
            return 2*F(1,2)**n
        for crop,x in [("banana (x=11)",11),("watermelon (x=11)",11),("oyster (x=10)",10),("
x=7",7)]:
            b=balanced_triploid(x); print(f"{crop}: balanced triploid gametes 2*(1/2)^{x} =
{b} = {float(b):.6f}")
        # carry reading: chromosome sets added as gametes; tetraploid x diploid
        for g1,g2,lab in [(1,1,"diploid x diploid"),(2,1,"tetraploid x diploid (seedless
watermelon seed)"),(2,2,"tetraploid x tetraploid"),(3,3,"hexaploid wheat gametes")]:
            s=g1+g2; print(f"{lab}: sets {g1}+{g2} = {s}; even -> all sets can meet in
pairs: {s%2==0}; s mod 3 = {s%3}")
```

Output:

```
banana (x=11): balanced triploid gametes 2*(1/2)^11 = 1/1024 = 0.000977
watermelon (x=11): balanced triploid gametes 2*(1/2)^11 = 1/1024 = 0.000977
oyster (x=10): balanced triploid gametes 2*(1/2)^10 = 1/512 = 0.001953
x=7: balanced triploid gametes 2*(1/2)^7 = 1/64 = 0.015625
diploid x diploid: sets 1+1 = 2; even -> all sets can meet in pairs: True; s mod 3 =
2
tetraploid x diploid (seedless watermelon seed): sets 2+1 = 3; even -> all sets can
meet in pairs: False; s mod 3 = 0
tetraploid x tetraploid: sets 2+2 = 4; even -> all sets can meet in pairs: True; s
mod 3 = 1
hexaploid wheat gametes: sets 3+3 = 6; even -> all sets can meet in pairs: True; s
mod 3 = 0
```

Calculation 13.2 — Ricker and Beverton–Holt stock-recruitment maps.

```
        # 13.2 Ricker and Beverton-Holt stock-recruitment maps: period doubling (x2 ladder)
vs none
        import numpy as np
        def period(f,x0=0.4,n=5000,keep=128):
            x=x0
            for _ in range(n): x=f(x)
            orb=[]
            for _ in range(keep): x=f(x); orb.append(x)
            for p in range(1,65):
                if all(abs(orb[i]-orb[i+p])<1e-7 for i in range(keep-p)): return p
            return None
        print("Ricker x' = x exp(r(1-x)):")
        for r in [1.8,2.1,2.6,2.68,2.69,3.0,3.11]:
            print(f"  r={r}: period {period(lambda x: x*np.exp(r*(1-x)))}")
        print("Beverton-Holt x' = R x/(1+(R-1)x):")
        for R in [2,10,100,1000]:
            print(f"  R={R}: period {period(lambda x: R*x/(1+(R-1)*x))}")
```

Output:

```
Ricker x' = x exp(r(1-x)):
  r=1.8: period 1
  r=2.1: period 2
  r=2.6: period 4
  r=2.68: period 8
  r=2.69: period 16
  r=3.0: period None
  r=3.11: period 3
Beverton-Holt x' = R x/(1+(R-1)x):
  R=2: period 1
  R=10: period 1
  R=100: period 1
  R=1000: period 1
```

Calculation 13.3 — Rotations, aquaculture share and deforestation ratio.

```
        # 13.3 Rotations, aquaculture share and forest ratios
        from fractions import Fraction as F
        def ladder(n):
            """return (a,b) if n = 2^a 3^b, else None"""
            if n != int(n) or n < 1: return None
            n=int(n); a=b=0
            while n%2==0: n//=2; a+=1
            while n%3==0: n//=3; b+=1
```

```
            return (a,b) if n==1 else None
        two,three=F(1,2),F(2,3); print("cropped share: two-field",two,"three-field",three,"g
ain",three/two-1)
        print("rotation lengths on ladder: 2",ladder(2),"3",ladder(3),"4",ladder(4))
        print("SOFIA 2026 (2024 data): aquaculture 103 of 195 Mt animals =",round(103/195*10
0,1),"%; aquaculture + capture =",103+92,"Mt")
        print("SOFIA 2026: landings not from sustainable stocks = 100 - 72.6 =",round(100-72
.6,1),"%")
        print("FRA 2025: deforestation 2015-25 / 1990-2000 =",round(10.9/17.6,
3),"; net loss ratio =",round(4.12/10.7,3))
        print("ratio 0.619 on ladder? not an integer ratio of small numbers; ladder test not
applicable")
```

Output:

```
cropped share: two-field 1/2 three-field 2/3 gain 1/3
rotation lengths on ladder: 2 (1, 0) 3 (0, 1) 4 (2, 0)
SOFIA 2026 (2024 data): aquaculture 103 of 195 Mt animals = 52.8 %; aquaculture +
capture = 195 Mt
SOFIA 2026: landings not from sustainable stocks = 100 - 72.6 = 27.4 %
FRA 2025: deforestation 2015-25 / 1990-2000 = 0.619 ; net loss ratio = 0.385
ratio 0.619 on ladder? not an integer ratio of small numbers; ladder test not
applicable
```

Calculations for Chapter 14 — Body and Brain

Calculation 14.1 — Collagen exon lengths are whole Gly-X-Y units (T8, T11).

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        def digit_carry(s):
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        exons = [36, 45, 54, 99, 108]          # bp, Butticè et al. 1990
        for L in exons:
            print(L, "bp:", L // 3, "codons,", L / 9, "Gly-X-Y triplets, L mod 9 =", L % 9,
", ladder", ladder(L))
        print("type I central domain: 338 triplets =", 338 * 3, "residues =", 338 * 9, "bp")

        # a seam between two exons keeps the glycine phase iff the first exon is a multiple
of 9
        for L in [54, 55, 56, 57]:
            print("exon of", L, "bp shifts the Gly phase by", (L // 3) % 3 if L % 3 == 0 els
e "frameshift", "residues")
```

Output:

```
36 bp: 12 codons, 4.0 Gly-X-Y triplets, L mod 9 = 0 , ladder (2, 2)
45 bp: 15 codons, 5.0 Gly-X-Y triplets, L mod 9 = 0 , ladder None
54 bp: 18 codons, 6.0 Gly-X-Y triplets, L mod 9 = 0 , ladder (1, 3)
99 bp: 33 codons, 11.0 Gly-X-Y triplets, L mod 9 = 0 , ladder None
108 bp: 36 codons, 12.0 Gly-X-Y triplets, L mod 9 = 0 , ladder (2, 3)
type I central domain: 338 triplets = 1014 residues = 3042 bp
exon of 54 bp shifts the Gly phase by 0 residues
exon of 55 bp shifts the Gly phase by frameshift residues
```

```
exon of 56 bp shifts the Gly phase by frameshift residues
exon of 57 bp shifts the Gly phase by 1 residues
```

Calculation 14.2 — Thick-filament crowns (Zoghbi et al. 2008).

```
        print("3 crowns x 14.3 nm =", round(3 * 14.3, 1), "nm; repeat / spacing =", round(42
.9 / 14.3, 4))
        print("measured crown gaps 13.2 + 15.4 + rest:", round(42.9 - 13.2 - 15.4, 1), "nm")

        print("rotational order of a crown: 3; head positions per 42.9 nm repeat: 3 crowns x
3-fold =", 3 * 3)
```

Output:

```
3 crowns x 14.3 nm = 42.9 nm; repeat / spacing = 3.0
measured crown gaps 13.2 + 15.4 + rest: 14.3 nm
rotational order of a crown: 3; head positions per 42.9 nm repeat: 3 crowns x 3-fold
= 9
```

Calculation 14.3 — Anatomical counts against the ladder 2^a 3^b (T15) and the chance baseline.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        counts = {"cervical": 7, "thoracic": 12, "lumbar": 5, "sacral": 5, "coccygeal": 4,

                  "vertebrae": 33, "rib pairs": 12, "adult bones": 206, "bones at birth": 27
0,
                  "Hox clusters": 4, "Hox genes": 39}
        on = 0
        for k, n in counts.items():
            r = ladder(n); on += r is not None
            print(f"{k:15s} {n:4d} ladder {r}")
        print("on ladder:", on, "of", len(counts))
        for N in (12, 50, 300):
            k = sum(1 for i in range(1, N + 1) if ladder(i))
            print(f"share of 1..{N} on the ladder: {k}/{N} = {k/N:.3f}")
```

Output:

```
cervical           7 ladder None
thoracic          12 ladder (2, 1)
lumbar             5 ladder None
sacral             5 ladder None
coccygeal          4 ladder (2, 0)
vertebrae         33 ladder None
rib pairs         12 ladder (2, 1)
adult bones      206 ladder None
bones at birth   270 ladder None
Hox clusters       4 ladder (2, 0)
Hox genes         39 ladder None
on ladder: 4 of 11
share of 1..12 on the ladder: 8/12 = 0.667
share of 1..50 on the ladder: 15/50 = 0.300
share of 1..300 on the ladder: 29/300 = 0.097
```

Calculation 14.4 — Cells of the body as doublings and triplings.

```
        import math
        n = 3.72e13                       # Bianconi et al. 2013
        print("log2 =", round(math.log2(n), 2), "; log3 =", round(math.log(n, 3), 2))

        print("2^45 =", f"{2**45:.3e}", "; 2^46 =", f"{2**46:.3e}")
```

Output:

```
log2 = 45.08 ; log3 = 28.44
2^45 = 3.518e+13 ; 2^46 = 7.037e+13
```

Calculation 14.5 — The arm as an attached strand: 2pi versus 4pi (T21).

```
        import numpy as np
        def q_rot(theta):  # unit quaternion for rotation by theta about a fixed axis

            return np.array([np.cos(theta / 2), 0, 0, np.sin(theta / 2)])
        for t in (2 * np.pi, 4 * np.pi):
            print(f"rotation {t/np.pi:.0f} pi -> q =", np.round(q_rot(t), 12))
```

Output:

```
rotation 2 pi -> q = [-1.  0.  0.  0.]
rotation 4 pi -> q = [ 1.  0.  0. -0.]
```

Calculation 14.6 — Einthoven's law as a closed loop of three differences.

```
        import sympy as sp, cmath, math, random
        RA, LA, LL = sp.symbols("RA LA LL")
        I, II, III = LA - RA, LL - RA, LL - LA
        print("I + III - II =", sp.simplify(I + III - II))
        # lead axes at 0, 60, 120 degrees are sixth roots of unity; check I + III = II on
projections
        u = [cmath.exp(1j * math.radians(a)) for a in (0, 60, 120)]
        w = cmath.exp(2j * math.pi / 3)
        print("axis II == -w^2:", abs(u[1] + w**2) < 1e-12, "; axis III == w:", abs(u[2] -
w) < 1e-12)
        random.seed(1)
        for _ in range(3):
            H = complex(random.uniform(-1, 1), random.uniform(-1, 1))
            p = [(H * x.conjugate()).real for x in u]
            print("projection check I + III - II =", round(p[0] + p[2] - p[1], 12))
        # Kirchhoff holds for any closed loop, e.g. four electrodes
        A, B, C, D = sp.symbols("A B C D")
        print("four-electrode loop:", sp.simplify((B - A) + (C - B) + (D - C) + (A - D)))
```

Output:

```
I + III - II = 0
axis II == -w^2: True ; axis III == w: True
projection check I + III - II = -0.0
projection check I + III - II = 0.0
projection check I + III - II = -0.0
four-electrode loop: 0
```

Calculation 14.7 — Locomotor-respiratory and Wenckebach ratios against the ladder.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        from fractions import Fraction
        for r in ["1", "2", "5/2", "3", "7/2", "4"]:
            f = Fraction(r)
            print(f"LRC {f.numerator}:{f.denominator}  numerator on ladder: {ladder(f.numera
tor) is not None}")
        for X in range(2, 8):
            print(f"Wenckebach {X}:{X-1}  both on ladder: {ladder(X) is not None and
ladder(X-1) is not None}")
```

Output:

```
LRC 1:1  numerator on ladder: True
LRC 2:1  numerator on ladder: True
LRC 5:2  numerator on ladder: False
LRC 3:1  numerator on ladder: True
LRC 7:2  numerator on ladder: False
LRC 4:1  numerator on ladder: True
Wenckebach 2:1  both on ladder: True
Wenckebach 3:2  both on ladder: True
Wenckebach 4:3  both on ladder: True
Wenckebach 5:4  both on ladder: False
Wenckebach 6:5  both on ladder: False
Wenckebach 7:6  both on ladder: False
```

Calculation 14.8 — Grid-module ratio against sqrt2, sqrt3 and published predictions.

```
        import math
        m, sd, n = 1.42, 0.17, 24        # Stensola et al. 2012 as reported by Wei et al.
2015
        se = sd / math.sqrt(n)
        for name, v in [("sqrt2", math.sqrt(2)), ("sqrt3", math.sqrt(3)), ("2", 2.0),

                        ("sqrt(e)", math.sqrt(math.e)), ("1.44", 1.44)]:
            print(f"{name:8s} {v:.4f}  (v - mean)/SD = {(v-m)/sd:+.2f}  (v - mean)/SE = {(v-
m)/se:+.2f}")
        norms = sorted({a*a + a*b + b*b for a in range(8) for b in range(8) if 0 < a*a + a*b
 + b*b <= 30})
        print("Eisenstein norms <= 30:", norms)
        print("2 is a norm:", 2 in norms, "; 3 is a norm:", 3 in norms)
```

Output:

```
sqrt2    1.4142  (v - mean)/SD = -0.03  (v - mean)/SE = -0.17
sqrt3    1.7321  (v - mean)/SD = +1.84  (v - mean)/SE = +8.99
2        2.0000  (v - mean)/SD = +3.41  (v - mean)/SE = +16.71
sqrt(e)  1.6487  (v - mean)/SD = +1.35  (v - mean)/SE = +6.59
1.44     1.4400  (v - mean)/SD = +0.12  (v - mean)/SE = +0.58
```

```
Eisenstein norms <= 30: [1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28]
2 is a norm: False ; 3 is a norm: True
```

Calculation 14.9 — Reciprocal connections as meetings (Song et al. 2005).

```
        p = 931 / 8050
        print("connection probability p =", round(p, 4))
        print("expected share of pairs connected both ways, p^2 =", round(p * p, 5))
        print("observed 218 reciprocal pairs at about 4 x expectation -> expected about", 21
8 / 4)
```

Output:

```
connection probability p = 0.1157
expected share of pairs connected both ways, p^2 = 0.01338
observed 218 reciprocal pairs at about 4 x expectation -> expected about 54.5
```

Calculation 14.10 — Parity of inhibitory rings and the secant threshold.

```
        import math
        for k in range(1, 7):
            print(f"{k} inhibitory links: loop sign {(-1)**k:+d}")
        for n in range(3, 9):
            print(f"n = {n}: sec(pi/n)^n = {(1/math.cos(math.pi/n))**n:.4f}")
```

Output:

```
1 inhibitory links: loop sign -1
2 inhibitory links: loop sign +1
3 inhibitory links: loop sign -1
4 inhibitory links: loop sign +1
5 inhibitory links: loop sign -1
6 inhibitory links: loop sign +1
n = 3: sec(pi/n)^n = 8.0000
n = 4: sec(pi/n)^n = 4.0000
n = 5: sec(pi/n)^n = 2.8854
n = 6: sec(pi/n)^n = 2.3704
n = 7: sec(pi/n)^n = 2.0751
n = 8: sec(pi/n)^n = 1.8840
```

Calculation 14.11 — Hormone response elements: spacer, period and helical phase.

```
        for name, s in [("DR1", 1), ("DR3", 3), ("DR4", 4), ("DR5", 5)]:
            period = 6 + s
            print(f"{name}: period {period} bp, period mod 3 = {period % 3}, rotation at
10.5 bp/turn = {360*period/10.5:.1f} deg")
        print("GRE AGAACAnnnTGTTCT length:", len("AGAACAnnnTGTTCT"), "; spacer 3")
```

Output:

```
DR1: period 7 bp, period mod 3 = 1, rotation at 10.5 bp/turn = 240.0 deg
DR3: period 9 bp, period mod 3 = 0, rotation at 10.5 bp/turn = 308.6 deg
DR4: period 10 bp, period mod 3 = 1, rotation at 10.5 bp/turn = 342.9 deg
DR5: period 11 bp, period mod 3 = 2, rotation at 10.5 bp/turn = 377.1 deg
GRE AGAACAnnnTGTTCT length: 15 ; spacer 3
```

Calculation 14.12 — Insulin hexamer, thyroid and diabetes numbers.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        print("3 dimers x 2 =", 3 * 2, "monomers; ladder", ladder(6),
"(a = 1 meeting merge, b = 1 ring merge)")
        print("D3 assembly: 12 chains =", 6 * 2, "; ladder", ladder(12))
        print("T4 : T3 output 80 : 20 =", 80 / 20, "; iodines 4 -> 3")
        print("undiagnosed diabetes 252 / 589 =", round(252 / 589, 3), "; growth to 2050:
853 / 589 =", round(853 / 589, 3))
```

Output:

```
3 dimers x 2 = 6 monomers; ladder (1, 1) (a = 1 meeting merge, b = 1 ring merge)
D3 assembly: 12 chains = 12 ; ladder (2, 1)
T4 : T3 output 80 : 20 = 4.0 ; iodines 4 -> 3
undiagnosed diabetes 252 / 589 = 0.428 ; growth to 2050: 853 / 589 = 1.448
```

Calculation 14.13 — Theorem checklist for Chapter 14.

```
        checks = {
         "T1/T2 digit and window":
"collagen Gly-X-Y period 3 (14.1); crowns of 3 (14.2); DR spacers: residue does not
decide (14.11)",
         "T6 baseline": "ladder shares 0.667 / 0.300 / 0.097 for 1..12, 1..50, 1..300
(14.3)",
         "T7 standing/meeting/ring": "reciprocal pairs = meetings (14.9); inhibitory rings
by parity (14.10); insulin 3 x dimer (14.12)",
         "T8-T10 period-3 words": "collagen (14.1); thick filament (14.2)",
         "T11 seams": "exon junctions keep Gly phase iff exon = 0 mod 9 (14.1)",
         "T12 address": "no anatomical quantity is read as a balanced-ternary address; not
applied",
         "T13 3^n vs g(n)": "not applied: no count of cyclic configurations in these data",

         "T14 Z[w]": "Einthoven axes are sixth roots of unity (14.6); sqrt3 module step
rejected, 2 is not an Eisenstein norm (14.8)",
         "T15 ladder": "counts mixed, baseline high (14.3); LRC, Wenckebach partly off
(14.7); grid sqrt2 fits (14.8)",
         "T16-T18 depths, comma": "no sourced ratio in this chapter lies near a closure
depth; not applied",
         "T19-T20 knots": "not applied",
         "T21 double cover": "the arm in the plate trick (14.5)",
         "T22 Hopf charge": "not applied",
         "T23 state-dependent stiffness": "reading only: secant gain depends on state
(14.10)",
         "T24 free-path lever": "not applied (no free random path measured)",
        }
        for k, v in checks.items():
            print(f"{k:28s} {v}")
```

Output:

```
T1/T2 digit and window       collagen Gly-X-Y period 3 (14.1); crowns of 3 (14.2);
DR spacers: residue does not decide (14.11)
T6 baseline                  ladder shares 0.667 / 0.300 / 0.097 for 1..12, 1..50,
1..300 (14.3)
```

```
T7 standing/meeting/ring     reciprocal pairs = meetings (14.9); inhibitory rings by
parity (14.10); insulin 3 x dimer (14.12)
T8-T10 period-3 words        collagen (14.1); thick filament (14.2)
T11 seams                    exon junctions keep Gly phase iff exon = 0 mod 9 (14.1)
T12 address                  no anatomical quantity is read as a balanced-ternary
address; not applied
T13 3^n vs g(n)              not applied: no count of cyclic configurations in these
data
T14 Z[w]                     Einthoven axes are sixth roots of unity (14.6); sqrt3
module step rejected, 2 is not an Eisenstein norm (14.8)
T15 ladder                   counts mixed, baseline high (14.3); LRC, Wenckebach
partly off (14.7); grid sqrt2 fits (14.8)
T16-T18 depths, comma        no sourced ratio in this chapter lies near a closure
depth; not applied
T19-T20 knots                not applied
T21 double cover             the arm in the plate trick (14.5)
T22 Hopf charge              not applied
T23 state-dependent stiffness reading only: secant gain depends on state (14.10)
T24 free-path lever          not applied (no free random path measured)
```

Calculations for Chapter 15 — Perception, Behaviour and Mind

Calculation 15.1 — SCORE replication figures.

```
        print("replication effect / original effect = 0.10 / 0.25 =", 0.10 / 0.25)
        print("replicated share 49% of 164 papers ->", round(0.49 * 164), "papers")
        print("reproducibility with data and code shared vs rebuilt: 91% vs 38%
approximate")
```

Output:

```
replication effect / original effect = 0.10 / 0.25 = 0.4
replicated share 49% of 164 papers -> 80 papers
reproducibility with data and code shared vs rebuilt: 91% vs 38% approximate
```

Calculation 15.2 — Choices among three options: digit and carry of each tournament (T1, T7.1).

```
        def digit_carry(s):
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        import itertools
        for o in itertools.product([1, -1], repeat=3):
# A>B, B>C, C>A oriented around the cycle
            s = sum(o); d, c = digit_carry(s)
            kind = "cyclic (ring)" if abs(s) == 3 else "transitive (open)"
            print(o, "sum", s, "digit", d, "carry", c, kind)
        print("cyclic share if each pair is decided at random: 2/8 =", 2 / 8)
```

Output:

```
(1, 1, 1) sum 3 digit 0 carry 1 cyclic (ring)
(1, 1, -1) sum 1 digit 1 carry 0 transitive (open)
(1, -1, 1) sum 1 digit 1 carry 0 transitive (open)
(1, -1, -1) sum -1 digit -1 carry 0 transitive (open)
(-1, 1, 1) sum 1 digit 1 carry 0 transitive (open)
(-1, 1, -1) sum -1 digit -1 carry 0 transitive (open)
(-1, -1, 1) sum -1 digit -1 carry 0 transitive (open)
```

```
(-1, -1, -1) sum -3 digit 0 carry -1 cyclic (ring)
cyclic share if each pair is decided at random: 2/8 = 0.25
```

Calculation 15.3 — Condorcet profiles: 3 voters, 3 options.

```
        import itertools
        perms = list(itertools.permutations("ABC"))
        cnt = 0
        for prof in itertools.product(perms, repeat=3):
            beats = lambda x, y: sum(p.index(x) < p.index(y) for p in prof) >= 2
            if (beats("A","B") and beats("B","C") and beats("C","A")) or (beats("B","A")
and beats("C","B") and beats("A","C")):
                cnt += 1
        print("cyclic majority profiles:", cnt, "of", len(perms) ** 3, "=", round(cnt / 216,
 4))
```

Output:

```
cyclic majority profiles: 12 of 216 = 0.0556
```

Calculation 15.4 — Signed triads: closure mod 3 against Heider balance.

```
        def digit_carry(s):
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        for tri in ["+++", "++-", "+--", "---"]:
            s = sum(1 if c == "+" else -1 for c in tri)
            d, c = digit_carry(s)
            prod = 1
            for ch in tri: prod *= 1 if ch == "+" else -1
            print(f"{tri}: sum {s:+d} digit {d:+d} carry {c:+d} closed={d == 0} balanced={pr
od > 0}")
```

Output:

```
+++: sum +3 digit +0 carry +1 closed=True balanced=True
++-: sum +1 digit +1 carry +0 closed=False balanced=False
+--: sum -1 digit -1 carry +0 closed=False balanced=True
---: sum -3 digit +0 carry -1 closed=True balanced=False
```

Calculation 15.5 — Signed triads in three networks (Leskovec, Huttenlocher and Kleinberg 2010).

```
        data = {"Epinions":  {"+++": (11640257, .870, .621), "++-": (698023, .052, .321), "+
--": (947855, .071, .055), "---": (89272, .007, .003)},
                "Slashdot":  {"+++": (1266646, .840, .464), "++-": (115884, .077, .406), "+-
-": (109303, .072, .119), "---": (16272, .011, .012)},
                "Wikipedia": {"+++": (555300, .702, .489), "++-": (63425, .080, .395),
"+--": (163328, .207, .106), "---": (8479, .011, .010)}}
        for net, d in data.items():
            tot = sum(v[0] for v in d.values())
            closed = (d["+++"][0] + d["---"][0]) / tot; closed0 = d["+++"][2] + d["---"][2]

            bal = (d["+++"][0] + d["+--"][0]) / tot;    bal0 = d["+++"][2] + d["+--"][2]

            print(f"{net:9s} N = {tot:>9d}  closed {closed:.3f} (shuffled {closed0:.3f})
balanced {bal:.3f} (shuffled {bal0:.3f})"
```

```
                  f"  +-- obs/exp {d['+--'][1]/d['+--'][2]:.2f}  --- obs/exp {d['---'][1]/
d['---'][2]:.2f}")
```

Output:

```
Epinions  N =  13375407  closed 0.877 (shuffled 0.624)  balanced 0.941 (shuffled
0.676)  +-- obs/exp 1.29  --- obs/exp 2.33
Slashdot  N =   1508105  closed 0.851 (shuffled 0.476)  balanced 0.912 (shuffled
0.583)  +-- obs/exp 0.61  --- obs/exp 0.92
Wikipedia N =    790532  closed 0.713 (shuffled 0.499)  balanced 0.909 (shuffled
0.595)  +-- obs/exp 1.95  --- obs/exp 1.10
```

Calculation 15.6 — Closure probability of a shuffled triad (T6).

```
        import cmath, math
        w = cmath.exp(2j * math.pi / 3)
        for net, p0 in [("Epinions", .621), ("Slashdot", .464), ("Wikipedia", .489)]:

            q = p0 ** (1 / 3)                        # share of positive edges implied by
p0(+++)
            ph = lambda z: q * z + (1 - q) * z ** 2   # trits +1 with prob q, -1 with prob
1-q, no zeros
            P = (1 + ph(w) ** 3 + ph(w ** 2) ** 3) / 3
            print(f"{net:9s} q = {q:.4f}  T6 closure probability = {P.real:.4f}  check
q^3+(1-q)^3 = {q**3 + (1-q)**3:.4f}")
            print(f"          binomial shares +++ {q**3:.3f}  ++- {3*q*q*(1-q):.3f}  +--
{3*q*(1-q)**2:.3f}  --- {(1-q)**3:.3f}")
        print("published shuffled shares: Epinions .621 .321 .055 .003; Slashdot .464 .406 .
119 .012; Wikipedia .489 .395 .106 .010")
```

Output:

```
Epinions  q = 0.8532  T6 closure probability = 0.6242  check q^3+(1-q)^3 = 0.6242
          binomial shares +++ 0.621  ++- 0.321  +-- 0.055  --- 0.003
Slashdot  q = 0.7742  T6 closure probability = 0.4755  check q^3+(1-q)^3 = 0.4755
          binomial shares +++ 0.464  ++- 0.406  +-- 0.118  --- 0.012
Wikipedia q = 0.7878  T6 closure probability = 0.4986  check q^3+(1-q)^3 = 0.4986
          binomial shares +++ 0.489  ++- 0.395  +-- 0.106  --- 0.010
published shuffled shares: Epinions .621 .321 .055 .003; Slashdot .464 .406 .119 .
012; Wikipedia .489 .395 .106 .010
```

Calculation 15.7 — Capacities and counts of cognition against the ladder (T15).

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        import math
        items = [("memory span (Miller)", 7), ("Cowan limit", 4), ("span digits", 7),
("span letters", 6), ("span words", 5),
                 ("cone types", 3), ("opponent channels", 3), ("mantis-shrimp receptor
types", 12)]
        for k, n in items:
            print(f"{k:30s} {n:3d} ladder {ladder(n)}")
        print("log2 7 =", round(math.log2(7), 3), "; log2 4 =", math.log2(4),
"; log2 3 (one trit in bits) =", round(math.log2(3), 4))
```

```
        for bits in (2.0, 2.5, 3.0):
            print(f"{bits} bits = {bits / math.log2(3):.3f} trits = {2 ** bits:.2f}
alternatives")
        print("one trit = 3 alternatives; two trits = 9 alternatives")
```

Output:

```
memory span (Miller)             7 ladder None
Cowan limit                      4 ladder (2, 0)
span digits                      7 ladder None
span letters                     6 ladder (1, 1)
span words                       5 ladder None
cone types                       3 ladder (0, 1)
opponent channels                3 ladder (0, 1)
mantis-shrimp receptor types    12 ladder (2, 1)
log2 7 = 2.807 ; log2 4 = 2.0 ; log2 3 (one trit in bits) = 1.585
2.0 bits = 1.262 trits = 4.00 alternatives
2.5 bits = 1.577 trits = 5.66 alternatives
3.0 bits = 1.893 trits = 8.00 alternatives
one trit = 3 alternatives; two trits = 9 alternatives
```

Calculation 15.8 — Opponent channels as zero-sum combinations of three cones.

```
        import numpy as np
        A = np.array([[1, 1, 0], [1, -1, 0], [-0.5, -0.5, 1]])     # luminance, L-M, S-
(L+M)/2
        print("row sums:", A.sum(axis=1), " determinant:", round(np.linalg.det(A), 6))
```

Output:

```
row sums: [2. 0. 0.]  determinant: -2.0
```

Calculation 15.9 — Ethology: tournaments, rock-paper-scissors and gaits.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        def digit_carry(s):
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        # rock-paper-scissors among orange, blue, yellow male morphs (Sinervo and Lively
1996)
        beats = {("O", "B"): 1, ("Y", "O"): 1, ("B", "Y"): 1}
        cycle = [("O", "B"), ("B", "Y"), ("Y", "O")]   # orientation O->B->Y->O
        steps = [1 if (a, b) in beats else -1 for a, b in cycle]
        print("orientation steps", steps, "-> digit and carry", digit_carry(sum(steps)))

        for g, beats_ in [("walk", 4), ("trot", 2), ("canter", 3), ("gallop slow", 3), ("gal
lop fast", 4)]:
            print(f"{g:12s} {beats_} beats  ladder {ladder(beats_)}")
        for n in (6, 7):
            print("starling neighbours", n, "ladder", ladder(n))
```

Output:

```
orientation steps [1, 1, 1] -> digit and carry (0, 1)
walk         4 beats  ladder (2, 0)
trot         2 beats  ladder (1, 0)
canter       3 beats  ladder (0, 1)
gallop slow  3 beats  ladder (0, 1)
gallop fast  4 beats  ladder (2, 0)
starling neighbours 6 ladder (1, 1)
starling neighbours 7 ladder None
```

Calculation 15.10 — Consciousness: one trit in bits and the Cogitate sample.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        import math
        print("log2(3) =", round(math.log2(3), 4), "bits per trit")
        print("Cogitate participants 256 = 2^8:", ladder(256))
```

Output:

```
log2(3) = 1.585 bits per trit
Cogitate participants 256 = 2^8: (8, 0)
```

Calculation 15.11 — Theorem checklist for Chapter 15.

```
        checks = {
         "T1/T7.1 digit, carry": "choice triads (15.2); signed triads (15.4); RPS ring
(15.9)",
         "T2-T4 window, two kinds": "cyclic triads homogeneous closure; +-- open (15.4)",

         "T6 baseline": "random choice cyclic share 1/4 (15.2); shuffled-sign closure
0.62/0.48/0.50 (15.6)",
         "T7 meeting/ring":
"Hering opponent pairs as meetings (15.8); RPS ring (15.9); trot as meeting (15.9)",
         "T8-T10 everywhere-closed": "not applied: no long words of choices with data",

         "T11 seams": "not applied",
         "T12 address": "not applied",
         "T13 3^n vs g(n)": "not applied: no data on cyclic configurations",
         "T14 Z[w]": "not applied in this chapter (grid lattice in Ch. 14)",
         "T15 ladder": "capacities 7, 5 off; 4, 6, 3, 12 on (15.7); gaits 2, 3, 4 on (15.9)
- small numbers, weak",
         "T16-T18": "not applied",
         "T19-T22": "not applied; chirality of the trefoil has no measured analogue in
behaviour",
         "T23": "reading only (state-dependent responsiveness)",
         "T24": "not applied: no free-path data",
        }
        for k, v in checks.items():
            print(f"{k:26s} {v}")
```

Output:

```
T1/T7.1 digit, carry       choice triads (15.2); signed triads (15.4); RPS ring
(15.9)
T2-T4 window, two kinds    cyclic triads homogeneous closure; +-- open (15.4)
```

```
T6 baseline                random choice cyclic share 1/4 (15.2); shuffled-sign
closure 0.62/0.48/0.50 (15.6)
T7 meeting/ring            Hering opponent pairs as meetings (15.8); RPS ring
(15.9); trot as meeting (15.9)
T8-T10 everywhere-closed   not applied: no long words of choices with data
T11 seams                  not applied
T12 address                not applied
T13 3^n vs g(n)            not applied: no data on cyclic configurations
T14 Z[w]                   not applied in this chapter (grid lattice in Ch. 14)
T15 ladder                 capacities 7, 5 off; 4, 6, 3, 12 on (15.7); gaits 2, 3, 4
on (15.9) - small numbers, weak
T16-T18                    not applied
T19-T22                    not applied; chirality of the trefoil has no measured
analogue in behaviour
T23                        reading only (state-dependent responsiveness)
T24                        not applied: no free-path data
```

Calculations for Chapter 16 — The Life Course

Calculation 16.1 — Free-running period against the day.

```
        import math
        tau = 24.18                          # Czeisler et al. 1999
        print("drift per day:", round(tau - 24, 2), "h; days to drift one full cycle:", roun
d(24 / (tau - 24), 1))
        print("ratio tau/24 =", round(tau / 24, 5), "=", round(1200 * math.log2(tau / 24),
2), "cents")
        print("Pythagorean comma 3^12/2^19 =", round(3**12 / 2**19, 6), "=", round(1200 *
math.log2(3**12 / 2**19), 2), "cents")
```

Output:

```
drift per day: 0.18 h; days to drift one full cycle: 133.3
ratio tau/24 = 1.0075 = 12.94 cents
Pythagorean comma 3^12/2^19 = 1.013643 = 23.46 cents
```

Calculation 16.2 — The clock loop: secant thresholds and loop parity.

```
        import math
        for n in (2, 3, 4, 5):
            b = float("inf") if n == 2 else (1 / math.cos(math.pi / n)) ** n
            print(f"{n}-stage negative ring: oscillation needs gain > {b:.4f}")
        for k in range(1, 5):
            print(f"{k} repressions in the loop: sign {(-1)**k:+d}")
```

Output:

```
2-stage negative ring: oscillation needs gain > inf
3-stage negative ring: oscillation needs gain > 8.0000
4-stage negative ring: oscillation needs gain > 4.0000
5-stage negative ring: oscillation needs gain > 2.8854
1 repressions in the loop: sign -1
2 repressions in the loop: sign +1
3 repressions in the loop: sign -1
4 repressions in the loop: sign +1
```

Calculation 16.3 — Sleep cycles and internal desynchrony as locked ratios.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        from fractions import Fraction
        print("24 h / 90 min =", 24 * 60 / 90, "; ladder", ladder(16))
        print("7.5 h / 90 min =", 7.5 * 60 / 90, "cycles; ladder of 5:", ladder(5))
        print("24 h / 60 min =", 24, "; ladder", ladder(24), "; 24 h / 50 min =", round(24
* 60 / 50, 2))
        for sw in (12.5, 25, 50):   # sleep-wake periods within Wever's 12-68 h range,
temperature ~25 h
            print(f"sleep-wake {sw} h : temperature 25 h = {Fraction(sw / 25).limit_denomina
tor(6)}")
```

Output:

```
24 h / 90 min = 16.0 ; ladder (4, 0)
7.5 h / 90 min = 5.0 cycles; ladder of 5: None
24 h / 60 min = 24 ; ladder (3, 1) ; 24 h / 50 min = 28.8
sleep-wake 12.5 h : temperature 25 h = 1/2
sleep-wake 25 h : temperature 25 h = 1
sleep-wake 50 h : temperature 25 h = 2
```

Calculation 16.4 — Menstrual cycle against the synodic month.

```
        m, sd = 29.3, 5.2                 # Bull et al. 2019
        L = 29.530589                     # mean synodic month, days
        print("follicular 16.9 + luteal 12.4 =", round(16.9 + 12.4, 1))
        print("month - mean cycle =", round(L - m, 3), "d =", round((L - m) / sd, 3), "SD")

        print("share of 28-day cycles: 81,605 / 612,613 =", round(81605 / 612613, 3))

        print("decline 25 -> 45 years at 0.18 d/yr:", round(0.18 * 20, 1),
"d (paper reports 3.2 d)")
```

Output:

```
follicular 16.9 + luteal 12.4 = 29.3
month - mean cycle = 0.231 d = 0.044 SD
share of 28-day cycles: 81,605 / 612,613 = 0.133
decline 25 -> 45 years at 0.18 d/yr: 3.6 d (paper reports 3.2 d)
```

Calculation 16.5 — Gompertz doubling and the record ages.

```
        for a0, a1 in [(30, 110), (40, 90), (30, 122)]:
            print(f"hazard factor {a0} -> {a1}: 2^({a1 - a0}/8) = {2 ** ((a1 - a0) / 8):.
0f}")
        print("annual factor 2^(1/8) =", round(2 ** (1 / 8), 4))
        print("122 - 115 =", 122 - 115, "years =", 7 / 8, "hazard doublings")
```

Output:

```
hazard factor 30 -> 110: 2^(80/8) = 1024
hazard factor 40 -> 90: 2^(50/8) = 76
hazard factor 30 -> 122: 2^(92/8) = 2896
```

```
annual factor 2^(1/8) = 1.0905
122 - 115 = 7 years = 0.875 hazard doublings
```

Calculation 16.6 — Cell counts, divisions and telomeres.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        import math
        print("log2(3.72e13) =", round(math.log2(3.72e13), 2), "; log3 =", round(math.log(3.
72e13, 3), 2))
        print("2^40 =", f"{2**40:.3e}", "; 2^60 =", f"{2**60:.3e}")
        print("telomere repeat TTAGGG:", len("TTAGGG"), "bp; ladder", ladder(6))
        print("loss 50-100 bp per division =", round(50 / 6, 1), "to", round(100 / 6, 1), "r
epeats")
```

Output:

```
log2(3.72e13) = 45.08 ; log3 = 28.44
2^40 = 1.100e+12 ; 2^60 = 1.153e+18
telomere repeat TTAGGG: 6 bp; ladder (1, 1)
loss 50-100 bp per division = 8.3 to 16.7 repeats
```

Calculation 16.7 — Stage counts of the life course against the ladder.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        for name, n in [("Piaget stages", 4), ("Erikson stages", 8), ("Erikson + Joan
Erikson", 9),
                        ("hallmarks of ageing", 12), ("Horvath clock CpGs", 353), ("Horvath
tissues", 51),
                        ("ABCD research sites", 21)]:
            print(f"{name:24s} {n:4d} ladder {ladder(n)}")
```

Output:

```
Piaget stages               4 ladder (2, 0)
Erikson stages              8 ladder (3, 0)
Erikson + Joan Erikson      9 ladder (0, 2)
hallmarks of ageing        12 ladder (2, 1)
Horvath clock CpGs        353 ladder None
Horvath tissues            51 ladder None
ABCD research sites        21 ladder None
```

Calculation 16.8 — Balanced-ternary addresses of ages (T12).

```
        def bt(n):
            s = ""
            while n != 0:
                d = ((n + 1) % 3) - 1
                s = "T01"[d + 1] + s
                n = (n - d) // 3
            return s or "0"
```

```
        for age in (1, 4, 13, 40, 73, 115, 122):
            print(age, "->", bt(age), "digits:", len(bt(age)))
        print("capacities (3^k - 1)/2:", [(3**k - 1) // 2 for k in range(1, 6)])
```

Output:

```
1 -> 1 digits: 1
4 -> 11 digits: 2
13 -> 111 digits: 3
40 -> 1111 digits: 4
73 -> 10T01 digits: 5
115 -> 111T1 digits: 5
122 -> 1TTTTT digits: 6
capacities (3^k - 1)/2: [1, 4, 13, 40, 121]
```

Calculation 16.9 — Meiosis: meetings and the triploid failure.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        for n, name in [(23, "human haploid set"), (11, "banana"), (30, "cattle")]:
            print(f"{name}: balanced gamete from a triploid 2*(1/2)^{n} = {2 * 0.5 ** n:.
3e}")
        print("one meiosis -> 4 haploid products, ladder", ladder(4))
```

Output:

```
human haploid set: balanced gamete from a triploid 2*(1/2)^23 = 2.384e-07
banana: balanced gamete from a triploid 2*(1/2)^11 = 9.766e-04
cattle: balanced gamete from a triploid 2*(1/2)^30 = 1.863e-09
one meiosis -> 4 haploid products, ladder (2, 0)
```

Calculation 16.10 — Aneuploidy and triploidy figures as rates.

```
        print("aneuploidy: newborns ~0.3%, stillbirths ~4%, spontaneous abortions >35%
(Nagaoka et al. 2012)")
        print("ratio abortions / newborns >=", round(35 / 0.3))
        print("triploidy: 2-3% of pregnancies, ~15% of miscarriages")
```

Output:

```
aneuploidy: newborns ~0.3%, stillbirths ~4%, spontaneous abortions >35% (Nagaoka et
al. 2012)
ratio abortions / newborns >= 117
triploidy: 2-3% of pregnancies, ~15% of miscarriages
```

Calculation 16.11 — IVF and population figures.

```
        print("IVF births 1978-2018: 10-13 million; to 2024: 13-17 million -> added", 13 - 1
0, "to", 17 - 13, "million")
        print("world population 8.2 bn (2024) -> peak 10.3 bn: factor", round(10.3 / 8.2, 3)
)
        print("life expectancy 73.3 y =", round(73.3 * 365.2425), "days =", round(73.3 * 365
.2425 * 24 / 24.18), "free-running cycles")
```

Output:

```
IVF births 1978-2018: 10-13 million; to 2024: 13-17 million -> added 3 to 4 million
world population 8.2 bn (2024) -> peak 10.3 bn: factor 1.256
life expectancy 73.3 y = 26772 days = 26573 free-running cycles
```

Calculation 16.12 — Theorem checklist for Chapter 16.

```
        checks = {
         "T1/T7.1 digit, carry": "circadian drift accumulates as carry (16.1); clock loop
parity (16.2)",
         "T2/T7 ring of three": "minimal oscillating transcriptional loop n = 3, gain 8
(16.2)",
         "T6 baseline": "ladder share of small integers (Ch. 14, 14.3)",
         "T8-T10": "not applied",
         "T11 seams": "not applied",
         "T12 address": "ages in balanced ternary (16.8): descriptive only",
         "T13": "not applied",
         "T14": "not applied",
         "T15 ladder": "16 sleep cycles/day on, 5 per night off (16.3); stage counts 4, 8,
9, 12 on, 51, 353 off (16.7); meiosis 4 (16.9)",
         "T16-T18 comma": "tau/24 = 12.94 cents vs comma 23.46 cents: no match (16.1)",

         "T19-T22": "not applied",
         "T23 coupling as state function": "reading only (entrainment strength depends on
light state)",
         "T24": "not applied",
         "period doubling": "Wever desynchrony 1:2 and 2:1 locks (16.3)",
        }
        for k, v in checks.items():
            print(f"{k:32s} {v}")
```

Output:

```
T1/T7.1 digit, carry             circadian drift accumulates as carry (16.1); clock
loop parity (16.2)
T2/T7 ring of three              minimal oscillating transcriptional loop n = 3,
gain 8 (16.2)
T6 baseline                      ladder share of small integers (Ch. 14, 14.3)
T8-T10                           not applied
T11 seams                        not applied
T12 address                      ages in balanced ternary (16.8): descriptive only
T13                              not applied
T14                              not applied
T15 ladder                       16 sleep cycles/day on, 5 per night off (16.3);
stage counts 4, 8, 9, 12 on, 51, 353 off (16.7); meiosis 4 (16.9)
T16-T18 comma                    tau/24 = 12.94 cents vs comma 23.46 cents: no match
(16.1)
T19-T22                          not applied
T23 coupling as state function   reading only (entrainment strength depends on light
state)
T24                              not applied
period doubling                  Wever desynchrony 1:2 and 2:1 locks (16.3)
```

Calculations for Chapter 17 — Healing

Calculation 17.1 — The frame rule as digit and carry (T1, T7.1).

```
        def digit_carry(s):
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        for L in range(1, 10):
            d, c = digit_carry(L)
            print(f"deletion of {L} nt: digit {d:+d}, carry {c:+d} ->", "in frame" if d ==
0 else "frameshift")
        print("1+1+1 -> digit, carry:", digit_carry(3), "; 1 + (-1) ->", digit_carry(0))
```

Output:

```
deletion of 1 nt: digit +1, carry +0 -> frameshift
deletion of 2 nt: digit -1, carry +1 -> frameshift
deletion of 3 nt: digit +0, carry +1 -> in frame
deletion of 4 nt: digit +1, carry +1 -> frameshift
deletion of 5 nt: digit -1, carry +2 -> frameshift
deletion of 6 nt: digit +0, carry +2 -> in frame
deletion of 7 nt: digit +1, carry +2 -> frameshift
deletion of 8 nt: digit -1, carry +3 -> frameshift
deletion of 9 nt: digit +0, carry +3 -> in frame
1+1+1 -> digit, carry: (0, 1) ; 1 + (-1) -> (0, 0)
```

Calculation 17.2 — Exon skipping restores closure.

```
        def digit_carry(s):
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        # residues of a deletion and of a skipped exon (in nucleotides mod 3)
        for r_del in (1, 2):
            for r_skip in (0, 1, 2):
                total = r_del + r_skip
                d, _ = digit_carry(total)
                print(f"deletion residue {r_del}, skipped exon residue {r_skip}: total
digit {d:+d} ->", "restored" if d == 0 else "still shifted")
```

Output:

```
deletion residue 1, skipped exon residue 0: total digit +1 -> still shifted
deletion residue 1, skipped exon residue 1: total digit -1 -> still shifted
deletion residue 1, skipped exon residue 2: total digit +0 -> restored
deletion residue 2, skipped exon residue 0: total digit -1 -> still shifted
deletion residue 2, skipped exon residue 1: total digit +0 -> restored
deletion residue 2, skipped exon residue 2: total digit +1 -> still shifted
```

Calculation 17.3 — How much the frame rule explains beyond chance.

```
        # chance agreement if phenotype were independent of frame, with 1/3 of deletions in
frame (T6, uniform)
        for p_becker in (0.1, 0.2, 0.3):
            print(f"P(Becker) = {p_becker}: chance agreement = {p_becker/3 + (1 -
p_becker)*2/3:.3f}  (observed 0.91)")
```

Output:

```
P(Becker) = 0.1: chance agreement = 0.633  (observed 0.91)
P(Becker) = 0.2: chance agreement = 0.600  (observed 0.91)
P(Becker) = 0.3: chance agreement = 0.567  (observed 0.91)
```

Calculation 17.4 — Gene editing by NHEJ: in-frame share of random indels.

```
        from fractions import Fraction
        print("uniform residues: in-frame", Fraction(1, 3), "frameshift", Fraction(2, 3), "=
", round(2 / 3, 4))
        for k in (1, 2):
            print(f"two independent alleles knocked out (both frameshift): (2/3)^2 = {(2/3)*
*2:.4f}" if k == 2 else "one allele frameshift: 2/3 = 0.6667")
```

Output:

```
uniform residues: in-frame 1/3 frameshift 2/3 = 0.6667
one allele frameshift: 2/3 = 0.6667
two independent alleles knocked out (both frameshift): (2/3)^2 = 0.4444
```

Calculation 17.5 — Wenckebach and 2:1 block as ratios.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        from fractions import Fraction
        for X in range(2, 7):
            print(f"{X}:{X-1}  conducted fraction {Fraction(X - 1, X)}  group length {X}
ladder {ladder(X)}")
```

Output:

```
2:1  conducted fraction 1/2  group length 2 ladder (1, 0)
3:2  conducted fraction 2/3  group length 3 ladder (0, 1)
4:3  conducted fraction 3/4  group length 4 ladder (2, 0)
5:4  conducted fraction 4/5  group length 5 ladder None
6:5  conducted fraction 5/6  group length 6 ladder (1, 1)
```

Calculation 17.6 — Chirality: two enantiomers and their interconversion.

```
        # a ring of three has two orientations (T4: the two non-zero homogeneous closures)

        for ring in [(1, 1, 1), (-1, -1, -1)]:
            s = sum(ring); d = ((s + 1) % 3) - 1; c = (s - d) // 3
            print(ring, "digit", d, "carry", c)
        print("racemate: 1:1 mix of the two -> net carry", (1 + -1))
```

Output:

```
(1, 1, 1) digit 0 carry 1
(-1, -1, -1) digit 0 carry -1
racemate: 1:1 mix of the two -> net carry 0
```

Calculation 17.7 — Insulin assembly and its undoing.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
```

```
        print("hexamer = 3 x dimer = 6 monomers:", ladder(6), "; lispro returns to the
monomer: 1 =", ladder(1))
```

Output:

```
hexamer = 3 x dimer = 6 monomers: (1, 1) ; lispro returns to the monomer: 1 = (0, 0)
```

Calculation 17.8 — AAV capsid, the Eisenstein norms (T14) and the dystrophin payload.

```
        norms = sorted({a*a + a*b + b*b for a in range(6) for b in range(6) if 0 < a*a + a*b
 + b*b <= 13})
        print("triangulation numbers T = h^2 + hk + k^2 up to 13:", norms)
        print("AAV: 60 subunits = 60 x T with T =", 60 // 60)
        print("VP1:VP2:VP3 = 1:1:10 -> 60 subunits as 5:5:50 =", 5 + 5 + 50)
        print("dystrophin coding bases 3,685 aa x 3 =", 3685 * 3, "(+3 for the stop codon
=", 3685 * 3 + 3, ") vs AAV genome ~4,700 nt: ratio", round(3685 * 3 / 4700, 2))
```

Output:

```
triangulation numbers T = h^2 + hk + k^2 up to 13: [1, 3, 4, 7, 9, 12, 13]
AAV: 60 subunits = 60 x T with T = 1
VP1:VP2:VP3 = 1:1:10 -> 60 subunits as 5:5:50 = 60
dystrophin coding bases 3,685 aa x 3 = 11055 (+3 for the stop codon = 11058 ) vs AAV
genome ~4,700 nt: ratio 2.35
```

Calculation 17.9 — Einthoven: three leads, one closure, twelve leads.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        print("I + III - II = 0 for every heart vector (Calculation 14.6)")
        print("12 leads:", ladder(12), "; 10 electrodes:", ladder(10), "; 6 limb leads:",
ladder(6), "; 6 chest leads:", ladder(6))
```

Output:

```
I + III - II = 0 for every heart vector (Calculation 14.6)
12 leads: (2, 1) ; 10 electrodes: None ; 6 limb leads: (1, 1) ; 6 chest leads: (1,
1)
```

Calculation 17.10 — Veterinary and dairy numbers.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        print("milk 524 Mt (1992) -> 930 Mt (2022): factor", round(930 / 524, 3), "; annual
growth", round(100 * ((930 / 524) ** (1 / 30) - 1), 2), "%")
        print("ruminant stomach compartments 4:", ladder(4), "; bovine oestrous cycle 21
d:", ladder(21))
        print("Lyon 1761 -> rinderpest eradication 2011:", 2011 - 1761, "years")
```

Output:

```
milk 524 Mt (1992) -> 930 Mt (2022): factor 1.775 ; annual growth 1.93 %
ruminant stomach compartments 4: (2, 0) ; bovine oestrous cycle 21 d: None
Lyon 1761 -> rinderpest eradication 2011: 250 years
```

Calculation 17.11 — Nutrition numbers.

```
        def ladder(n):
            a = b = 0
            while n % 2 == 0: n //= 2; a += 1
            while n % 3 == 0: n //= 3; b += 1
            return (a, b) if n == 1 else None
        from fractions import Fraction
        print("essential amino acids 9 of 20:", Fraction(9, 20), "; 9 on ladder", ladder(9))

        print("Atwater 4 / 9 / 4 kcal/g: fat / carbohydrate =", 9 / 4)
        print("WHO 2022: obese share of overweight adults 890 / 2500 =", round(890 / 2500,
3))
        print("children stunted 149 M, wasted 45 M: ratio", round(149 / 45, 2))
```

Output:

```
essential amino acids 9 of 20: 9/20 ; 9 on ladder (0, 2)
Atwater 4 / 9 / 4 kcal/g: fat / carbohydrate = 2.25
WHO 2022: obese share of overweight adults 890 / 2500 = 0.356
children stunted 149 M, wasted 45 M: ratio 3.31
```

Calculation 17.12 — Psychiatric burden and the drama triangle.

```
        def digit_carry(s):
            d = ((s + 1) % 3) - 1
            return d, (s - d) // 3
        print("anxiety 470 M + depression 322 M =", 470 + 322, "M; of 1,200 M with any
disorder (overlap not removed)")
        idx = {"P": 0, "R": 1, "V": 2}
        for cyc in [("V", "R", "P"), ("V", "P", "R")]:
            steps = []
            for i in range(3):
                st = (idx[cyc[(i + 1) % 3]] - idx[cyc[i]]) % 3
                steps.append(st if st < 2 else -1)
            print("role cycle", cyc, "steps", steps, "digit, carry", digit_carry(sum(steps))
)
        print("rapid cycling: >= 4 episodes per year; ladder of 4 is (2, 0)")
```

Output:

```
anxiety 470 M + depression 322 M = 792 M; of 1,200 M with any disorder (overlap not
removed)
role cycle ('V', 'R', 'P') steps [-1, -1, -1] digit, carry (0, -1)
role cycle ('V', 'P', 'R') steps [1, 1, 1] digit, carry (0, 1)
rapid cycling: >= 4 episodes per year; ladder of 4 is (2, 0)
```

Calculation 17.13 — Free-path lever as a reading only (T24).

```
        P3 = 0.3405373
        print("lever of a free 3D path:", round(-(1 + P3) / P3, 4))
```

Output:

```
lever of a free 3D path: -3.9365
```

Calculation 17.14 — Theorem checklist for Chapter 17.

```
        checks = {
         "T1/T7.1 digit, carry": "frame rule (17.1); exon skipping (17.2); NHEJ (17.4)",

         "T2-T4 window, two kinds": "codon window; enantiomer pair as the two homogeneous
rings (17.6)",
         "T6 baseline": "chance agreement 0.57-0.63 vs observed 0.91 (17.3); 1/3 in-frame
indels (17.4)",
         "T7 standing/meeting/ring": "pulse vs continuous dosing (text); insulin hexamer
(17.7); drama triangle ring (17.12)",
         "T8-T10": "collagen glycine (Ch. 14) behind osteogenesis imperfecta; not
recomputed",
         "T11 seams": "exon junctions in skipping (17.2)",
         "T12 address": "not applied",
         "T13": "not applied",
         "T14 Z[w]": "capsid T-numbers are Eisenstein norms (17.8)",
         "T15 ladder": "Wenckebach groups 2, 3, 4 on, 5 off (17.5); ECG 12 on, 10 off
(17.9); dairy and nutrition counts mixed (17.10, 17.11)",
         "T16-T18": "not applied",
         "T19-T20 chirality": "thalidomide enantiomers: chirality not conserved in vivo
(17.6, text)",
         "T21-T22": "not applied",
         "T23 coupling": "receptor desensitisation under continuous agonist (text): reading
only",
         "T24 lever": "computed for reference only (17.13); no clinical data compared",

        }
        for k, v in checks.items():
            print(f"{k:26s} {v}")
```

Output:

```
T1/T7.1 digit, carry       frame rule (17.1); exon skipping (17.2); NHEJ (17.4)
T2-T4 window, two kinds    codon window; enantiomer pair as the two homogeneous
rings (17.6)
T6 baseline                chance agreement 0.57-0.63 vs observed 0.91 (17.3); 1/3
in-frame indels (17.4)
T7 standing/meeting/ring   pulse vs continuous dosing (text); insulin hexamer
(17.7); drama triangle ring (17.12)
T8-T10                     collagen glycine (Ch. 14) behind osteogenesis imperfecta;
not recomputed
T11 seams                  exon junctions in skipping (17.2)
T12 address                not applied
T13                        not applied
T14 Z[w]                   capsid T-numbers are Eisenstein norms (17.8)
T15 ladder                 Wenckebach groups 2, 3, 4 on, 5 off (17.5); ECG 12 on, 10
off (17.9); dairy and nutrition counts mixed (17.10, 17.11)
T16-T18                    not applied
T19-T20 chirality          thalidomide enantiomers: chirality not conserved in vivo
(17.6, text)
T21-T22                    not applied
T23 coupling               receptor desensitisation under continuous agonist (text):
reading only
```

```
T24 lever                  computed for reference only (17.13); no clinical data
compared
```

Calculations for Chapter 18 — Social Structure

Every number below was computed for this chapter; the code ran on 27 October 2026 with Python 3.13 and the output shown is what it printed. Data files are named relative to a working folder; their public origin is stated with each calculation.

Calculation 18.1 — Closure mod 3 against balance on the four complete signed triangles. Digit and carry by the rule of Foundations T1; strong balance (Heider; Cartwright and Harary) and weak balance (Davis).

```
        # Signed complete triangles: closure mod 3 (Foundations T1, T2, T4, T7.1) against
Heider/Cartwright-Harary
        # balance (product of signs > 0) and Davis weak balance (no cycle with exactly one
negative edge).
        for t in [(1,1,1),(1,1,-1),(1,-1,-1),(-1,-1,-1)]:
            s=sum(t); d=((s+1)%3)-1; carry=(s-d)//3
            print(t, "sum",s,"digit",d,"carry",carry,
                  "| mod3:", "closed" if d==0 else "open",
                  "| strong balance:", "yes" if t[0]*t[1]*t[2]>0 else "no",
                  "| weak balance:", "no" if t.count(-1)==1 else "yes")
```

Output:

```
(1, 1, 1) sum 3 digit 0 carry 1 | mod3: closed | strong balance: yes | weak balance:
yes
(1, 1, -1) sum 1 digit 1 carry 0 | mod3: open | strong balance: no | weak balance:
no
(1, -1, -1) sum -1 digit -1 carry 0 | mod3: open | strong balance: yes | weak
balance: yes
(-1, -1, -1) sum -3 digit 0 carry -1 | mod3: closed | strong balance: no | weak
balance: yes
```

Calculation 18.2 — Signed triangle census of five public networks. Data: epinions.csv, slashdot.csv and wikirfa.csv from the datasets folder of the GitHub repository SherylHYX/pytorch_geometric_signed_directed (signed edge lists originally published by SNAP, Stanford); btc_alpha.csv and btc_otc.csv from the input folder of benedekrozemberczki/SGCN (Bitcoin Alpha and OTC trust networks, also SNAP), with the header and empty lines removed. Run as python3 02_triads.py epinions.csv slashdot.csv wikirfa.csv btc_alpha.csv btc_otc.csv. Edges are made undirected; pairs with opposite signs in the two directions are dropped. The null model is a shuffle of signs, as in Leskovec et al. (2010): expected share C(3,k)p^k(1-p)^(3-k), surprise (observed − expected)/sqrt(Δ·p0·(1−p0)).

```
        import sys, collections, math, random
        def load(path):
            d={}
            for line in open(path):
                a,b,s=line.strip().split(',')[:3]
                a=int(a);b=int(b);s=1 if float(s)>0 else -1
                if a==b: continue
                k=(min(a,b),max(a,b))
```

```
                d.setdefault(k,set()).add(s)
            und={k:next(iter(v)) for k,v in d.items() if len(v)==1}
            conflict=sum(1 for v in d.values() if len(v)>1)
            return und,conflict
        def triads(und):
            adj=collections.defaultdict(dict)
            for (a,b),s in und.items(): adj[a][b]=s; adj[b][a]=s
            cnt=collections.Counter()
            deg={n:len(v) for n,v in adj.items()}
            order=lambda n:(deg[n],n)
            for u in adj:
                nu=[v for v in adj[u] if order(v)>order(u)]
                su=set(nu)
                for v in nu:
                    for w in adj[v]:
                        if order(w)>order(v) and w in su:
                            npos=(adj[u][v]>0)+(adj[v][w]>0)+(adj[u][w]>0)
                            cnt[npos]+=1
            return cnt
        for name in sys.argv[1:]:
            und,conf=load(name)
            E=len(und); p=sum(1 for s in und.values() if s>0)/E
            c=triads(und); D=sum(c.values())
            print(name.split('/')[-1], "edges",E,"conflicting pairs dropped",conf,"p+=%.
4f"%p,"triangles",D)
            for k,lab in [(3,'+++'),(2,'++-'),(1,'+--'),(0,'---')]:
                pk=math.comb(3,k)*p**k*(1-p)**(3-k)
                obs=c[k]; exp=pk*D
                s=(obs-exp)/math.sqrt(D*pk*(1-pk))
                print("  %s obs %8d frac %.4f  expected frac %.4f  surprise %+9.1f  mod3
%s  balance %s"%(lab,obs,obs/D,pk,s,
                      'closed' if (k*1+(3-k)*(-1))%3==0 else 'open','yes' if (3-k)%2==0
else 'no'))
```

Output:

```
epinions.csv edges 708507 conflicting pairs dropped 2703 p+=0.8326 triangles 4770102
  +++ obs  3939584 frac 0.8259  expected frac 0.5771  surprise   +1099.8  mod3
closed  balance yes
  ++- obs   397612 frac 0.0834  expected frac 0.3482  surprise   -1214.0  mod3 open
balance no
  +-- obs   377150 frac 0.0791  expected frac 0.0700  surprise     +77.5  mod3 open
balance yes
  --- obs    55756 frac 0.0117  expected frac 0.0047  surprise    +223.6  mod3
closed  balance no
slashdot.csv edges 498532 conflicting pairs dropped 1949 p+=0.7641 triangles 571127
  +++ obs   417000 frac 0.7301  expected frac 0.4461  surprise    +431.8  mod3
closed  balance yes
  ++- obs    64224 frac 0.1125  expected frac 0.4132  surprise    -461.6  mod3 open
balance no
  +-- obs    77876 frac 0.1364  expected frac 0.1276  surprise     +19.9  mod3 open
balance yes
  --- obs    12027 frac 0.0211  expected frac 0.0131  surprise     +52.7  mod3
closed  balance no
wikirfa.csv edges 170675 conflicting pairs dropped 887 p+=0.7763 triangles 1185668
  +++ obs   766708 frac 0.6466  expected frac 0.4678  surprise    +390.2  mod3
closed  balance yes
  ++- obs   283760 frac 0.2393  expected frac 0.4044  surprise    -366.3  mod3 open
balance no
  +-- obs   116767 frac 0.0985  expected frac 0.1165  surprise     -61.3  mod3 open
balance yes
```

```
  --- obs    18433 frac 0.0155  expected frac 0.0112  surprise     +45.1  mod3
closed  balance no
btc_alpha.csv edges 14081 conflicting pairs dropped 0 p+=0.9068 triangles 21677
  +++ obs    16838 frac 0.7768  expected frac 0.7457  surprise     +10.5  mod3
closed  balance yes
  ++- obs     2973 frac 0.1371  expected frac 0.2299  surprise     -32.4  mod3 open
balance no
  +-- obs     1727 frac 0.0797  expected frac 0.0236  surprise     +54.3  mod3 open
balance yes
  --- obs      139 frac 0.0064  expected frac 0.0008  surprise     +29.0  mod3
closed  balance no
btc_otc.csv edges 21434 conflicting pairs dropped 0 p+=0.8529 triangles 32944
  +++ obs    23365 frac 0.7092  expected frac 0.6204  surprise     +33.2  mod3
closed  balance yes
  ++- obs     3875 frac 0.1176  expected frac 0.3210  surprise     -79.1  mod3 open
balance no
  +-- obs     5378 frac 0.1632  expected frac 0.0554  surprise     +85.6  mod3 open
balance yes
  --- obs      326 frac 0.0099  expected frac 0.0032  surprise     +21.6  mod3
closed  balance no
```

Calculation 18.3 — T6 baselines for complete triangles. T6 with p0 = 0 gives a closure probability $p^3 + q^3$; the balance null is $p^3 + 3pq^2$. Observed shares from Calculation 18.2.

```
        # T6 closure probability for a full signed triangle with independent signs (p0=0),
versus the balance null,
        # and the observed shares from Calculation 18.1.
        import math
        data=[("Epinions",0.8326,0.8259,0.0791,0.0117),("Slashdot",0.7641,0.7301,0.1364,0.02
11),
              ("WikiRfA",0.7763,0.6466,0.0985,0.0155),("BTC-Alpha",0.9068,0.7768,0.0797,0.00
64),("BTC-OTC",0.8529,0.7092,0.1632,0.0099)]
        w=complex(-0.5,math.sqrt(3)/2)
        for name,p,ppp,pmm,mmm in data:
            ph=lambda z:p*z+(1-p)*z*z
            t6=((1+ph(w)**3+ph(w*w)**3)/3).real
            print("%-9s T6 closed %.4f observed closed %.4f | balance null %.4f observed
balanced %.4f"%(
                name,t6,ppp+mmm,p**3+3*p*(1-p)**2,ppp+pmm))
```

Output:

```
Epinions  T6 closed 0.5819 observed closed 0.8376 | balance null 0.6472 observed
balanced 0.9050
Slashdot  T6 closed 0.4592 observed closed 0.7512 | balance null 0.5737 observed
balanced 0.8665
WikiRfA   T6 closed 0.4790 observed closed 0.6621 | balance null 0.5844 observed
balanced 0.7451
BTC-Alpha T6 closed 0.7465 observed closed 0.7832 | balance null 0.7693 observed
balanced 0.8565
BTC-OTC   T6 closed 0.6236 observed closed 0.7191 | balance null 0.6758 observed
balanced 0.8724
```

Calculation 18.4 — The decisive statistic: sign of the third edge opposite two negative edges. Same data and command line as Calculation 18.2. The statistic and both null models were fixed in the code comment before the run.

```
        # Decisive statistic, fixed before running.
        # For every triangle and every apex whose two edges are both negative, record the
sign of the third edge.
        # Strong balance (Heider; Cartwright-Harary): third edge positive ("enemy of my
enemy is my friend").
        # Closure mod 3 (Foundations T2-T4): (-,-,-) sums to -3 = 0 mod 3 (closed); (-,-,+)
sums to -1 (open): third edge negative.
        # Null 1: global share of negative edges. Null 2: node-propensity null -- each third
edge (a,b) gets the empirical
        # negative rate of all edges whose endpoints fall in the same (decile of a's
negativity, decile of b's negativity) cell,
        # with the edge itself excluded from its endpoints' rates.
        import sys, collections, math
        import numpy as np
        def load(path):
            d={}
            for line in open(path):
                p=line.strip().split(',')
                if len(p)<3 or p[2]=='': continue
                try: a=int(p[0]);b=int(p[1]);s=1 if float(p[2])>0 else -1
                except: continue
                if a==b: continue
                d.setdefault((min(a,b),max(a,b)),set()).add(s)
            return {k:next(iter(v)) for k,v in d.items() if len(v)==1}
        for name in sys.argv[1:]:
            und=load(name)
            adj=collections.defaultdict(dict)
            neg=collections.Counter(); deg=collections.Counter()
            for (a,b),s in und.items():
                adj[a][b]=s; adj[b][a]=s; deg[a]+=1; deg[b]+=1
                if s<0: neg[a]+=1; neg[b]+=1
            def rate(x,e_neg):
                d=deg[x]-1
                return (neg[x]-e_neg)/d if d>0 else 0.0
            # decile edges by excluded rates
            keys=list(und.keys())
            ra=np.array([rate(a,und[(a,b)]<0) for a,b in keys]); rb=np.array([rate(b,und[(a,
b)]<0) for a,b in keys])
            allr=np.concatenate([ra,rb]); qs=np.quantile(allr,np.linspace(0,1,11)[1:-1])

            ba=np.searchsorted(qs,ra,side='right'); bb=np.searchsorted(qs,rb,side='right')

            lo=np.minimum(ba,bb); hi=np.maximum(ba,bb)
            isneg=np.array([und[k]<0 for k in keys])
            cell_n=collections.Counter(); cell_neg=collections.Counter()
            for l,h,n in zip(lo,hi,isneg): cell_n[(l,h)]+=1; cell_neg[(l,h)]+=n
            cellp={k:cell_neg[k]/cell_n[k] for k in cell_n}
            eidx={k:i for i,k in enumerate(keys)}
            order=lambda n:(deg[n],n)
            obs_nn=0; tot_nn=0; exp_nn=0.0; var_nn=0.0
            obs_pn=0; tot_pn=0; exp_pn=0.0
            for u in adj:
                nu=[v for v in adj[u] if order(v)>order(u)]; su=set(nu)
                for v in nu:
                    for w in adj[v]:
                        if order(w)>order(v) and w in su:
                            tri=[((u,v),adj[u][v]),((v,w),adj[v][w]),((u,w),adj[u][w])]

                            for i in range(3):
                                e3,s3=tri[i]; s1=tri[(i+1)%3][1]; s2=tri[(i+2)%3][1]
                                k=(min(e3),max(e3)); j=eidx[k]; p=cellp[(lo[j],hi[j])]
```

```
                                if s1<0 and s2<0:
                                    tot_nn+=1; obs_nn+=(s3<0); exp_nn+=p; var_nn+=p*(1-p)

                                elif s1*s2<0:
                                    tot_pn+=1; obs_pn+=(s3<0); exp_pn+=p
            g=sum(isneg)/len(isneg)
            print(name.split('/')[-1])
            print("  (-,-) wedges closed by a triangle: %d ; third edge negative: observed %
.4f, global null %.4f, node-propensity null %.4f, z = %.1f"%(tot_nn,obs_nn/tot_nn,g,
exp_nn/tot_nn,(obs_nn-exp_nn)/math.sqrt(var_nn)))
            print("  (+,-) wedges closed by a triangle: %d ; third edge negative: observed %
.4f, global null %.4f, node-propensity null %.4f"%(tot_pn,obs_pn/tot_pn,g,exp_pn/tot
_pn))
```

Output:

```
epinions.csv
  (-,-) wedges closed by a triangle: 544418 ; third edge negative: observed 0.3072,
global null 0.1674, node-propensity null 0.4341, z = -253.0
  (+,-) wedges closed by a triangle: 1549524 ; third edge negative: observed 0.4868,
global null 0.1674, node-propensity null 0.4075
slashdot.csv
  (-,-) wedges closed by a triangle: 113957 ; third edge negative: observed 0.3166,
global null 0.2359, node-propensity null 0.4813, z = -133.6
  (+,-) wedges closed by a triangle: 284200 ; third edge negative: observed 0.5480,
global null 0.2359, node-propensity null 0.4161
wikirfa.csv
  (-,-) wedges closed by a triangle: 172066 ; third edge negative: observed 0.3214,
global null 0.2237, node-propensity null 0.3819, z = -59.3
  (+,-) wedges closed by a triangle: 801054 ; third edge negative: observed 0.2915,
global null 0.2237, node-propensity null 0.2923
btc_alpha.csv
  (-,-) wedges closed by a triangle: 2144 ; third edge negative: observed 0.1945,
global null 0.0932, node-propensity null 0.3142, z = -13.6
  (+,-) wedges closed by a triangle: 9400 ; third edge negative: observed 0.3674,
global null 0.0932, node-propensity null 0.2478
btc_otc.csv
  (-,-) wedges closed by a triangle: 6356 ; third edge negative: observed 0.1539,
global null 0.1471, node-propensity null 0.5312, z = -68.9
  (+,-) wedges closed by a triangle: 18506 ; third edge negative: observed 0.5812,
global null 0.1471, node-propensity null 0.4109
```

Calculation 18.5 — Wedge completion by sign type. Same data, run in the order btc_alpha, btc_otc, wikirfa, slashdot, epinions. Nodes of degree above 3,000 are skipped as wedge centres to bound the run time.

```
        # Prediction fixed before running: under closure mod 3 the wedge (+,-,absent) has
sum 0 (closed),
        # (+,+,absent) sum 2 and (-,-,absent) sum -2 (open). So the share of wedges that are
closed
        # by a third edge should be lowest for (+,-) wedges.
        import sys, collections
        def load(path):
            d={}
            for line in open(path):
                p=line.strip().split(',')
                if len(p)<3 or p[2]=='' : continue
                try: a=int(p[0]);b=int(p[1]);s=1 if float(p[2])>0 else -1
```

```
                except: continue
                if a==b: continue
                d.setdefault((min(a,b),max(a,b)),set()).add(s)
            return {k:next(iter(v)) for k,v in d.items() if len(v)==1}
        for name in sys.argv[1:]:
            und=load(name)
            adj=collections.defaultdict(dict)
            for (a,b),s in und.items(): adj[a][b]=s; adj[b][a]=s
            W=collections.Counter(); C=collections.Counter()
            for v,nb in adj.items():
                items=list(nb.items())
                if len(items)>3000: continue  # skip hubs to bound runtime
                for i in range(len(items)):
                    a,sa=items[i]; na=adj[a]
                    for j in range(i+1,len(items)):
                        b,sb=items[j]; key=sa+sb
                        W[key]+=1
                        if b in na: C[key]+=1
            print(name.split('/')[-1])
            for key,lab in [(2,'(+,+)'),(0,'(+,-)'),(-2,'(-,-)')]:
                print("  wedge %s n=%10d closed share %.4f"%(lab,W[key],C[key]/W[key]))
```

Output:

```
btc_alpha.csv
  wedge (+,+) n=    693137 closed share 0.0772
  wedge (+,-) n=    127666 closed share 0.0736
  wedge (-,-) n=     23616 closed share 0.0908
btc_otc.csv
  wedge (+,+) n=   1271110 closed share 0.0582
  wedge (+,-) n=    320183 closed share 0.0578
  wedge (-,-) n=     94459 closed share 0.0673
wikirfa.csv
  wedge (+,+) n=  17725085 closed share 0.1458
  wedge (+,-) n=   7997006 closed share 0.1002
  wedge (-,-) n=   1774910 closed share 0.0969
slashdot.csv
  wedge (+,+) n=  49705833 closed share 0.0265
  wedge (+,-) n=  16251401 closed share 0.0175
  wedge (-,-) n=   7139824 closed share 0.0160
epinions.csv
  wedge (+,+) n= 122023499 closed share 0.0997
  wedge (+,-) n=  35900559 closed share 0.0429
  wedge (-,-) n=  15369502 closed share 0.0354
```

Calculation 18.6 — Triad classes and the necklace count. Holland–Leinhardt classes by brute force; T13 necklace count g(3).

```
        # Orbit counts: Holland-Leinhardt directed triad classes; signed triangles up to
symmetry; necklaces g(3) (T13).
        import itertools
        from math import gcd
        nodes=range(3); arcs=[(i,j) for i in nodes for j in nodes if i!=j]; seen=set();
classes=0
        for bits in itertools.product([0,1],repeat=6):
            g=frozenset(a for a,b in zip(arcs,bits) if b)
            if g in seen: continue
            classes+=1
            for p in itertools.permutations(nodes): seen.add(frozenset((p[i],p[j]) for i,j i
n g))
```

```
        print("directed triad isomorphism classes:",classes)
        cl={tuple(sorted(s)) for s in itertools.product([-1,0,1],repeat=3)}
        print("triangles with edges in {-,absent,+} up to
symmetry:",len(cl),"; complete (no absent edge):",sum(0 not in c for c in cl))
        phi=lambda m:sum(1 for i in range(1,m+1) if gcd(i,m)==1)
        g=lambda n,k:sum(phi(n//d)*k**d for d in range(1,n+1) if n%d==0)//n
        print("necklaces of length 3 over 3 values g(3) =",g(3,3))
```

Output:

```
directed triad isomorphism classes: 16
triangles with edges in {-,absent,+} up to symmetry: 10 ; complete (no absent edge):
4
necklaces of length 3 over 3 values g(3) = 11
```

Calculation 18.7 — Rotation classes against full symmetry. The two cyclic orders of (T, 0, 1).

```
        # Rotation classes of 3-trit cyclic words (T13: g(3) = 11) against classes under the
full symmetric group (10):
        # the two cyclic orders of (-1, 0, +1) are distinct under rotation and merge under
reflection.
        import itertools
        rot={min(t[i:]+t[:i] for i in range(3)) for t in itertools.product([-1,0,
1],repeat=3)}
        full={tuple(sorted(t)) for t in itertools.product([-1,0,1],repeat=3)}
        print('rotation classes',len(rot),'; full-symmetry classes',len(full))
        print('chiral pair:',sorted(t for t in rot if len(set(t))==3))
```

Output:

```
rotation classes 11 ; full-symmetry classes 10
chiral pair: [(-1, 0, 1), (-1, 1, 0)]
```

Calculation 18.8 — Restricted and generalized exchange. A cycle of k groups returns a gift after k steps; only k = 2 is direct reciprocity. Digit and carry by T1 and T7.1.

```
        # Marriage exchange among k groups, each group giving to the next (i -> i+1 mod k).

        # Restricted exchange = the gift returns at once (a meeting); generalized exchange =
it returns only round the cycle (a ring).
        for k in range(2,7):
            reciprocal = all(((i+1)%k+1)%k==i for i in range(k))
            print("k=%d groups: first return after %d steps; direct reciprocity: %s"%(k,k,re
ciprocal))
        # Trit reading: a gift along the cycle orientation is +1; the return trip of a
meeting is -1.
        for name,w in [("meeting A->B, B->A",(1,-1)),("ring A->B->C->A",(1,1,1))]:
            s=sum(w); d=((s+1)%3)-1; print(name,"sum",s,"digit",d,"carry",(s-d)//3)
```

Output:

```
k=2 groups: first return after 2 steps; direct reciprocity: True
k=3 groups: first return after 3 steps; direct reciprocity: False
k=4 groups: first return after 4 steps; direct reciprocity: False
k=5 groups: first return after 5 steps; direct reciprocity: False
```

```
k=6 groups: first return after 6 steps; direct reciprocity: False
meeting A->B, B->A sum 0 digit 0 carry 0
ring A->B->C->A sum 3 digit 0 carry 1
```

Calculation 18.9 — Dunbar layers and the ladder. Figures 5, 15 and 148 (with interval 100–230) from the Wikipedia page on Dunbar's number.

```
        # Verified Dunbar figures (Wikipedia 'Dunbar's number'): core about 5, sympathy
group about 15, mean group size 148 (~150).
        # Check against the ring ladder x3 (T15, a=0) and the full ladder 2^a 3^b.
        lad={2**a*3**b for a in range(12) for b in range(8)}
        layers=[5,15,148]
        print("ratios:",[round(b/a,3) for a,b in zip(layers,layers[1:])])
        print("15/5 =",15/5,"; 148/15 = %.2f = 3^%.2f"%(148/15,__import__('math').log(148/
15,3)))
        print("on 2^a3^b:",[x for x in layers if x in lad],"; nearest ladder values:",[min(l
ad,key=lambda v:abs(v-x)) for x in layers])
        print("95%% CI of 148: 100-230 contains ladder values:",sorted(v for v in lad if
100<=v<=230))
```

Output:

```
ratios: [3.0, 9.867]
15/5 = 3.0 ; 148/15 = 9.87 = 3^2.08
on 2^a3^b: [] ; nearest ladder values: [4, 16, 144]
95%% CI of 148: 100-230 contains ladder values: [108, 128, 144, 162, 192, 216]
```

Calculation 18.10 — A Leslie matrix with one fertile age. An illustrative matrix with fertility 6 in the third age class and survival 1/2 and 1/3; exact arithmetic. The period of an imprimitive Leslie matrix is the gcd of the fertile ages.

```
        # Bernardelli (1941) beetle matrix: only age class 3 reproduces (6 offspring),
survival 1/2 and 1/3.
        # A Leslie matrix whose only fertile age is k is imprimitive with index k: the
population returns after k steps.
        from fractions import Fraction as F
        L=[[F(0),F(0),F(6)],[F(1,2),F(0),F(0)],[F(0),F(1,3),F(0)]]
        mul=lambda A,B:[[sum(A[i][k]*B[k][j] for k in range(3)) for j in range(3)] for i in
range(3)]
        P=L
        for n in range(2,4): P=mul(P,L); print("L^%d ="%n,[[str(x) for x in r] for r in P])

        x=[F(60),F(0),F(0)]
        for t in range(7):
            print("t=%d"%t,[str(v) for v in x]); x=[sum(L[i][j]*x[j] for j in range(3)) for
i in range(3)]
        from math import gcd
        for ages in [(3,),(2,4),(2,3),(3,6),(4,6)]:
            g=0
            for a in ages: g=gcd(g,a)
            print("fertile ages",ages,"-> imprimitivity index (period)",g)
```

Output:

```
L^2 = [['0', '2', '0'], ['0', '0', '3'], ['1/6', '0', '0']]
L^3 = [['1', '0', '0'], ['0', '1', '0'], ['0', '0', '1']]
t=0 ['60', '0', '0']
t=1 ['0', '30', '0']
```

```
t=2 ['0', '0', '10']
t=3 ['60', '0', '0']
t=4 ['0', '30', '0']
t=5 ['0', '0', '10']
t=6 ['60', '0', '0']
fertile ages (3,) -> imprimitivity index (period) 3
fertile ages (2, 4) -> imprimitivity index (period) 2
fertile ages (2, 3) -> imprimitivity index (period) 1
fertile ages (3, 6) -> imprimitivity index (period) 3
fertile ages (4, 6) -> imprimitivity index (period) 2
```

Calculation 18.11 — Pedigree doubling. Nominal ancestors 2^n against 8.2 billion.

```
        # Genealogical doubling (meeting ladder x2, T15 with b=0): nominal ancestors 2^n in
generation n.
        pop=8.2e9
        n=next(n for n in range(1,100) if 2**n>pop); print("first n with 2^n > 8.2e9:",n,"2^
n =",2**n)
        print("at 25-30 years per generation: %d to %d years"%(25*n,30*n))
```

Output:

```
first n with 2^n > 8.2e9: 33 2^n = 8589934592
at 25-30 years per generation: 825 to 990 years
```

Calculation 18.12 — Arithmetic on the UN figures. World Population Prospects 2024 and World Urbanization Prospects 2025 as cited in 18.2 and 18.3.

```
        # UN WPP 2024 and WUP 2025 arithmetic.
        print("growth to peak: %.3f (10.3/8.2)"%(10.3/8.2))
        print("countries past peak lose 14%% in 30 years: annual rate %.4f"%(0.86**(1/30)-1)
)
        print("urban population 1950 %.2f bn, 2025 %.2f bn, ratio %.2f"%(0.20*2.5,0.45*8.2,0
.45*8.2/(0.20*2.5)))
        print("megacities 8 (1975) -> 33 (2025): ratio %.3f, annual growth %.4f"%(33/8,(33/
8)**(1/50)-1))
```

Output:

```
growth to peak: 1.256 (10.3/8.2)
countries past peak lose 14% in 30 years: annual rate -0.0050
urban population 1950 0.50 bn, 2025 3.69 bn, ratio 7.38
megacities 8 (1975) -> 33 (2025): ratio 4.125, annual growth 0.0287
```

Calculation 18.13 — Löschian numbers as norms of Z[ω] and the ladder. Norm $N(a + b\omega) = a^2 - ab + b^2$.

```
        # Loesch market-area sizes K are the norms N(a+b*w)=a^2-ab+b^2 of Eisenstein
integers (OEIS A003136).
        lad={2**a*3**b for a in range(12) for b in range(8)}
        N=lambda a,b:a*a-a*b+b*b
        L=sorted({N(a,b) for a in range(-10,11) for b in range(-10,11)}-{0}); L=[x for x in
L if x<=50]
        print("K values <= 50:",L)
        print("on ladder 2^a3^b:",[x for x in L if x in lad]); print("off ladder:",[x for x
in L if x not in lad])
```

```
        print("N(1-w) =",N(1,-1),"(Christaller K=3, the prime above 3)")
        print("N(2) =",N(2,0),"(K=4)","; N(3+w) =",N(3,1),"(K=7)")
```

Output:

```
K values <= 50: [1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31, 36, 37, 39, 43,
48, 49]
on ladder 2^a3^b: [1, 3, 4, 9, 12, 16, 27, 36, 48]
off ladder: [7, 13, 19, 21, 25, 28, 31, 37, 39, 43, 49]
N(1-w) = 3 (Christaller K=3, the prime above 3)
N(2) = 4 (K=4) ; N(3+w) = 7 (K=7)
```

Calculation 18.14 — Kinship section counts and the ladder. Moieties, sections, subsections and the Lardil generation cycles as given on the Wikipedia page on Australian Aboriginal kinship.

```
        # Lardil subsections: men alternate between 2 subsections, women cycle through 4.
Moieties 2, sections 4, subsections 8.
        lad={2**a*3**b for a in range(6) for b in range(4)}
        for name,n in [("moieties",2),("sections",4),("subsections",8),("male cycle",2),("fe
male cycle",4)]:
            print(name,n,"= 2^%d"%(n.bit_length()-1),"on ladder" if n in lad else "off")

        print("powers of 3 among them:",[n for n in (2,4,8) if n%3==0])
```

Output:

```
moieties 2 = 2^1 on ladder
sections 4 = 2^2 on ladder
subsections 8 = 2^3 on ladder
male cycle 2 = 2^1 on ladder
female cycle 4 = 2^2 on ladder
powers of 3 among them: []
```

Calculation 18.15 — Clustering of the ring lattice. Formula checked against networkx 3.7.

```
        # Watts-Strogatz ring lattice: n nodes, each joined to K/2 neighbours on each side.
Closed formula 3(K-2)/(4(K-1)),
        # checked against a direct count with networkx; the limit is 3/4.
        import networkx as nx
        for K in (4,6,10,20):
            G=nx.watts_strogatz_graph(1000,K,0.0)
            print("K",K,"direct %.4f formula %.4f"%(nx.average_clustering(G),3*(K-2)/
(4*(K-1))))
        print("K 10^6 formula %.6f"%(3*(10**6-2)/(4*(10**6-1))))
```

Output:

```
K 4 direct 0.5000 formula 0.5000
K 6 direct 0.6000 formula 0.6000
K 10 direct 0.6667 formula 0.6667
K 20 direct 0.7105 formula 0.7105
K 10^6 formula 0.749999
```

Calculation 18.16 — The lever of T24 for walks on the square lattice. Exact return series of the two-dimensional simple random walk; F is the return probability within L steps.

```
        # T24 lever -(1+F)/F for a walk on the square lattice Z^2 (recurrent, Polya 1921),
finite length L.
        # Expected visits to the origin in L steps: U = sum_{n<=L/2} (C(2n,n)/4^n)^2 ;
return probability within the walk F = 1 - 1/U.
        from math import comb
        from fractions import Fraction
        for L in (10**2,10**3,10**4):
            U=sum(float(Fraction(comb(2*n,n),4**n)**2) for n in range(0,L//2+1))
            F=1-1/U
            print("L=%d U=%.4f F=%.4f lever=%.4f"%(L,U,F,-(1+F)/F))
```

Output:

```
L=100 U=2.3163 F=0.5683 lever=-2.7597
L=1000 U=3.0449 F=0.6716 lever=-2.4890
L=10000 U=3.7774 F=0.7353 lever=-2.3600
```

Calculations for Chapter 19 — Markets, Power and Communication

Every number below was computed for this chapter; the code ran on 27 October 2026 with Python 3.13 and the output shown is what it printed. Data files are named relative to a working folder; their public origin is stated with each calculation.

Calculation 19.1 — Majority cycles among all three-voter profiles. All $6^3 = 216$ profiles of strict rankings of A, B, C; digit and carry by T1 and T7.1; the impartial-culture limit $1 - (3/4 + (3/(2\pi))\cdot\arcsin(1/3))$.

```
        # Three voters, three alternatives, all 6^3 = 216 strict-order profiles: how many
give a majority cycle?
        # Trit reading: orient A->B->C->A; each pairwise majority agrees (+1) or disagrees
(-1) with the orientation.
        import itertools, math
        orders=list(itertools.permutations('ABC')); cyc=0; words={}
        for prof in itertools.product(orders,repeat=3):
            beats=lambda x,y: sum(o.index(x)<o.index(y) for o in prof)>=2
            t=tuple(1 if beats(x,y) else -1 for x,y in (('A','B'),('B','C'),('C','A')))

            words[t]=words.get(t,0)+1
            if abs(sum(t))==3: cyc+=1
        print("cyclic profiles:",cyc,"of 216 =",round(cyc/216,5))
        for t,n in sorted(words.items()):
            s=sum(t); d=((s+1)%3)-1
            print(t,"profiles",n,"sum",s,"digit",d,"carry",(s-d)//3,"closed" if d==0 else "o
pen")
        print("impartial-culture limit, 3 alternatives: %.5f"%(1-(0.75+3/(2*math.pi)*math.as
in(1/3))))
```

Output:

```
cyclic profiles: 12 of 216 = 0.05556
(-1, -1, -1) profiles 6 sum -3 digit 0 carry -1 closed
(-1, -1, 1) profiles 34 sum -1 digit -1 carry 0 open
```

```
(-1, 1, -1) profiles 34 sum -1 digit -1 carry 0 open
(-1, 1, 1) profiles 34 sum 1 digit 1 carry 0 open
(1, -1, -1) profiles 34 sum -1 digit -1 carry 0 open
(1, -1, 1) profiles 34 sum 1 digit 1 carry 0 open
(1, 1, -1) profiles 34 sum 1 digit 1 carry 0 open
(1, 1, 1) profiles 6 sum 3 digit 0 carry 1 closed
impartial-culture limit, 3 alternatives: 0.08774
```

Calculation 19.2 — Majority cycles in 68 real ranked-ballot elections. Data: PrefLib (GitHub repository PrefLib/PrefLib-Data), folders 00001 irish, 00002 debian, 00008 glasgow, 00016 aspen, 00017 berkley, 00018 minneapolis, 00019 oakland, 00020 pierce, 00021 sf, 00022 sl, 00023 takomapark, all .toc files. Run as python3 07_preflib_cycles.py preflib/datasets. Unranked candidates count as ranked below all ranked ones; elections with more than 30 candidates are skipped in the triple count. Prediction fixed in the first edition of this book.

```
        import sys, glob, itertools, math, os
        def parse(path):
            m=None; ballots=[]
            for line in open(path):
                if line.startswith('#'):
                    if 'NUMBER ALTERNATIVES' in line: m=int(line.split(':')[1])
                    continue
                if ':' not in line: continue
                cnt,order=line.split(':',1); cnt=int(cnt)
                ranks={}; r=0; s=order.strip(); i=0
                groups=[]
                while i<len(s):
                    if s[i]=='{':
                        j=s.index('}',i); groups.append([int(x) for x in s[i+1:j].split(',')
 if x.strip()]); i=j+1
                    elif s[i] in ', ': i+=1
                    else:
                        j=i
                        while j<len(s) and s[j] not in ', ': j+=1
                        groups.append([int(s[i:j])]); i=j
                for g in groups:
                    for a in g: ranks[a]=r
                    r+=1
                ballots.append((cnt,ranks))
            return m,ballots
        files=sorted(glob.glob(sys.argv[1]+'/*/*.toc'))+[f for f in sorted(glob.glob(sys.arg
v[1]+'/*/*.soc'))]
        tot=cw=0; trip=cyc=0
        for f in files:
            m,b=parse(f)
            W=[[0]*(m+1) for _ in range(m+1)]
            for cnt,rk in b:
                for a in range(1,m+1):
                    ra=rk.get(a,10**6)
                    for c in range(1,m+1):
                        if a!=c and ra<rk.get(c,10**6): W[a][c]+=cnt
            beats=lambda a,c: W[a][c]>W[c][a]
            has=any(all(beats(a,c) for c in range(1,m+1) if c!=a) for a in range(1,m+1))

            tot+=1; cw+=has
            if m>30: continue
            for a,c,d in itertools.combinations(range(1,m+1),3):
                trip+=1
```

```
                if (beats(a,c) and beats(c,d) and beats(d,a)) or (beats(c,a) and beats(d,c)
and beats(a,d)): cyc+=1; print("  3-cycle in",os.path.basename(f),"m=",m)
            if not has: print("no Condorcet winner:",os.path.basename(f),"m=",m)
        print("elections",tot,"with Condorcet winner",cw)
        print("triples",trip,"majority 3-cycles",cyc,"share %.5f"%(cyc/trip))
        print("baseline 3 voters 12/216 = %.
5f ; impartial culture n->inf for 3 alternatives = %.5f"%(12/216, 1-(0.75+3/(2*math.
pi)*math.asin(1/3))))
```

Output:

```
  3-cycle in 00008-00000009.toc m= 11
no Condorcet winner: 00008-00000009.toc m= 11
  3-cycle in 00008-00000021.toc m= 10
elections 68 with Condorcet winner 67
triples 9238 majority 3-cycles 2 share 0.00022
baseline 3 voters 12/216 = 0.05556 ; impartial culture n->inf for 3 alternatives =
0.08774
```

Calculation 19.3 — Closed windows in monthly S&P returns. Data: data/data.csv of the GitHub repository datasets/s-and-p-500 (Shiller's monthly series to June 2023, FRED prices after). Trit band ±1 % on monthly log changes; bear peaks are months after which the index fell 20 % before regaining the peak. Prediction fixed in the first edition of this book.

```
        # Prediction fixed before running (first edition of this book, V.2): closed-window
rate exceeds the T6 baseline in calm
        # regimes and falls below it in the 12 months before a bear-market peak (>=20%
fall). Equality everywhere refutes.
        import csv, numpy as np, cmath, random
        rows=[r for r in csv.DictReader(open('./data/datasets_s-and-
p-500_main_data_data.csv'))]
        P=np.array([float(r['SP500']) for r in rows]); D=[r['Date'] for r in rows]
        ret=np.diff(np.log(P))
        h=0.01
        t=np.where(ret>h,1,np.where(ret<-h,-1,0))
        def T6(trits):
            n=len(trits); p0=(trits==0).mean(); pp=(trits==1).mean(); pm=(trits==-1).mean()

            w=cmath.exp(2j*cmath.pi/3)
            ph=lambda z: p0+pp*z+pm*z*z
            return ((1+ph(w)**3+ph(w*w)**3)/3).real
        def closed(trits,idx=None):
            s=[(trits[k]+trits[k+1]+trits[k+2])%3==0 for k in range(len(trits)-2)]
            return np.array(s)
        cw=closed(t)
        print("months",len(t),"freq +,0,-: %.3f %.3f %.3f"%((t==1).mean(),(t==0).mean(),(t==
-1).mean()))
        print("all: closed share %.4f, T6 baseline %.4f"%(cw.mean(),T6(t)))
        # bear-market peaks: running max, decline >=20% from peak
        peaks=[]; i=0; n=len(P)
        while i<n:
            j=i
            # find peak i where later price falls 20% below before exceeding P[i]
            k=i+1; fell=False
            while k<n and P[k]<=P[i]:
                if P[k]<=0.8*P[i]: fell=True; break
                k+=1
            if fell:
```

```
                peaks.append(i)
                # move to recovery above peak
                while k<n and P[k]<P[i]: k+=1
                i=k
            else: i=max(i+1,k)
        print("bear peaks:",len(peaks),[D[p][:7] for p in peaks])
        pre=np.zeros(len(cw),bool); near=np.zeros(len(cw),bool)
        for p in peaks:
            for k in range(max(0,p-12),p):  # window starting k uses returns k..k+2 (months
k+1..k+3)
                if k<len(cw): pre[k]=True
            for k in range(max(0,p-24),min(len(cw),p+24)): near[k]=True
        calm=~near
        for name,m in [("pre-peak 12m",pre),("calm (>24m from any peak)",calm)]:
            idx=np.where(m)[0]
            sub=t[np.unique(np.concatenate([idx,idx+1,idx+2]))]
            print("%s: windows %d closed share %.4f  T6 baseline from same months %.4f"%(nam
e,len(idx),cw[idx].mean(),T6(sub)))
        # permutation: shuffle months, closed share distribution
        random.seed(1); arr=list(t); sims=[]
        for _ in range(2000):
            random.shuffle(arr); a=np.array(arr); sims.append(closed(a).mean())
        sims=np.array(sims); print("shuffle mean %.4f sd %.4f z(all)=%.2f"%(sims.mean(),sims
.std(),(cw.mean()-sims.mean())/sims.std()))
        # Decomposition and a first-order Markov baseline (does serial correlation alone
explain the excess?)
        hom=np.mean([t[k]==t[k+1]==t[k+2] for k in range(len(t)-2)])
        het=np.mean([len({t[k],t[k+1],t[k+2]})==3 for k in range(len(t)-2)])
        print("homogeneous share %.4f heterogeneous share %.4f"%(hom,het))
        S=[-1,0,1]; M=np.zeros((3,3))
        for a,b in zip(t[:-1],t[1:]): M[a+1,b+1]+=1
        pi=M.sum(1)/M.sum(); M=M/M.sum(1,keepdims=True)
        pm=sum(pi[i]*M[i,j]*M[j,k] for i in range(3) for j in range(3) for k in range(3) if
(S[i]+S[j]+S[k])%3==0)
        print("Markov(1) baseline %.4f ; lag-1 autocorrelation of monthly log returns %.3f"%
(pm,np.corrcoef(ret[:-1],ret[1:])[0,1]))
        # Significance of the pre-peak deficit against the same-month T6 baseline (binomial
approximation)
        import math
        idx=np.where(pre)[0]; obs=cw[idx].mean(); base=0.3483
        print("pre-peak deficit %.4f, binomial sd %.4f, z = %.2f"%(obs-
base,math.sqrt(base*(1-base)/len(idx)),(obs-base)/math.sqrt(base*(1-base)/len(idx)))
)
        # Markov(1) baseline estimated inside each regime
        def markov_base(mask):
            idx=np.where(mask)[0]; M=np.zeros((3,3))
            for k in idx:
                for a,b in ((t[k],t[k+1]),(t[k+1],t[k+2])): M[a+1,b+1]+=1
            pi=M.sum(1)/M.sum(); M=M/M.sum(1,keepdims=True)
            return sum(pi[i]*M[i,j]*M[j,k] for i in range(3) for j in range(3) for k in rang
e(3) if (S[i]+S[j]+S[k])%3==0)
        print("Markov(1) baseline: pre-peak %.4f, calm %.4f"%(markov_base(pre),markov_base(c
alm)))
```

Output:

```
months 1868 freq +,0,-: 0.468 0.240 0.291
all: closed share 0.3800, T6 baseline 0.3381
bear peaks: 13 ['1872-04', '1881-06', '1902-09', '1906-09', '1909-12', '1929-09',
'1961-12', '1968-12', '1973-01', '1987-08', '2000-08', '2007-10', '2021-12']
pre-peak 12m: windows 156 closed share 0.3397  T6 baseline from same months 0.3483
```

```
calm (>24m from any peak): windows 1260 closed share 0.3841  T6 baseline from same
months 0.3386
shuffle mean 0.3379 sd 0.0114 z(all)=3.67
homogeneous share 0.2117 heterogeneous share 0.1683
Markov(1) baseline 0.3756 ; lag-1 autocorrelation of monthly log returns 0.274
pre-peak deficit -0.0086, binomial sd 0.0381, z = -0.22
Markov(1) baseline: pre-peak 0.3531, calm 0.3735
```

Calculation 19.4 — Autocorrelation created by monthly averaging. A daily Gaussian random walk averaged over 21-day months.

```
        # Monthly AVERAGES of a daily random walk have autocorrelated changes. Simulate 21
trading days per month.
        import numpy as np
        rng=np.random.default_rng(0); days=21; months=200000
        x=np.cumsum(rng.standard_normal(days*months)); m=x.reshape(months,days).mean(1); d=n
p.diff(m)
        print("lag-1 autocorrelation of changes in monthly averages: %.
3f"%np.corrcoef(d[:-1],d[1:])[0,1])
```

Output:

```
lag-1 autocorrelation of changes in monthly averages: 0.248
```

Calculation 19.5 — The cobweb as a meeting in time. Deviation from equilibrium multiplied by −s/d each season.

```
        # Cobweb model p_t = (a - s*p_{t-1})/d with linear demand/supply: the deviation is
multiplied by -s/d each period,
        # a meeting (sign flip, period 2). Converges iff s/d < 1.
        for r in (0.8,1.0,1.2):
            x=1.0; xs=[]
            for t in range(6): xs.append(round(x,3)); x=-r*x
            print("s/d=%.1f deviations"%r,xs)
```

Output:

```
s/d=0.8 deviations [1.0, -0.8, 0.64, -0.512, 0.41, -0.328]
s/d=1.0 deviations [1.0, -1.0, 1.0, -1.0, 1.0, -1.0]
s/d=1.2 deviations [1.0, -1.2, 1.44, -1.728, 2.074, -2.488]
```

Calculation 19.6 — Exchange cycles without a double coincidence of wants. Digit and carry by T1 and T7.1.

```
        # An economy of k types: type i consumes good i and produces good i+1 (mod k).

        # A double coincidence of wants between types i and j: i holds what j consumes and j
holds what i consumes.
        for k in range(2,8):
            pairs=[(i,j) for i in range(k) for j in range(i+1,k) if (i+1)%k==j and (j+1)
%k==i]
            print("k=%d types: pairs with double coincidence: %d; trade closes after %d
hand-overs"%(k,len(pairs),k))
        for name,w in [("barter A<->B",(1,-1)),("three-type ring",(1,1,1))]:
            s=sum(w); d=((s+1)%3)-1; print(name,"sum",s,"digit",d,"carry",(s-d)//3)
```

Output:

```
k=2 types: pairs with double coincidence: 1; trade closes after 2 hand-overs
k=3 types: pairs with double coincidence: 0; trade closes after 3 hand-overs
k=4 types: pairs with double coincidence: 0; trade closes after 4 hand-overs
k=5 types: pairs with double coincidence: 0; trade closes after 5 hand-overs
k=6 types: pairs with double coincidence: 0; trade closes after 6 hand-overs
k=7 types: pairs with double coincidence: 0; trade closes after 7 hand-overs
barter A<->B sum 0 digit 0 carry 0
three-type ring sum 3 digit 0 carry 1
```

Calculation 19.7 — Terms of office and the ladder. Terms from the Wikipedia page ‘Term of office’; binomial null with p = 4/6, the share of integers 2–7 on $2^a \cdot 3^b$.

```
        # Terms of office (years) from the Wikipedia 'Term of office' page, checked against
the ladder 2^a3^b (T15).
        lad={2**a*3**b for a in range(6) for b in range(4)}
        terms={"US President":4,"US House":2,"US Senate":6,"French President":5,"German
Bundestag":4,"UK Parliament max":5,
        "Mexico President":6,"Russia President":6,"Philippines President":6,"Australia
House":3,"New Zealand Parliament":3,
        "India Lok Sabha":5,"Italy Parliament":5,"Netherlands chambers":4,"Sweden Riksdag":
4,"Brazil President":4,"Japan Councillors":6}
        on=[k for k,v in terms.items() if v in lad]; off=[(k,v) for k,v in terms.items() if
v not in lad]
        print("on ladder: %d of %d"%(len(on),len(terms))); print("off ladder:",off)
        import itertools
        print("ladder values 1..10:",sorted(v for v in lad if v<=10),"; share of integers
2..7 on ladder: %d/6"%sum(1 for v in range(2,8) if v in lad))
        print("US Senate: 3 classes x 2 years =",3*2,"; seats 33+33+34 =",33+33+34)
        from math import comb
        p=4/6; n=len(terms); k=len(on)
        print("binomial P(>= %d of %d on ladder | p=4/6) = %.3f"%(k,n,sum(comb(n,j)*p**j*(1-
p)**(n-j) for j in range(k,n+1))))
```

Output:

```
on ladder: 13 of 17
off ladder: [('French President', 5), ('UK Parliament max', 5), ('India Lok Sabha',
5), ('Italy Parliament', 5)]
ladder values 1..10: [1, 2, 3, 4, 6, 8, 9] ; share of integers 2..7 on ladder: 4/6
US Senate: 3 classes x 2 years = 6 ; seats 33+33+34 = 100
binomial P(>= 13 of 17 on ladder | p=4/6) = 0.281
```

Calculation 19.8 — V-Dem arithmetic. Figures from the Democracy Report 2026.

```
        print("V-Dem 2026: countries %d; autocratizing share %.3f; democratizing share %.
3f"%(87+92,44/179,18/179))
        print("liberal democracies 45 (2009) -> 31 (2025): change %.3f"%(31/45-1))
```

Output:

```
V-Dem 2026: countries 179; autocratizing share 0.246; democratizing share 0.101
liberal democracies 45 (2009) -> 31 (2025): change -0.311
```

Calculation 19.9 — Reuters Institute arithmetic. Figures from the Digital News Report 2026 executive summary.

```
        print("AI chatbot weekly news use 7%% -> 10%%: relative rise %.3f"%(10/7-1))
        print("interest in news 59%% -> 46%%: %d points"%(59-46))
        print("social as main source 22%% -> 30%%; sole reliance 6%% -> 12%%: ratio %.1f"%(1
2/6))
```

Output:

```
AI chatbot weekly news use 7% -> 10%: relative rise 0.429
interest in news 59% -> 46%: 13 points
social as main source 22% -> 30%; sole reliance 6% -> 12%: ratio 2.0
```

Calculation 19.10 — Where the two cycles lie. Pairwise wins of the candidates in the two Glasgow wards with a cycle.

```
        # Where do the two majority cycles lie? Pairwise wins (Copeland score) of each
candidate in the two Glasgow wards.
        import itertools, os
        D='./preflib/datasets/00008 - glasgow'
        def parse(path):
            m=None; ballots=[]
            for line in open(path):
                if line.startswith('#'):
                    if 'NUMBER ALTERNATIVES' in line: m=int(line.split(':')[1])
                    continue
                cnt,order=line.split(':',1); ranks={}; r=0; s=order.strip(); i=0
                while i<len(s):
                    if s[i]=='{':
                        j=s.index('}',i); g=[int(x) for x in s[i+1:j].split(',') if
x.strip()]; i=j+1
                    elif s[i] in ', ': i+=1; continue
                    else:
                        j=i
                        while j<len(s) and s[j] not in ', ': j+=1
                        g=[int(s[i:j])]; i=j
                    for a in g: ranks[a]=r
                    r+=1
                ballots.append((int(cnt),ranks))
            return m,ballots
        for fn in ['00008-00000009.toc','00008-00000021.toc']:
            m,b=parse(os.path.join(D,fn))
            W=[[0]*(m+1) for _ in range(m+1)]
            for cnt,rk in b:
                for a in range(1,m+1):
                    for c in range(1,m+1):
                        if a!=c and rk.get(a,10**6)<rk.get(c,10**6): W[a][c]+=cnt
            beats=lambda a,c: W[a][c]>W[c][a]
            cope={a:sum(beats(a,c) for c in range(1,m+1) if c!=a) for a in range(1,m+1)}

            for a,c,d in itertools.combinations(range(1,m+1),3):
                if (beats(a,c) and beats(c,d) and beats(d,a)) or (beats(c,a) and beats(d,c)
and beats(a,d)):
                    print(fn,"cycle among candidates",(a,c,d),"pairwise wins",[cope[x] for
x in (a,c,d)],"of",m-1,"; max in ward",max(cope.values()))
```

Output:

```
00008-00000009.toc cycle among candidates (3, 4, 6) pairwise wins [9, 9, 9] of 10 ;
max in ward 9
```

```
00008-00000021.toc cycle among candidates (1, 5, 9) pairwise wins [4, 4, 4] of 9 ;
max in ward 9
```

Calculations for Chapter 20 — The Cycles of Societies

Every number below was computed for this chapter; the code ran on 27 October 2026 with Python 3.13 and the output shown is what it printed. Data files are named relative to a working folder; their public origin is stated with each calculation.

Calculation 20.1 — Dominant measles periods in twenty towns of England and Wales, 1944–1964. Data: twentymeas.RData from the CRAN package tsiR (GitHub mirror cran/tsiR, folder data), read with the Python package rdata. Spectrum of log(1 + cases) with a Hann window and eightfold zero padding; highest peak between 0.67 and 8 years. Prediction fixed in the first edition of this book.

```
        # Prediction fixed before running (first edition of this book, III.3; Foundations
T15): dominant inter-epidemic periods
        # lie on 2^a*3^b years (1, 2, 3, 4, 6 ...); a stable dominant period of 5 or 7 years
counts against.
        import rdata, numpy as np
        c=rdata.conversion.convert(rdata.parser.parse_file('./tsir/data/twentymeas.RData'))
['twentymeas']
        def ladder(x,tol=0.15):
            for a in range(0,4):
                for b in range(0,3):
                    v=2**a*3**b
                    if abs(x-v)/v<=tol: return v
            return None
        onl=0; ntown=0
        for town,df in c.items():
            if not hasattr(df,'shape'): continue
            y=np.log1p(df['cases'].values.astype(float)); t=df['time'].values
            dt=np.median(np.diff(t)); y=y-y.mean()
            n=len(y); f=np.fft.rfftfreq(n*8,d=dt); P=np.abs(np.fft.rfft(y*np.hanning(n),n*8)
)**2
            m=(f>1/8)&(f<1.5)  # periods between 0.67 and 8 years
            per=1/f[m][np.argsort(P[m])[::-1]]
            top=per[0]
            # second distinct peak
            sec=[p for p in per if abs(p-top)/top>0.3][0]
            ntown+=1; onl+= ladder(top) is not None
            print("%-18s pop1950 %9.0f  dominant %.2f y (ladder %s)  next %.2f y (ladder %s)
"%(town, df['pop'].values[len(df)//3], top, ladder(top), sec, ladder(sec)))
        # coverage of the search band [0.667, 8] years by ladder windows of +-15% (log
measure) = chance of a random period landing on the ladder
        import numpy as np
        lo,hi=1/1.5,8.0; xs=np.exp(np.linspace(np.log(lo),np.log(hi),200001))
        cov=np.mean([ladder(x) is not None for x in xs[::50]])
        print("log-uniform chance a period in the band is within 15%% of a ladder value: %.
3f"%cov)
        from math import comb
        k=onl; n=ntown; p=cov; pv=sum(comb(n,j)*p**j*(1-p)**(n-j) for j in range(k,n+1)); pr
int("dominant periods on ladder: %d of %d; binomial P(>=%d of %d) = %.4f"%(k,n,k,n,p
v))
        # Sharper null: seasonal forcing locks periods to whole years; of the whole-year
periods 1..8, six (1,2,3,4,6,8) are on the ladder.
```

```
        p2=6/8; pv2=sum(comb(n,j)*p2**j*(1-p2)**(n-j) for j in range(k,n+1)); print("integer
-year null p=6/8: P(>=%d of %d) = %.3f"%(k,n,pv2))
```

Output:

```
Bedwellty          pop1950     28707  dominant 2.07 y (ladder 2)  next 0.88 y
(ladder 1)
Birmingham         pop1950   1113319  dominant 2.05 y (ladder 2)  next 1.01 y
(ladder 1)
Bradford           pop1950    291355  dominant 2.02 y (ladder 2)  next 0.99 y
(ladder 1)
Bristol            pop1950    441978  dominant 2.05 y (ladder 2)  next 1.01 y
(ladder 1)
Cardiff            pop1950    243860  dominant 2.05 y (ladder 2)  next 1.00 y
(ladder 1)
Consett            pop1950     39058  dominant 2.07 y (ladder 2)  next 0.99 y
(ladder 1)
Dalton.in.Furness  pop1950     10383  dominant 1.62 y (ladder None)  next 2.47 y
(ladder None)
Halesworth         pop1950      2170  dominant 3.65 y (ladder 4)  next 1.85 y
(ladder 2)
Hastings           pop1950     65297  dominant 2.00 y (ladder 2)  next 1.40 y
(ladder None)
Hull               pop1950    299482  dominant 2.02 y (ladder 2)  next 2.67 y
(ladder 3)
Leeds              pop1950    505335  dominant 2.05 y (ladder 2)  next 0.99 y
(ladder 1)
Lees               pop1950      4221  dominant 2.02 y (ladder 2)  next 1.01 y
(ladder 1)
Liverpool          pop1950    790847  dominant 1.00 y (ladder 1)  next 1.79 y
(ladder 2)
London             pop1950   3368925  dominant 2.02 y (ladder 2)  next 1.00 y
(ladder 1)
Manchester         pop1950    701489  dominant 2.05 y (ladder 2)  next 1.00 y
(ladder 1)
Mold               pop1950      6474  dominant 2.40 y (ladder None)  next 0.75 y
(ladder None)
Northwich          pop1950     17564  dominant 2.02 y (ladder 2)  next 1.00 y
(ladder 1)
Nottingham         pop1950    306738  dominant 2.02 y (ladder 2)  next 1.01 y
(ladder 1)
Oswestry           pop1950     10918  dominant 2.07 y (ladder 2)  next 1.08 y
(ladder 1)
Sheffield          pop1950    511728  dominant 2.07 y (ladder 2)  next 1.00 y
(ladder 1)
log-uniform chance a period in the band is within 15% of a ladder value: 0.662
dominant periods on ladder: 18 of 20; binomial P(>=18 of 20) = 0.0158
integer-year null p=6/8: P(>=18 of 20) = 0.091
```

Calculation 20.2 — NBER cycle lengths and the ladder. Durations transcribed from the NBER table of US business cycle expansions and contractions (contraction, expansion) in months, 1857–2020. Uniform null over the observed range, 20,000 draws, seed 0.

```
        # NBER US reference cycles 1854-2020 (trough to trough, months) against the ladder
2^a3^b in years (T15).
        import statistics, random
        rows=[(18,30),(8,22),(32,46),(18,18),(65,34),(38,36),(13,22),(10,27),(17,20),(18,18)
,(18,24),(23,21),(13,33),(24,19),(23,12),(7,44),(18,10),(14,22),(13,27),(43,21),(13,
```

```
50),(8,80),(11,37),(10,45),(8,39),(10,24),(11,106),(16,36),(6,58),(16,12),(8,92),(8,
120),(18,73),(2,128)]
        tt=[a+b for a,b in rows]; yrs=[x/12 for x in tt]
        lad=sorted({2**a*3**b for a in range(8) for b in range(5)})
        near=lambda x,tol: min(abs(x-v)/v for v in lad)<=tol
        print("cycles %d; mean %.1f months (%.2f y); median %.1f months"%(len(tt),statistics
.mean(tt),statistics.mean(tt)/12,statistics.median(tt)))
        print("mean contraction %.1f, expansion %.1f"%(statistics.mean(a for a,b in
rows),statistics.mean(b for a,b in rows)))
        obs=sum(near(y,0.1) for y in yrs); random.seed(0)
        sims=[sum(near(random.uniform(min(yrs),max(yrs)),0.1) for _ in yrs) for _ in range(2
0000)]
        print("within 10%% of a ladder value: %d/%d; uniform null mean %.2f; P(null>=obs)=%.
3f"%(obs,len(yrs),statistics.mean(sims),sum(s>=obs for s in sims)/len(sims)))
        early=[a+b for a,b in rows[:16]]   # cycles ending 1858-1914 (NBER period 1854-1919)

        post=[a+b for a,b in rows[22:]]    # cycles ending 1945-2020 (NBER period 1945-2020)

        print("1854-1919 mean %.1f m; 1945-2020 mean %.1f m; ratio %.2f"%(statistics.mean(ea
rly),statistics.mean(post),statistics.mean(post)/statistics.mean(early)))
```

Output:

```
cycles 34; mean 58.4 months (4.86 y); median 47.5 months
mean contraction 17.0, expansion 41.4
within 10% of a ladder value: 23/34; uniform null mean 21.35; P(null>=obs)=0.346
1854-1919 mean 48.2 m; 1945-2020 mean 74.5 m; ratio 1.55
```

Calculation 20.3 — Ratios implied by the conventional ranges of long waves. Ranges as given on the Wikipedia pages ‘Business cycle’ and ‘Kondratiev wave’.

```
        # Ratios implied by the period ranges Kitchin 3-5, Juglar 7-11, Kuznets 15-25,
Kondratiev 45-60 years.
        for name,(a,b),(c,d) in [("Juglar/Kitchin",(7,11),(3,5)),("Kuznets/Juglar",(15,25),
(7,11)),("Kondratiev/Kuznets",(45,60),(15,25)),("Kondratiev/Juglar",(45,60),(7,11))]
:
            print("%-20s %.2f - %.2f"%(name,a/d,b/c))
        # Closure depths and ladder: is any long-wave length a 2^a3^b number of years?

        lad=sorted({2**a*3**b for a in range(8) for b in range(5)}); print("ladder 40-60:",
[v for v in lad if 40<=v<=60])
```

Output:

```
Juglar/Kitchin       1.40 - 3.67
Kuznets/Juglar       1.36 - 3.57
Kondratiev/Kuznets   1.80 - 4.00
Kondratiev/Juglar    4.09 - 8.57
ladder 40-60: [48, 54]
```

Calculation 20.4 — Pandemic intervals and the measles threshold. Cholera pandemic start years and Florence plague years from the Wikipedia pages cited in 20.3; R_0 range 12–18.

```
        starts=[1817,1826,1846,1863,1881,1899,1961]; iv=[b-a for a,b in zip(starts,starts[1:
])]
        lad=sorted({2**a*3**b for a in range(8) for b in range(5)})
        print("cholera pandemic start intervals:",iv,"ladder:",[x in lad for x in iv])
```

```
        fl=[1348,1362,1371,1382]; print("Florence plague intervals:",[b-a for a,b in zip(fl,
fl[1:])])
        print("Black Death mortality range 30-60%: survivors 0.70-0.40")
        print("measles herd immunity 1-1/R0 for R0 12..18:",[round(1-1/r,3) for r in (12,15,
18)])
```

Output:

```
cholera pandemic start intervals: [9, 20, 17, 18, 18, 62] ladder: [True, False,
False, True, True, False]
Florence plague intervals: [14, 9, 11]
Black Death mortality range 30-60%: survivors 0.70-0.40
measles herd immunity 1-1/R0 for R0 12..18: [0.917, 0.933, 0.944]
```

Calculation 20.5 — Urban growth, megacities and stage counts. UN World Urbanization Prospects 2025 figures; stage counts of Polybius and Strauss–Howe; the US electoral terms.

```
        import math
        print("urban 1950: %.2f bn; 2025: %.2f bn; ratio %.2f; doublings %.2f; annual
growth %.4f"%(0.2*2.5,0.45*8.2,0.45*8.2/0.5,math.log2(0.45*8.2/0.5),(0.45*8.2/
0.5)**(1/75)-1))
        print("megacities 8 (1975) -> 33 (2025) -> 37 (2050): annual %.4f then %.4f"%((33/
8)**(1/50)-1,(37/33)**(1/25)-1))
        lad={2**a*3**b for a in range(8) for b in range(5)}
        print("H3 aperture 7 on ladder:",7 in lad,"; Christaller K 3,4,7:",[k in lad for k i
n (3,4,7)])
        print("Polybius forms 3 x 2 = 6; Strauss-Howe 4 x 21 = %d; Senate 3 x 2 = 6"%(4*21))

        import math
        print("US House 2, President 4, Senate 6: all three terms end together every lcm =",
math.lcm(2,4,6),"years")
```

Output:

```
urban 1950: 0.50 bn; 2025: 3.69 bn; ratio 7.38; doublings 2.88; annual growth 0.0270
megacities 8 (1975) -> 33 (2025) -> 37 (2050): annual 0.0287 then 0.0046
H3 aperture 7 on ladder: False ; Christaller K 3,4,7: [True, True, False]
Polybius forms 3 x 2 = 6; Strauss-Howe 4 x 21 = 84; Senate 3 x 2 = 6
US House 2, President 4, Senate 6: all three terms end together every lcm = 12 years
```

Calculation 20.6 — Period doubling and windows in the logistic map. Attractor period of $x \to r \cdot x \cdot (1 - x)$ after 4,000 iterations, tolerance 10^{-6}.

```
        # Period doubling in the logistic map x -> r x (1-x) (May 1976): attractor periods
at sample r values.
        def period(r,n=4000,tol=1e-6):
            x=0.5
            for _ in range(n): x=r*x*(1-x)
            orb=[x]
            for k in range(1,65):
                x=r*x*(1-x)
                if abs(x-orb[0])<tol: return k
                orb.append(x)
            return None
```

```
        for r in (2.9,3.2,3.5,3.55,3.566,3.83,3.845,3.739,3.702,3.628):
            print("r=%.3f period %s"%(r,period(r)))
```

Output:

```
r=2.900 period 1
r=3.200 period 2
r=3.500 period 4
r=3.550 period 8
r=3.566 period 16
r=3.830 period 3
r=3.845 period 6
r=3.739 period 5
r=3.702 period 7
r=3.628 period 6
```

Calculations for Chapter 21 — Ordering Society

Every number below was computed for this chapter; the code ran on 27 October 2026 with Python 3.13 and the output shown is what it printed. Data files are named relative to a working folder; their public origin is stated with each calculation.

Calculation 21.1 — Deontic status as trit and mirror. Von Wright's three statuses as T, 0, 1; the status of not-p; the pair as a meeting (T1, T7.1).

```
        # Deontic status as a trit (von Wright 1951): O p = +1, P-and-not-O (optional) = 0,
F p = -1.
        # The status of not-p is the trit-wise mirror (T12.1). Check the standard square: O
p <-> F not-p; optional p <-> optional not-p.
        st={'obligatory':1,'optional':0,'forbidden':-1}; inv={v:k for k,v in st.items()}

        for k,v in st.items(): print("%-11s %+d -> not-p: %s (%+d)"%(k,v,inv[-v],-v))

        # A normative triple over the three exhaustive acts {p and q, p and not-q, not-p}?
Instead check consistency counts:
        import itertools
        # For a single act p and its negation, the pair (s(p), s(not p)) must be (x, -x): a
meeting, closed with carry 0.
        for x in (1,0,-1):
            s=x+(-x); d=((s+1)%3)-1; print("pair",(x,-x),"sum",s,"digit",d,"carry",(s-d)//3)
```

Output:

```
obligatory  +1 -> not-p: forbidden (-1)
optional    +0 -> not-p: optional (+0)
forbidden   -1 -> not-p: obligatory (+1)
pair (1, -1) sum 0 digit 0 carry 0
pair (0, 0) sum 0 digit 0 carry 0
pair (-1, 1) sum 0 digit 0 carry 0
```

Calculation 21.2 — Hohfeld's square as the Klein four-group. Correlative and opposite on the first square, as given on the Wikipedia page 'Hohfeldian analysis'.

```
        # Hohfeld (1913, 1917): 8 jural relations in 2 squares; each square = 4 terms joined
by correlatives (between parties)
        # and opposites (same party). Check the group structure: correlative C and opposite
O are involutions that commute,
```

```
        # so they generate the Klein group V4 = Z2 x Z2 (order 4 = 2^2), not Z3.
        sq1={'right':('duty','no-right'),'duty':('right','privilege'),'privilege':('no-
right','duty'),'no-right':('privilege','right')}
        C=lambda x:sq1[x][0]; O=lambda x:sq1[x][1]
        for x in sq1: assert C(C(x))==x and O(O(x))==x and C(O(x))==O(C(x))
        print("C, O involutions, commute: True; CO of right =",C(O('right')))
        print("group order",len({x for x in sq1}),"; terms per square 4 = 2^2; total 8 =
2^3")
```

Output:

```
C, O involutions, commute: True; CO of right = privilege
group order 4 ; terms per square 4 = 2^2; total 8 = 2^3
```

Calculation 21.3 — Jury sizes and the ladder. Sizes from the Wikipedia page ‘Jury’; binomial null with the ladder share of the integers 6–15.

```
        lad={2**a*3**b for a in range(6) for b in range(4)}
        J=[("England & Wales",12),("Scotland",15),("US civil minimum",6),("Canada",12),("Aus
tralia",12),("Belgium",12),("Brazil",7),("Austria (jurors)",8),("Norway",10)]
        for n,s in J: print("%-18s %2d %s"%(n,s,"on ladder" if s in lad else "off ladder"))

        print("on ladder: %d of %d"%(sum(s in lad for n,s in J),len(J)))
        print("Scotland: simple majority 8/15 = %.3f; two-thirds of 15 = %d jurors"%(8/15,
15*2//3))
        print("Not proven ~1/3 of jury acquittals (1999): if split evenly into the two
remaining verdicts, 1/6 of acquittals move each way")
        from math import comb
        p=sum(1 for v in range(6,16) if v in lad)/10; k=sum(s in lad for n,s in J); n=len(J)

        print("ladder share of sizes 6..15: %.1f; P(>=%d of %d) = %.3f"%
(p,k,n,sum(comb(n,j)*p**j*(1-p)**(n-j) for j in range(k,n+1))))
```

Output:

```
England & Wales    12 on ladder
Scotland           15 off ladder
US civil minimum    6 on ladder
Canada             12 on ladder
Australia          12 on ladder
Belgium            12 on ladder
Brazil              7 off ladder
Austria (jurors)    8 on ladder
Norway             10 off ladder
on ladder: 6 of 9
Scotland: simple majority 8/15 = 0.533; two-thirds of 15 = 10 jurors
Not proven ~1/3 of jury acquittals (1999): if split evenly into the two remaining
verdicts, 1/6 of acquittals move each way
ladder share of sizes 6..15: 0.4; P(>=6 of 9) = 0.099
```

Calculation 21.4 — Graicunas’s count of relationships. $R(n) = n(2^{n-1} + n - 1)$ for n subordinates.

```
        # Graicunas (1933): potential relationships for n subordinates R(n) = n(2^(n-1) + n
- 1).
        lad={2**a*3**b for a in range(14) for b in range(9)}
```

```
        for n in range(1,9):
            R=n*(2**(n-1)+n-1); print(n,R,"on ladder" if R in lad else "")
```

Output:

```
1 1 on ladder
2 6 on ladder
3 18 on ladder
4 44
5 100
6 222
7 490
8 1080
```

Calculation 21.5 — PISA rounds and AI Act delays. PISA years and figures from the Wikipedia page on PISA; AI Omnibus dates from the Future of Privacy Forum.

```
        import datetime as dt
        r=[2000,2003,2006,2009,2012,2015,2018,2022,2025]; print("PISA rounds held:",r,"gaps:
",[b-a for a,b in zip(r,r[1:])])
        print("2022 drops: maths %d, reading %d points"%(489-472,487-476))
        print("PISA 2025 students per country-economy: %.0f"%(760000/91))
        d=lambda a,b:(dt.date(*b)-dt.date(*a)).days
        print("AI Act high-risk Annex III delay: %d days; Annex I delay: %d days"%(d((2026,
8,2),(2027,12,2)),d((2027,8,2),(2028,8,2))))
```

Output:

```
PISA rounds held: [2000, 2003, 2006, 2009, 2012, 2015, 2018, 2022, 2025] gaps: [3,
3, 3, 3, 3, 3, 4, 3]
2022 drops: maths 17, reading 11 points
PISA 2025 students per country-economy: 8352
AI Act high-risk Annex III delay: 487 days; Annex I delay: 366 days
```

Calculation 21.6 — Epidemiological arithmetic. Herd-immunity threshold $1 - 1/R_0$; WHO tuberculosis, COVID-19 and measles figures cited in 21.3.

```
        # SIR threshold and measles; COVID excess deaths range; TB case fatality.
        for R0 in (12,15,18): print("R0 %d herd threshold %.3f"%(R0,1-1/R0))
        print("WHO TB 2024: deaths/cases %.3f"%(1.23/10.7))
        print("COVID excess 14.9 (13.3-16.6) million: half-width %.2f"%((16.6-13.3)/2))

        print("Measles coverage MCV1 84%%, MCV2 77%%: gap to 95%% = %d and %d points"%
(95-84,95-77))
```

Output:

```
R0 12 herd threshold 0.917
R0 15 herd threshold 0.933
R0 18 herd threshold 0.944
WHO TB 2024: deaths/cases 0.115
COVID excess 14.9 (13.3-16.6) million: half-width 1.65
Measles coverage MCV1 84%, MCV2 77%: gap to 95% = 11 and 18 points
```

Calculation 21.7 — Engagement and workforce arithmetic. Gallup State of the Global Workplace 2026; New York Times counts as reported on the Wikipedia page on the Department of Government Efficiency.

```
        print("Gallup manager engagement 27 -> 22: %d points, relative %.3f"%(27-22,22/27-1)
)
        print("NYT May 2025 planned reductions 58,500+76,000+149,000 = %d; share of 2.4m = %
.3f"%(58500+76000+149000,(58500+76000+149000)/2.4e6))
```

Output:

```
Gallup manager engagement 27 -> 22: 5 points, relative -0.185
NYT May 2025 planned reductions 58,500+76,000+149,000 = 283500; share of 2.4m =
0.118
```

Calculation 21.8 — Named counts and the ladder. Counts of named stages, steps and categories cited in this chapter; binomial null with the ladder share of the integers 3–9.

```
        # Named-stage counts in the practical social sciences, checked against the ladder
2^a 3^b (T15).
        from math import comb
        lad={2**a*3**b for a in range(6) for b in range(4)}
        counts={"trivium":3,"quadrivium":4,"liberal arts":7,"Bloom levels":6,"Piaget
stages":4,"Hill viewpoints":9,
        "Omran ages":3,"Omran + Barrett ages":5,"Fayol functions":5,"Tuckman stages (1965)":
4,"Tuckman stages (1977)":5,
        "Shewhart steps":3,"PDCA steps":4,"Hohfeld relations":8,"Montesquieu powers":3,

        "Gulick POSDCORB":7,"Six Sigma DMAIC":5,"Maslow levels":5,"Balanced scorecard":4,"Os
trom principles":8,"Weber authority types":3,"Bologna cycles":3}
        on=[k for k,v in counts.items() if v in lad]
        print("on ladder: %d of %d; off:"%(len(on),len(counts)),[(k,v) for k,v in
counts.items() if v not in lad])
        p=sum(1 for v in range(3,10) if v in lad)/7
        n=len(counts); k=len(on)
        print("share of 3..9 on ladder p = %.3f; P(>= %d of %d) = %.3f"%
(p,k,n,sum(comb(n,j)*p**j*(1-p)**(n-j) for j in range(k,n+1))))
```

Output:

```
on ladder: 15 of 22; off: [('liberal arts', 7), ('Omran + Barrett ages', 5), ('Fayol
functions', 5), ('Tuckman stages (1977)', 5), ('Gulick POSDCORB', 7), ('Six Sigma
DMAIC', 5), ('Maslow levels', 5)]
share of 3..9 on ladder p = 0.714; P(>= 15 of 22) = 0.724
```

Calculation 21.9 — Łukasiewicz negation as the trit mirror. Values {0, ½, 1} shifted to {−1, 0, +1}; exact fractions.

```
        # Lukasiewicz three-valued logic (1920): values 0, 1/2, 1; negation 1 - x;
conjunction min; disjunction max.
        # Shift to balanced form b = 2x - 1 in {-1, 0, +1}: negation becomes b -> -b, the
trit-wise mirror of T12.1.
        from fractions import Fraction as F
        vals=[F(0),F(1,2),F(1)]
        bal=lambda x:int(2*x-1)
```

```
        for x in vals: print("x=%s  not x=%s  | balanced %+d -> %+d"%(x,1-x,bal(x),bal(1-x))
)
        ok=all(bal(min(x,y))==min(bal(x),bal(y)) and bal(max(x,y))==max(bal(x),bal(y)) for
x in vals for y in vals)
        print("min/max commute with the shift:",ok)
        # Lukasiewicz implication x->y = min(1, 1-x+y); its balanced form
        for x in vals:
            print(" ".join("%+d"%bal(min(F(1),1-x+y)) for y in vals), " <- row x =",x)
```

Output:

```
x=0  not x=1  | balanced -1 -> +1
x=1/2  not x=1/2  | balanced +0 -> +0
x=1  not x=0  | balanced +1 -> -1
min/max commute with the shift: True
+1 +1 +1  <- row x = 0
+0 +1 +1  <- row x = 1/2
-1 +0 +1  <- row x = 1
```

Calculations for Chapter 22 — Language and Sign

All scripts were run with Python 3.13 on 27 October 2026. Texts for 22.1–22.3 are the TEI files of the Perseus Digital Library (PerseusDL canonical-greekLit and canonical-latinLit on GitHub): Iliad and Odyssey (Monro–Allen), Herodotus and Thucydides (Perseus editions), Aeneid, and Caesar's *Bellum Gallicum*. Accents are removed and words lower-cased. The helper files are printed once, under 22.1.

Calculation 22.1. Trit rule on six classical texts; closed-window rate against T6 and 200 shuffles.

```
        # Shared text loader. Texts: PerseusDL canonical-greekLit / canonical-latinLit on
GitHub (raw files).
        import re, unicodedata
        from lxml import etree
        D='/tmp/claude-0/-home-claude/bb88f9f7-7ac8-502c-9b68-538e6785632c/scratchpad/dl/'

        FILES=[('Iliad (verse)','iliad.xml','g'),('Odyssey (verse)','tlg0012.tlg002.perseus-
grc2.xml','g'),
        ('Herodotus (prose)','tlg0016.tlg001.perseus-grc2.xml','g'),('Thucydides (prose)','t
lg0003.tlg001.perseus-grc2.xml','g'),
        ('Aeneid (verse)','phi0690.phi003.perseus-lat2.xml','l'),('Caesar BG (prose)','phi04
48.phi001.perseus-lat2.xml','l')]
        def words(fn,lang):
            body=etree.parse(D+fn).find('.//{http://www.tei-c.org/ns/1.0}body')
            for tag in ('note','bibl','head'):
                for n in list(body.iter('{http://www.tei-c.org/ns/1.0}'+tag)):
n.getparent().remove(n)
            s=unicodedata.normalize('NFD',' '.join(body.itertext()))
            s=''.join(c for c in s if unicodedata.category(c)!='Mn').lower()
            if lang=='l': return re.findall(r'[a-z]+',s.replace('j','i').replace('v','u'))

            return re.findall(r'[α-ωςϲ]+',s)
        def trits(w):   # the trit rule of Foundations III.1, applied to a word sequence

            first={}; out=[]
            for i,x in enumerate(w):
                if x in first:
                    if i<len(w)-1:
```

```
                        f=first[x]
                        if w[i+1]==w[f+1]: out.append(1)
                        elif f>0 and w[i+1]==w[f-1]: out.append(-1)
                        else: out.append(0)
                else: first[x]=i
            return out
        def rate(t): return sum((t[k]+t[k+1]+t[k+2])%3==0 for k in range(len(t)-2))/
(len(t)-2)
        import random, statistics as st, cmath
        from common_text import *
        w3=cmath.exp(2j*cmath.pi/3)
        random.seed(2026)
        for name,fn,lang in FILES:
            t=trits(words(fn,lang)); n=len(t)
            p0,pp,pm=t.count(0)/n,t.count(1)/n,t.count(-1)/n
            ph=lambda z:p0+pp*z+pm*z*z
            t6=((1+ph(w3)**3+ph(w3**2)**3)/3).real        # Foundations T6
            r=rate(t); sh=[]
            for _ in range(200):
                ts=t[:]; random.shuffle(ts); sh.append(rate(ts))
            m,s=st.mean(sh),st.stdev(sh)
            print(f"{name:19s} trits={n:6d} p0={p0:.4f} p+={pp:.4f} p-={pm:.4f} closed={r:.
4f} T6={t6:.4f} shuffled={m:.4f}±{s:.4f} z={(r-m)/s:+.1f}")
```

Output:

```
Iliad (verse)       trits= 92046 p0=0.9136 p+=0.0831 p-=0.0033 closed=0.7775
T6=0.7646 shuffled=0.7646±0.0006 z=+21.4
Odyssey (verse)     trits= 71048 p0=0.8986 p+=0.0956 p-=0.0057 closed=0.7602
T6=0.7295 shuffled=0.7295±0.0008 z=+38.0
Herodotus (prose)   trits=156060 p0=0.9562 p+=0.0341 p-=0.0097 closed=0.8739
T6=0.8762 shuffled=0.8763±0.0004 z=-6.8
Thucydides (prose)  trits=128857 p0=0.9633 p+=0.0289 p-=0.0078 closed=0.8942
T6=0.8952 shuffled=0.8952±0.0003 z=-3.5
Aeneid (verse)      trits= 47126 p0=0.9757 p+=0.0221 p-=0.0021 closed=0.9322
T6=0.9293 shuffled=0.9293±0.0003 z=+9.9
Caesar BG (prose)   trits= 40372 p0=0.9624 p+=0.0322 p-=0.0054 closed=0.8914
T6=0.8925 shuffled=0.8926±0.0005 z=-2.5
```

The shuffled rate equals the T6 value to within sampling error, as T6 requires for independent trits.

Calculation 22.2. Closed-window rate against a first-order Markov null (100 simulated sequences per text).

```
        import numpy as np
        from common_text import *
        rng=np.random.default_rng(7); vals=np.array([-1,0,1]); idx={-1:0,0:1,1:2}
        for name,fn,lang in FILES:
            t=trits(words(fn,lang)); n=len(t); a=np.array([idx[x] for x in t])
            C=np.zeros((3,3)); np.add.at(C,(a[:-1],a[1:]),1); cum=(C/C.sum(1,keepdims=True))
.cumsum(1)
            sims=[]
            for _ in range(100):
                s=np.empty(n,dtype=int); s[0]=a[0]; u=rng.random(n)
                for k in range(1,n): s[k]=min(2,np.searchsorted(cum[s[k-1]],u[k]))
                v=vals[s]; sims.append(((v[:-2]+v[1:-1]+v[2:])%3==0).mean())
            m,sd=np.mean(sims),np.std(sims,ddof=1); r=rate(t)
```

```
            print(f"{name:19s} closed={r:.4f} first-order Markov null={m:.4f}±{sd:.4f}
z={(r-m)/sd:+.1f}")
```

Output:

```
Iliad (verse)       closed=0.7775 first-order Markov null=0.7725±0.0023 z=+2.2
Odyssey (verse)     closed=0.7602 first-order Markov null=0.7489±0.0026 z=+4.3
Herodotus (prose)   closed=0.8739 first-order Markov null=0.8737±0.0016 z=+0.1
Thucydides (prose)  closed=0.8942 first-order Markov null=0.8936±0.0013 z=+0.4
Aeneid (verse)      closed=0.9322 first-order Markov null=0.9313±0.0018 z=+0.5
Caesar BG (prose)   closed=0.8914 first-order Markov null=0.8917±0.0029 z=-0.1
```

Calculation 22.3. Lever of T24 applied to the count of uses per word type.

```
        from collections import Counter
        import numpy as np
        from common_text import *
        for name,fn,lang in FILES:
            c=np.array(list(Counter(words(fn,lang)).values()),dtype=float)
            m=c.mean(); k2=c.var(); k3=((c-m)**3).mean(); P=1-1/m
            print(f"{name:19s} types={len(c):6d} mean V={m:5.2f} lever={-m*k3/k2**2:6.1f}
geometric lever, same mean={-(1+P)/P:6.3f} share V=1: {np.mean(c==1):.3f}")
```

Output:

```
Iliad (verse)       types= 19848 mean V= 5.64 lever=  -5.8 geometric lever, same
mean=-2.216 share V=1: 0.544
Odyssey (verse)     types= 16144 mean V= 5.40 lever=  -5.5 geometric lever, same
mean=-2.227 share V=1: 0.547
Herodotus (prose)   types= 27377 mean V= 6.70 lever=  -4.4 geometric lever, same
mean=-2.175 share V=1: 0.585
Thucydides (prose)  types= 21266 mean V= 7.06 lever=  -5.3 geometric lever, same
mean=-2.165 share V=1: 0.593
Aeneid (verse)      types= 16588 mean V= 3.84 lever= -12.9 geometric lever, same
mean=-2.352 share V=1: 0.555
Caesar BG (prose)   types= 10927 mean V= 4.69 lever=  -5.2 geometric lever, same
mean=-2.271 share V=1: 0.569
```

Calculation 22.4. Word orders (WALS 81A) as heterogeneous windows: rotation classes (T9).

```
        n={'SOV':564,'SVO':488,'VSO':95,'VOS':25,'OVS':11,'OSV':4}   # WALS 81A (Dryer)

        rot=lambda s:min(s[i:]+s[:i] for i in range(3))
        cls={}
        for o in n: cls.setdefault(rot(o),[]).append(o)
        for v in cls.values(): print("rotation class",v,"languages:",sum(n[x] for x in v))

        tot=sum(n.values()); sbo=sum(v for o,v in n.items() if o.index('S')<o.index('O'))

        print("dominant order:",tot,"| S before O:",sbo,f"({sbo/tot:.4f}) | O before
S:",tot-sbo)
```

Output:

```
rotation class ['SOV', 'VSO', 'OVS'] languages: 670
rotation class ['SVO', 'VOS', 'OSV'] languages: 517
dominant order: 1187 | S before O: 1147 (0.9663) | O before S: 40
```

Calculation 22.5. Ladder test 2^a·3^b (T15) on counts from grammar and script.

```
        def lad(k):
            """True if k = 2^a * 3^b (the ladder of Foundations T15)."""
            if k < 1: return False
            while k % 2 == 0: k //= 2
            while k % 3 == 0: k //= 3
            return k == 1
        def ladder(k):
            while k%2==0: k//=2
            while k%3==0: k//=3
            return k==1
        print("ladder numbers 2^a*3^b up to 100:",[k for k in range(1,101) if ladder(k)])

        for lab,vals in [("demonstrative distance contrasts (WALS 41A)",[2,3,4,5]),("gender
counts (WALS 30A)",[2,3,4]),
          ("modal vowel count (WALS 2A)",[5]),("numeral bases (WALS 131A)",[10,20]),
          ("alphabets: Phoenician 22, Greek 24, Latin 26, Arabic 28, Russian 33",[22,24,26,2
8,33]),
          ("Unicode 17.0 scripts; Glottolog families",[172,246])]:
            print(lab,[(v,ladder(v)) for v in vals])
        print("2-way + 3-way demonstratives: (126+88)/234 =",round((126+88)/234,4))
        print("five-vowel systems: 188/564 =",round(188/564,4))
```

Output:

```
ladder numbers 2^a*3^b up to 100: [1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36,
48, 54, 64, 72, 81, 96]
demonstrative distance contrasts (WALS 41A) [(2, True), (3, True), (4, True), (5,
False)]
gender counts (WALS 30A) [(2, True), (3, True), (4, True)]
modal vowel count (WALS 2A) [(5, False)]
numeral bases (WALS 131A) [(10, False), (20, False)]
alphabets: Phoenician 22, Greek 24, Latin 26, Arabic 28, Russian 33 [(22, False),
(24, True), (26, False), (28, False), (33, False)]
Unicode 17.0 scripts; Glottolog families [(172, False), (246, False)]
2-way + 3-way demonstratives: (126+88)/234 = 0.9145
five-vowel systems: 188/564 = 0.3333
```

Calculation 22.6. Balanced residue with carry (T1) at radix ten: Yoruba 11–19.

```
        # digit rule of Foundations T1 transposed to radix 10 (balanced residue, ties to
-5), Yoruba 11-19
        for s in range(11,20):
            d=s%10; d=d-10 if d>=5 else d
            print(f"{s} = {s-d} {'+' if d>=0 else '-'} {abs(d)}")
```

Output:

```
11 = 10 + 1
12 = 10 + 2
13 = 10 + 3
14 = 10 + 4
15 = 20 - 5
```

```
16 = 20 - 4
17 = 20 - 3
18 = 20 - 2
19 = 20 - 1
```

Calculation 22.7. Peirce’s sign classes against the closed windows of T3–T4, and the counts for ten trichotomies.

```
        from itertools import product
        from math import comb
        P=[t for t in product([1,2,3],repeat=3) if t[0]>=t[1]>=t[2]]
        print("Peirce classes (non-increasing triples):",len(P),"of 27 = C(5,3) =",comb(5,3)
)
        print("closed windows of the register: ",sum(1 for t in product([-1,0,1],repeat=3) i
f sum(t)%3==0),"of 27")
        print("Peirce classes that are all-equal or all-different (T4 closure, any
labelling):",[t for t in P if len(set(t)) in (1,3)])
        print("non-increasing words over 3 values: n=3 ->",comb(5,2),"; n=10 ->",comb(12,
2),"| open words 3^10 =",3**10)
        from math import gcd
        phi=lambda m:sum(1 for k in range(1,m+1) if gcd(k,m)==1)
        g=lambda n:sum(phi(n//d)*3**d for d in range(1,n+1) if n%d==0)//n
        print("closed configurations g(10) (T13) =",g(10))
```

Output:

```
Peirce classes (non-increasing triples): 10 of 27 = C(5,3) = 10
closed windows of the register:  9 of 27
Peirce classes that are all-equal or all-different (T4 closure, any labelling): [(1,
1, 1), (2, 2, 2), (3, 2, 1), (3, 3, 3)]
non-increasing words over 3 values: n=3 -> 10 ; n=10 -> 66 | open words 3^10 = 59049
closed configurations g(10) (T13) = 5934
```

Calculation 22.8. Three-consonant roots: open words, rotation classes (T13) and shape classes (T4).

```
        from math import gcd
        k=28  # consonants of Arabic
        print("open words of three consonants:",k**3,"| rotation classes:",sum(k**gcd(j,3) f
or j in range(3))//3)
        print("all different:",k*(k-1)*(k-2),"| exactly two equal:",3*k*(k-1),"| all
equal:",k)
```

Output:

```
open words of three consonants: 21952 | rotation classes: 7336
all different: 19656 | exactly two equal: 2268 | all equal: 28
```

Calculation 22.9. Unicode and Glottolog ratios.

```
        print("Unicode 17.0: 159801 - 4803 =",159801-4803,"| new share =",round(4803/159801,
4))
        print("Glottolog 5.3: 7674 languages / 246 families =",round(7674/246,
1),"| isolates share of families =",round(183/246,3))
```

Output:

```
Unicode 17.0: 159801 - 4803 = 154998 | new share = 0.0301
Glottolog 5.3: 7674 languages / 246 families = 31.2 | isolates share of families =
0.744
```

Calculations for Chapter 23 — The Arts

All scripts were run with Python 3.13 (mpmath for 40-digit logarithms) on 27 October 2026. `ladder.py`, imported as `lad`, is printed in Calculation 22.5.

Calculation 23.1. Error of the fifth in n-step equal tuning, and ladder membership of n.

```
        from mpmath import mp, mpf, log, nint
        mp.dps=40
        f=1200*log(mpf(3)/2)/log(2)
        from ladder import lad
        for n in [5,7,12,17,19,22,24,29,31,41,53]:
            k=int(nint(n*log(mpf(3)/2)/log(2)))
            print(f"n={n:2d} fifth={k:2d} steps error={float(1200*k/n-f):+8.3f} cents  on
ladder 2^a3^b: {lad(n)}")
```

Output:

```
n= 5 fifth= 3 steps error= +18.045 cents  on ladder 2^a3^b: False
n= 7 fifth= 4 steps error= -16.241 cents  on ladder 2^a3^b: False
n=12 fifth= 7 steps error=  -1.955 cents  on ladder 2^a3^b: True
n=17 fifth=10 steps error=  +3.927 cents  on ladder 2^a3^b: False
n=19 fifth=11 steps error=  -7.218 cents  on ladder 2^a3^b: False
n=22 fifth=13 steps error=  +7.136 cents  on ladder 2^a3^b: False
n=24 fifth=14 steps error=  -1.955 cents  on ladder 2^a3^b: True
n=29 fifth=17 steps error=  +1.493 cents  on ladder 2^a3^b: False
n=31 fifth=18 steps error=  -5.181 cents  on ladder 2^a3^b: False
n=41 fifth=24 steps error=  +0.484 cents  on ladder 2^a3^b: False
n=53 fifth=31 steps error=  -0.068 cents  on ladder 2^a3^b: False
```

Calculation 23.2. Best approximations of $\log_2 3$ of the second kind (closure depths, T17) and of the first kind, $q \le 60$.

```
        from mpmath import mp, mpf, log, nint
        mp.dps=40; th=log(3)/log(2); r2=r1=mpf(10)
        for q in range(1,61):
            p=int(nint(q*th)); d2=abs(q*th-p); d1=abs(th-mpf(p)/q); tag=[]
            if d2<r2: r2=d2; tag.append('second kind = closure depth (T17)')
            if d1<r1: r1=d1; tag.append('first kind')
            if tag: print(f"q={q:2d} p={p:2d} miss={float(1200*d2):8.3f} cents: {', '.join(t
ag)}")
```

Output:

```
q= 1 p= 2 miss= 498.045 cents: second kind = closure depth (T17), first kind
q= 2 p= 3 miss= 203.910 cents: second kind = closure depth (T17), first kind
q= 3 p= 5 miss= 294.135 cents: first kind
q= 5 p= 8 miss=  90.225 cents: second kind = closure depth (T17), first kind
q= 7 p=11 miss= 113.685 cents: first kind
q=12 p=19 miss=  23.460 cents: second kind = closure depth (T17), first kind
q=29 p=46 miss=  43.305 cents: first kind
```

```
q=41 p=65 miss=  19.845 cents: second kind = closure depth (T17), first kind
q=53 p=84 miss=   3.615 cents: second kind = closure depth (T17), first kind
```

Calculation 23.3. Pythagorean comma (T18), Holdrian comma and the 53-step fifth.

```
        from fractions import Fraction
        from mpmath import mpf, log
        c=Fraction(3**12,2**19); print("3^12/2^19 =",c,"=",float(c),"=",round(float(1200*log
(mpf(c.numerator)/c.denominator)/log(2)),3),"cents")
        print("Holdrian comma 1200/53 =",round(1200/53,3),"cents; 9 commas =",round(9*1200/5
3,3),"vs 9/8 =",round(float(1200*log(mpf(9)/8)/log(2)),3))
        print("53-EDO fifth, 31 steps =",round(31*1200/53,3),"cents")
```

Output:

```
3^12/2^19 = 531441/524288 = 1.0136432647705078 = 23.46 cents
Holdrian comma 1200/53 = 22.642 cents; 9 commas = 203.774 vs 9/8 = 203.91
53-EDO fifth, 31 steps = 701.887 cents
```

Calculation 23.4. Euclidean rhythms: placements against rotation classes (T13).

```
        from itertools import combinations
        from math import comb
        def classes(n,k):
            return len({min(s[i:]+s[:i] for i in range(n)) for s in (tuple(int(i in c) for
i in range(n)) for c in combinations(range(n),k))})
        for k,n in [(3,8),(5,8),(5,12),(7,12),(5,16),(7,16)]:
            pat=''.join('x' if (i*k)%n<k else '.' for i in range(n))
            print(f"E({k},{n}) one rotation: {pat}  placements C(n,k)={comb(n,k)}  rotation
classes={classes(n,k)}")
```

Output:

```
E(3,8) one rotation: x..x..x.  placements C(n,k)=56  rotation classes=7
E(5,8) one rotation: x.x.xx.x  placements C(n,k)=56  rotation classes=7
E(5,12) one rotation: x..x.x..x.x.  placements C(n,k)=792  rotation classes=66
E(7,12) one rotation: x.x.x.xx.x.x  placements C(n,k)=792  rotation classes=66
E(5,16) one rotation: x...x..x..x..x..  placements C(n,k)=4368  rotation classes=273
E(7,16) one rotation: x..x.x.x..x.x.x.  placements C(n,k)=11440  rotation
classes=715
```

Calculation 23.5. Neo-Riemannian P, L, R on the 24 triads: involutions and alternating cycle lengths.

```
        P=lambda t:(t[0],1-t[1])
        R=lambda t:((t[0]+9)%12,1) if t[1]==0 else ((t[0]+3)%12,0)
        L=lambda t:((t[0]+4)%12,1) if t[1]==0 else ((t[0]+8)%12,0)
        for a,b,nm in [(P,L,'PL'),(P,R,'PR'),(L,R,'LR')]:
            t=(0,0); i=0
            while True:
                t=(a if i%2==0 else b)(t); i+=1
                if t==(0,0) and i%2==0: break
            print(f"{nm} alternating cycle from C major: {i} steps")
        print("P,L,R are involutions:",all(X(X((r,q)))==(r,q) for X in (P,L,R) for r in rang
e(12) for q in (0,1)))
```

Output:

```
PL alternating cycle from C major: 6 steps
PR alternating cycle from C major: 8 steps
LR alternating cycle from C major: 24 steps
P,L,R are involutions: True
```

Calculation 23.6. The Tonnetz as the Eisenstein lattice Z[ω] (T14).

```
        import itertools
        f=lambda a,b:(4*a+3*b)%12   # step 1 = major third, step w = minor third on the
Eisenstein lattice
        print("pitch steps of 1, w, 1+w:",f(1,0),f(0,1),f(1,1))
        ker=[(a,b) for a in range(-15,16) for b in range(-15,16) if f(a,b)==0 and (a,b)!=(0,
0)]
        N=lambda a,b:a*a-a*b+b*b
        print("shortest kernel norms a^2-ab+b^2:",sorted({N(a,b) for a,b in ker})[:4])

        print("index of kernel = pitch classes:",min(abs(x[0]*y[1]-x[1]*y[0]) for x,y in
itertools.combinations(ker,2) if x[0]*y[1]-x[1]*y[0]))
        print("up triangle (0,1,1+w):",[f(0,0),f(1,0),f(1,1)],"down triangle (0,w,1+w):",
[f(0,0),f(0,1),f(1,1)])
```

Output:

```
pitch steps of 1, w, 1+w: 4 3 7
shortest kernel norms a^2-ab+b^2: [9, 13, 16, 28]
index of kernel = pitch classes: 12
up triangle (0,1,1+w): [0, 4, 7] down triangle (0,w,1+w): [0, 3, 7]
```

Calculation 23.7. Sestina order, Queneau numbers below 100, change-ringing counts.

```
        from math import factorial
        def order(p):
            cur=list(range(1,len(p)+1)); k=0
            while True:
                cur=[cur[x-1] for x in p]; k+=1
                if cur==list(range(1,len(p)+1)): return k
        def spiral(n):
            p=[]; lo,hi=1,n
            while lo<=hi:
                p.append(hi); hi-=1
                if lo<=hi: p.append(lo); lo+=1
            return p
        from ladder import lad
        print("sestina permutation 615243, order:",order([6,1,5,2,4,3]))
        q=[n for n in range(1,100) if order(spiral(n))==n]
        print("Queneau numbers below 100:",q)
        print("of which 2^a3^b:",[n for n in q if lad(n)])
        for n in range(3,9): print(f"bells n={n}: plain-hunt period {2*n}, extent n! = {fact
orial(n)}")
```

Output:

```
sestina permutation 615243, order: 6
Queneau numbers below 100: [1, 2, 3, 5, 6, 9, 11, 14, 18, 23, 26, 29, 30, 33, 35,
39, 41, 50, 51, 53, 65, 69, 74, 81, 83, 86, 89, 90, 95, 98, 99]
of which 2^a3^b: [1, 2, 3, 6, 9, 18, 81]
bells n=3: plain-hunt period 6, extent n! = 6
bells n=4: plain-hunt period 8, extent n! = 24
```

```
bells n=5: plain-hunt period 10, extent n! = 120
bells n=6: plain-hunt period 12, extent n! = 720
bells n=7: plain-hunt period 14, extent n! = 5040
bells n=8: plain-hunt period 16, extent n! = 40320
```

The Queneau list agrees with OEIS A054639 (opened 27 October 2026) up to 98; 99 is the next term found by the script.

Calculation 23.8. Market ratios from the cited reports.

```
        print("IFPI 2025: streaming share 22/31.7 =",round(22/31.7,3))
        print("Art market 2025: 34.8 + 20.7 + 4.2 =",round(34.8+20.7+4.2,1),"bn USD (report
total 59.6)")
        print("Deezer: 90,000 AI tracks/day x 365 =",90000*365)
        print("Broadway 2024-25: 1893 M USD / 14.66 M admissions =",round(1893/14.66,
2),"USD per admission")
```

Output:

```
IFPI 2025: streaming share 22/31.7 = 0.694
Art market 2025: 34.8 + 20.7 + 4.2 = 59.7 bn USD (report total 59.6)
Deezer: 90,000 AI tracks/day x 365 = 32850000
Broadway 2024-25: 1893 M USD / 14.66 M admissions = 129.13 USD per admission
```

Calculation 23.9. Hexameter patterns and morae.

```
        print("hexameter: 5 free feet, 2 forms each:",2**5,"| with final foot (2 forms):",2*
*6)
        print("morae: dactyl 2+1+1 =",2+1+1,"| spondee 2+2 =",2+2)
```

Output:

```
hexameter: 5 free feet, 2 forms each: 32 | with final foot (2 forms): 64
morae: dactyl 2+1+1 = 4 | spondee 2+2 = 4
```

Calculation 23.10. Crystallographic restriction and the ladder (T15).

```
        from math import cos, pi
        from ladder import lad
        ok=[n for n in range(1,101) if abs(2*cos(2*pi/n)-round(2*cos(2*pi/n)))<1e-12]

        print("orders n<=100 with 2cos(2pi/n) an integer:",ok,"| all on ladder:",all(lad(n)
for n in ok))
```

Output:

```
orders n<=100 with 2cos(2pi/n) an integer: [1, 2, 3, 4, 6] | all on ladder: True
```

Calculations for Chapter 24 — Memory of Culture

All scripts were run with Python 3.13 on 27 October 2026 with exact rational arithmetic (`fractions`). Period data: NASA (synodic 29.53059, draconic 27.21222, anomalistic 27.55455 days; Venus synodic 583.92, Saturn synodic 378.09, tropical year 365.242 days); the mean tropical year 365.24219 days is the conventional value. `24_1.py` defines the continued-fraction helpers reused by later scripts.

Calculation 24.1. Continued fraction and convergents of tropical year / synodic month; misses of the octaeteris, Metonic, Callippic and 334-year cycles.

```
        from fractions import Fraction as F
        def cf(x,n):
            a=[]
            for _ in range(n):
                i=x.numerator//x.denominator; a.append(i); x-=i
                if x==0: break
                x=1/x
            return a
        def conv(a):
            p0,q0,p1,q1=1,0,a[0],1; out=[(p1,q1)]
            for ai in a[1:]: p0,q0,p1,q1=p1,q1,ai*p1+p0,ai*q1+q0; out.append((p1,q1))

            return out
        Y=F('365.24219'); M=F('29.53059')
        x=Y/M; print("tropical year / synodic month =",float(x)); a=cf(x,9); print("continue
d fraction",a); print("convergents (months, years)",conv(a))
        for m,y in [(99,8),(235,19),(940,76),(4131,334)]: print(f"{m} months - {y} years =
{float(m*M-y*Y):+.4f} days")
        print("with NASA year 365.242: cf",cf(F('365.242')/M,9))
```

Output:

```
tropical year / synodic month = 12.36826592357281
continued fraction [12, 2, 1, 2, 1, 1, 17, 3, 1]
convergents (months, years) [(12, 1), (25, 2), (37, 3), (99, 8), (136, 11), (235,
19), (4131, 334), (12628, 1021), (16759, 1355)]
99 months - 8 years = +1.5909 days
235 months - 19 years = +0.0870 days
940 months - 76 years = +0.3482 days
4131 months - 334 years = -0.0242 days
with NASA year 365.242: cf [12, 2, 1, 2, 1, 1, 16, 1, 1]
```

Calculation 24.2. The Hebrew molad and its 19-year cycle; gaps between leap years.

```
        from fractions import Fraction as F
        mol=F(29)+F(12*1080+793,24*1080); Y=F('365.24219')
        print("molad =",mol,"=",float(mol),"days")
        print("235 molad =",float(235*mol),"| 19 tropical years =",float(19*Y),"| implied
year =",float(235*mol/19))
        print("leap years 3,6,8,11,14,17,19 -> 12*19+7 =",12*19+7)
        print("gaps between leap years:",[b-a for a,b in zip([3,6,8,11,14,17,19],[6,8,11,14,
17,19,22])])
```

Output:

```
molad = 765433/25920 = 29.53059413580247 days
235 molad = 6939.689621913581 | 19 tropical years = 6939.60161 | implied year =
365.24682220597794
leap years 3,6,8,11,14,17,19 -> 12*19+7 = 235
gaps between leap years: [3, 2, 3, 3, 3, 2, 3]
```

Calculation 24.3. The tabular Islamic 30-year cycle as a convergent; gaps between leap years.

```
        from fractions import Fraction as F
        exec(open('24_1.py').read().split('Y=F')[0])
        M=F('29.53059'); x=12*M-354
        print("lunar year excess over 354 days =",float(x),"; cf",cf(x,8),"; convergents",co
nv(cf(x,8)))
        print("tabular cycle 10631 days / 360 months =",round(10631/360,5))
        print("leap gaps 2,5,7,10,13,16,18,21,24,26,29:",[b-a for a,b in zip([2,5,7,10,13,
16,18,21,24,26,29],[5,7,10,13,16,18,21,24,26,29,32])])
```

Output:

```
lunar year excess over 354 days = 0.36708 ; cf [0, 2, 1, 2, 1, 1, 1, 2] ;
convergents [(0, 1), (1, 2), (1, 3), (3, 8), (4, 11), (7, 19), (11, 30), (29, 79)]
tabular cycle 10631 days / 360 months = 29.53056
leap gaps 2,5,7,10,13,16,18,21,24,26,29: [3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3]
```

Calculation 24.4. Solar leap rules: convergents of 0.24219; Julian and Gregorian errors.

```
        from fractions import Fraction as F
        exec(open('24_1.py').read().split('Y=F')[0])
        Y=F('365.24219'); x=Y-365
        print("fraction",float(x),"cf",cf(x,8),"convergents",conv(cf(x,8)))
        print("Gregorian 97/400 a convergent?",(97,400) in conv(cf(x,8)),"| error",float(F(9
7,400)-x),"days/yr -> 1 day per",round(1/float(F(97,400)-x)),"years")
        print("Julian 1/4 error",float(F(1,4)-x),"-> 1 day per",round(1/float(F(1,4)-x)),"ye
ars")
```

Output:

```
fraction 0.24219 cf [0, 4, 7, 1, 3, 24, 6, 2] convergents [(0, 1), (1, 4), (7, 29),
(8, 33), (31, 128), (752, 3105), (4543, 18758), (9838, 40621)]
Gregorian 97/400 a convergent? False | error 0.00031 days/yr -> 1 day per 3226 years
Julian 1/4 error 0.00781 -> 1 day per 128 years
```

Calculation 24.5. The Saros as a convergent of synodic/draconic month; the triple Saros.

```
        from fractions import Fraction as F
        exec(open('24_1.py').read().split('Y=F')[0])
        M=F('29.53059'); D=F('27.21222'); A=F('27.55455')
        print("223 synodic =",float(223*M),"| 242 draconic =",float(242*D),"| 239
anomalistic =",float(239*A))
        print("synodic/draconic cf",cf(M/D,8),"convergents",conv(cf(M/D,8)))
        fr=float(223*M)%1; print("Saros day fraction",round(fr,4),"| 3 Saros
=",round(float(669*M),4),"days; 3 x fraction =",round(3*fr,4))
```

Output:

```
223 synodic = 6585.32157 | 242 draconic = 6585.35724 | 239 anomalistic = 6585.53745
synodic/draconic cf [1, 11, 1, 2, 1, 4, 3, 4] convergents [(1, 1), (12, 11), (13,
12), (38, 35), (51, 47), (242, 223), (777, 716), (3350, 3087)]
Saros day fraction 0.3216 | 3 Saros = 19755.9647 days; 3 x fraction = 0.9647
```

Calculation 24.6. Venus and the Antikythera planetary relations against best approximations.

```
        from fractions import Fraction as F
        exec(open('24_1.py').read().split('Y=F')[0])
        Y=F('365.242'); V=F('583.92'); S=F('378.09')
        print("Venus synodic / year cf",cf(V/Y,8),"conv (synodic, years)->(years/periods):",
conv(cf(V/Y,8)))
        print("5 Venus synodic =",float(5*V),"| 8 years =",float(8*Y))
        for nm,P,yrs,c in [('Venus',V,462,289),('Saturn',S,442,427)]:
            x=Y/P; best=(1,None)
            for q in range(1,yrs+1):
                p=round(x*q); e=abs(x-F(p,q))
                if e<best[0]: best=(e,(p,q))
            print(f"{nm}: {yrs} yr = {float(yrs*Y/P):.3f} synodic; {c} syn - {yrs} yr = {flo
at(c*P-yrs*Y):+.2f} d; best rational periods/years with years<= {yrs}: {best[1]} ({f
loat(best[0]):.2e}) vs {c}/{yrs} ({float(abs(x-F(c,yrs))):.2e})")
```

Output:

```
Venus synodic / year cf [1, 1, 1, 2, 30, 1, 6, 1] conv (synodic, years)->(years/
periods): [(1, 1), (2, 1), (3, 2), (8, 5), (243, 152), (251, 157), (1749, 1094),
(2000, 1251)]
5 Venus synodic = 2919.6 | 8 years = 2921.936
Venus: 462 yr = 288.981 synodic; 289 syn - 462 yr = +11.08 d; best rational periods/
years with years<= 462: (157, 251) (2.06e-06) vs 289/462 (4.11e-05)
Saturn: 442 yr = 426.980 synodic; 427 syn - 442 yr = +7.47 d; best rational periods/
years with years<= 442: (199, 206) (7.45e-07) vs 427/442 (4.47e-05)
```

Calculation 24.7. Maya counts, the ladder test on counting units, Taixuanjing, Kuitems and Pew sums.

```
        from math import lcm
        from fractions import Fraction as F
        from ladder import lad
        print("lcm(260,365) =",lcm(260,365),"days =",lcm(260,365)//365,"haab")
        print("4400/149 =",round(4400/149,5),"| 405 lunations x 29.53059 =",round(405*29.530
59,2))
        print("on ladder:",[(u,lad(u)) for u in [7,12,19,20,24,30,52,60,64,81,260,360,7200,1
44000]])
        print("Taixuanjing 3^4 =",3**4,"| I Ching 2^6 =",2**6,"| balanced 4-trit range",-(3*
*4-1)//2,"to",(3**4-1)//2)
        print("Kuitems: 993 + 28 =",993+28)
        print("Pew 2020 shares sum:",round(28.8+25.6+24.2+14.9+4.1+0.2,1))
```

Output:

```
lcm(260,365) = 18980 days = 52 haab
4400/149 = 29.5302 | 405 lunations x 29.53059 = 11959.89
on ladder: [(7, False), (12, True), (19, False), (20, False), (24, True), (30,
False), (52, False), (60, False), (64, True), (81, True), (260, False), (360,
False), (7200, False), (144000, False)]
Taixuanjing 3^4 = 81 | I Ching 2^6 = 64 | balanced 4-trit range -40 to 40
Kuitems: 993 + 28 = 1021
Pew 2020 shares sum: 97.8
```

Calculation 24.8. Antikythera relations as sums of convergents, with a control for semiconvergents.

```
        # Can the Antikythera relations (years, synodic periods) be written as i*A + j*B
with A, B convergents?
```

```
        # And how many ways if semiconvergents are allowed (a control for how weak the test
is)?
        from fractions import Fraction as F
        exec(open('24_1.py').read().split('Y=F')[0])
        def semis(a):
            res=[]; p0,q0,p1,q1=1,0,a[0],1
            for ai in a[1:]:
                for t in range(1,ai+1): res.append((t*p1+p0,t*q1+q0))
                p0,q0,p1,q1=p1,q1,ai*p1+p0,ai*q1+q0
            return res
        Y=F('365.242')
        def sums(pool,yrs,cyc):
            return [(i,a,j,b) for a in pool for b in pool if a<b for i in range(1,60) for j
in range(1,60) if i*a[0]+j*b[0]==yrs and i*a[1]+j*b[1]==cyc]
        for nm,P,(yrs,cyc) in [('Saturn',F('378.09'),(442,427)),('Venus',F('583.92'),(462,28
9))]:
            a=cf(P/Y,10)
            cv=[c for c in conv(a) if c[0]<yrs]; sm=sorted({c for c in semis(a) if c[0]<yrs}
)
            print(nm,(yrs,cyc),"convergents (years, periods) below",yrs,":",cv)
            print("  sums of two convergents:",sums(cv,yrs,cyc) or "none")
            print("  number of decompositions if semiconvergents are allowed:",len(sums(sm,y
rs,cyc)))
```

Output:

```
Saturn (442, 427) convergents (years, periods) below 442 : [(1, 1), (29, 28), (59,
57), (147, 142), (206, 199)]
  sums of two convergents: [(7, (1, 1), 15, (29, 28)), (1, (1, 1), 3, (147, 142)),
(1, (29, 28), 7, (59, 57)), (5, (59, 57), 1, (147, 142)), (4, (59, 57), 1, (206,
199))]
  number of decompositions if semiconvergents are allowed: 10
Venus (462, 289) convergents (years, periods) below 462 : [(1, 1), (2, 1), (3, 2),
(8, 5), (243, 152), (251, 157)]
  sums of two convergents: [(2, (3, 2), 57, (8, 5))]
  number of decompositions if semiconvergents are allowed: 34
```

Calculation 24.9. Drift of a lunar month through the solar year.

```
        Y=365.24219; L=12*29.53059
        print("lunar year =",round(L,4),"days; shortfall",round(Y-L,4),"days; a fixed lunar
month runs through the solar year in",round(Y/(Y-L),2),"solar years")
```

Output:

```
lunar year = 354.3671 days; shortfall 10.8751 days; a fixed lunar month runs through
the solar year in 33.59 solar years
```

Calculation 24.10. Scaliger's Julian Period and the Calendar Round against the ladder.

```
        from ladder import lad
        print("Julian Period 28*19*15 =",28*19*15,"on ladder:",lad(7980))
        print("Calendar Round lcm(260,365) = 18980 on ladder:",lad(18980))
```

Output:

```
Julian Period 28*19*15 = 7980 on ladder: False
Calendar Round lcm(260,365) = 18980 on ladder: False
```

Calculations for Chapter 25 — Making Meaning

All scripts were run with Python 3.13 on 27 October 2026. `lad` is the ladder test printed in Calculation 22.5.

Calculation 25.1. Deontic statuses and Łukasiewicz values as trits; negation as the mirror of T12.1; closed status windows (T3, T4).

```
        from fractions import Fraction as F
        from itertools import product
        mirror={1:-1,0:0,-1:1}
        print("deontic digit of not-p (OB=+1, IM=-1, optional=0):",mirror)
        print("Lukasiewicz v -> 2v-1 and negation:",[(str(v),str(2*v-1),str(2*(1-v)-1)) for
v in (F(0),F(1,2),F(1))])
        W=list(product([1,0,-1],repeat=3)); c=[w for w in W if sum(w)%3==0]
        print("closed status windows:",len(c),"of",len(W))
        print("windows with an obligation and a prohibition and an option (all
different):",sum(1 for w in c if len(set(w))==3))
```

Output:

```
deontic digit of not-p (OB=+1, IM=-1, optional=0): {1: -1, 0: 0, -1: 1}
Lukasiewicz v -> 2v-1 and negation: [('0', '-1', '1'), ('1/2', '0', '0'), ('1', '1',
'-1')]
closed status windows: 9 of 27
windows with an obligation and a prohibition and an option (all different): 6
```

Calculation 25.2. Condorcet cycles: all 6^3 profiles of three voters over three options.

```
        from itertools import permutations, product
        from fractions import Fraction as F
        O=list(permutations('abc')); cyc=0
        for prof in product(O,repeat=3):
            m=lambda x,y:sum(o.index(x)<o.index(y) for o in prof)>=2
            if (m('a','b') and m('b','c') and m('c','a')) or (m('b','a') and m('c','b') and
m('a','c')): cyc+=1
        print("profiles with a majority cycle:",cyc,"of",6**3,"=",F(cyc,216))
```

Output:

```
profiles with a majority cycle: 12 of 216 = 1/18
```

Calculation 25.3. Loeschian numbers, Christaller K, Maxwell counts, E-series, Renard series, Pattern Language, Modulor, small ladder numbers.

```
        from ladder import lad
        print("Loeschian numbers x^2+xy+y^2 (<=40):",sorted({x*x+x*y+y*y for x in range(7) f
or y in range(7)} & set(range(41))))
        print("Christaller K on ladder:",[(k,lad(k)) for k in (3,4,7)])
        print("plane frame bars 2j-3:",[(j,2*j-3) for j in range(3,8)],"| polygon has j
bars, rigid only for j=3")
        print("space frame bars 3j-6:",[(j,3*j-6) for j in range(4,8)])
        print("E-series sizes on ladder:",[(n,lad(n)) for n in (3,6,12,24,48,96,
```

```
192)],"| E12 ratio",round(10**(1/12),4))
        print("A Pattern Language 253 =",11*23,"on ladder:",lad(253))
        print("Modulor: 2.26/1.13 =",round(2.26/1.13,3),"| 1.83/1.13 =",round(1.83/1.13,4),"
| golden ratio",round((1+5**0.5)/2,4))
        print("K=7 as a norm:",[(x,y) for x in range(4) for y in range(4) if x*x+x*y+y*y==7]
)
        print("Renard sizes 5,10,20,40,80 on ladder:",[(n,lad(n)) for n in (5,10,20,40,80)])

        print("ladder numbers 1..12:",[n for n in range(1,13) if lad(n)])
        print("12 divisible by:",[d for d in range(2,12) if 12%d==0])
```

Output:

```
Loeschian numbers x^2+xy+y^2 (<=40): [0, 1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27,
28, 31, 36, 37, 39]
Christaller K on ladder: [(3, True), (4, True), (7, False)]
plane frame bars 2j-3: [(3, 3), (4, 5), (5, 7), (6, 9), (7, 11)] | polygon has j
bars, rigid only for j=3
space frame bars 3j-6: [(4, 6), (5, 9), (6, 12), (7, 15)]
E-series sizes on ladder: [(3, True), (6, True), (12, True), (24, True), (48, True),
(96, True), (192, True)] | E12 ratio 1.2115
A Pattern Language 253 = 253 on ladder: False
Modulor: 2.26/1.13 = 2.0 | 1.83/1.13 = 1.6195 | golden ratio 1.618
K=7 as a norm: [(1, 2), (2, 1)]
Renard sizes 5,10,20,40,80 on ladder: [(5, False), (10, False), (20, False), (40,
False), (80, False)]
ladder numbers 1..12: [1, 2, 3, 4, 6, 8, 9, 12]
12 divisible by: [2, 3, 4, 6]
```

Calculation 25.4. Ratios from the PhilPapers 2020 survey, the Moral Machine and the buildings report.

```
        print("PhilPapers 2020, normative ethics inclusive: 36.99+32.05+30.56
=",round(36.99+32.05+30.56,2),"| exclusive: 25.04+19.70+21.42 =",round(25.04+19.70+2
1.42,2))
        print("Moral Machine: 40e6 decisions / 2e6 participants =",40e6/2e6)
        print("Buildings 32% energy, 34% CO2: ratio",round(34/32,3))
```

Output:

```
PhilPapers 2020, normative ethics inclusive: 36.99+32.05+30.56 = 99.6 | exclusive:
25.04+19.70+21.42 = 66.16
Moral Machine: 40e6 decisions / 2e6 participants = 20.0
Buildings 32% energy, 34% CO2: ratio 1.062
```

Calculation 25.5. Probability of a majority cycle for n random voters (exact multinomial sums) and Guilbaud’s limit.

```
        # Probability of a Condorcet cycle, three options, impartial culture: exact for
small odd n, Guilbaud limit
        from itertools import product
        from math import asin, pi, comb, factorial
        import numpy as np
        orders=[(0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0)]
        def pref(o,x,y): return o.index(x)<o.index(y)
        def cyc_prob(n):
            # sum over multinomial counts of the 6 orders
            tot=0; cyc=0
```

```
            def rec(k,left,cnt):
                nonlocal tot,cyc
                if k==5:
                    cnt=cnt+[left]; w=factorial(n)
                    for c in cnt: w//=factorial(c)
                    m=lambda x,y: sum(c for c,o in zip(cnt,orders) if pref(o,x,y))*2>n

                    tot+=w
                    if (m(0,1) and m(1,2) and m(2,0)) or (m(1,0) and m(2,1) and m(0,2)):
cyc+=w
                    return
                for c in range(left+1): rec(k+1,left-c,cnt+[c])
            rec(0,n,[]); return cyc,tot
        for n in (3,5,7,9,11,25):
            c,t=cyc_prob(n); print(f"n={n:2d} voters: cycle probability {c}/{t} = {c/t:.
5f}")
        print("Guilbaud limit 1/4 - (3/(2*pi))*asin(1/3) =",round(0.25-3/(2*pi)*asin(1/3),5)
)
```

Output:

```
n= 3 voters: cycle probability 12/216 = 0.05556
n= 5 voters: cycle probability 540/7776 = 0.06944
n= 7 voters: cycle probability 21000/279936 = 0.07502
n= 9 voters: cycle probability 785820/10077696 = 0.07798
n=11 voters: cycle probability 28956312/362797056 = 0.07981
n=25 voters: cycle probability 2397139928161319640/28430288029929701376 = 0.08432
Guilbaud limit 1/4 - (3/(2*pi))*asin(1/3) = 0.08774
```

Calculation 25.6. Łukasiewicz connectives in trit form: De Morgan, excluded middle, implication.

```
        # Three-valued logic: Lukasiewicz and Kleene connectives in trit form; which tables
preserve closure?
        from itertools import product
        T=[-1,0,1]
        neg=lambda x:-x
        conj=lambda x,y:min(x,y); disj=lambda x,y:max(x,y)
        luk_imp=lambda x,y:min(1,1-x+y)   # in trit form: min(1, y - x + 1) with values
-1,0,1
        # check mirror: neg(conj(x,y)) == disj(neg x, neg y)  (De Morgan)
        print("De Morgan holds on trits:",all(neg(conj(x,y))==disj(neg(x),neg(y)) for x in
T for y in T))
        print("excluded middle x OR not-x takes value:",{x:disj(x,neg(x)) for x in T})

        # closure of the window (x, y, x AND y): digit 0?
        c=sum(1 for x in T for y in T if (x+y+conj(x,y))%3==0)
        print("windows (x, y, x AND y) that close:",c,"of 9")
        c=sum(1 for x in T for y in T if (x+y+neg(x)+0)%3==0)
        print("Lukasiewicz implication table (trit form):",[[luk_imp(x,y) for y in T] for x
in T])
```

Output:

```
De Morgan holds on trits: True
excluded middle x OR not-x takes value: {-1: 1, 0: 0, 1: 1}
windows (x, y, x AND y) that close: 3 of 9
Lukasiewicz implication table (trit form): [[1, 1, 1], [0, 1, 1], [-1, 0, 1]]
```

Appendix B — References

The printed literature cited in each chapter. Web sources are linked inline in the chapter where they are used.

Chapter 1 — The Register: Mathematics, Logic and Computation

- Andrews, G.E. (2007). Euler's "De Partitio numerorum". *Bulletin of the American Mathematical Society* 44(4), 561–573.
- Arıkan, E. (2009). Channel polarization: a method for constructing capacity-achieving codes for symmetric binary-input memoryless channels. *IEEE Transactions on Information Theory* 55(7), 3051–3073.
- Campos, M., Griffiths, S., Morris, R. and Sahasrabudhe, J. (2023). An exponential improvement for diagonal Ramsey. arXiv:2303.09521.
- Eliahou, S. (1993). The 3x+1 problem: new lower bounds on nontrivial cycle lengths. *Discrete Mathematics* 118(1–3), 45–56.
- Ellenberg, J.S. and Gijswijt, D. (2017). On large subsets of F_q^n with no three-term arithmetic progression. *Annals of Mathematics* 185(1), 339–343.
- Feigenbaum, M.J. (1978). Quantitative universality for a class of nonlinear transformations. *Journal of Statistical Physics* 19(1), 25–52.
- Gleason, A.M. (1957). Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics (Indiana Univ. Math. J.)* 6(4), 885–893.
- Jeong, J.W., Choi, Y.E., Kim, W.S., Park, J.H., Kim, S., Shin, S., Lee, K., Chang, J., Kim, S.J. and Kim, K.R. (2019). Tunnelling-based ternary metal–oxide–semiconductor technology. *Nature Electronics* 2(7), 307–312.
- Knuth, D.E. (1997). *The Art of Computer Programming*, Vol. 2, *Seminumerical Algorithms*, 3rd ed. Addison-Wesley.
- Kochen, S. and Specker, E.P. (1967). The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics (Indiana Univ. Math. J.)* 17(1), 59–87.
- Li, F., Liu, B., Wang, X., Zhang, B. and Yan, J. (2016). Ternary weight networks. arXiv:1605.04711.
- Li, T.-Y. and Yorke, J.A. (1975). Period three implies chaos. *American Mathematical Monthly* 82(10), 985–992.
- Li, X. and Liao, S. (2025). Discovery of 10,059 new three-dimensional periodic orbits of general three-body problem. arXiv:2508.08568.
- Lovász, L. (1979). On the Shannon capacity of a graph. *IEEE Transactions on Information Theory* IT-25(1), 1–7.
- Ma, S., Wang, H., Ma, L., et al. (2024). The era of 1-bit LLMs: all large language models are in 1.58 bits. arXiv:2402.17764.
- Ma, S., et al. (2025). BitNet b1.58 2B4T technical report. arXiv:2504.12285.

- May, R.M. (1976). Simple mathematical models with very complicated dynamics. *Nature* 261(5560), 459–467.
- Mihăilescu, P. (2004). Primary cyclotomic units and a proof of Catalan’s conjecture. *Journal für die reine und angewandte Mathematik* 572, 167–195.
- Razborov, A.A. (1987). Lower bounds for the size of circuits of bounded depth with basis {⊕, ∨}. *Mathematical Notes of the Academy of Sciences of the USSR* 41(4), 333–338.
- Romera-Paredes, B., et al. (2024). Mathematical discoveries from program search with large language models. *Nature* 625, 468–475.
- Shannon, C.E. (1948). A mathematical theory of communication. *Bell System Technical Journal* 27(3), 379–423.
- Smolensky, R. (1987). Algebraic methods in the theory of lower bounds for Boolean circuit complexity. *Proceedings of the 19th ACM Symposium on Theory of Computing*, 77–82.
- Stifel, M. (1544). *Arithmetica Integra*. Nuremberg.
- von Wright, G.H. (1951). Deontic logic. *Mind* 60(237), 1–15.
- Konstapel, J. (2026). *The Vacuum.Net Theory: Foundations*, Ninth Edition (version 9.3). Constable Research, Leiden.

Chapter 2 — Particles and Fields

- Dirac, P.A.M. (1928). The quantum theory of the electron. *Proceedings of the Royal Society of London A* 117(778), 610–624.
- Efimov, V. (1970). Energy levels arising from resonant two-body forces in a three-body system. *Physics Letters B* 33(8), 563–564.
- Faddeev, L. and Niemi, A.J. (1997). Stable knot-like structures in classical field theory. *Nature* 387, 58–61.
- Fan, X., Myers, T.G., Sukra, B.A.D. and Gabrielse, G. (2023). Measurement of the electron magnetic moment. *Physical Review Letters* 130, 071801.
- Hopf, H. (1931). Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. *Mathematische Annalen* 104, 637–665.
- Hu, J., Webb, J.K., Ayres, T.R., et al. (2021). Measuring the fine-structure constant on a white dwarf surface; a detailed analysis of Fe V absorption in G191-B2B. *Monthly Notices of the Royal Astronomical Society*; arXiv:2007.10905.
- Kleckner, D. and Irvine, W.T.M. (2013). Creation and dynamics of knotted vortices. *Nature Physics* 9(4), 253–258.
- Lange, R., Huntemann, N., Rahm, J.M., Sanner, C., Shao, H., Lipphardt, B., Tamm, Chr., Weyers, S. and Peik, E. (2021). Improved limits for violations of local position invariance from atomic clock comparisons. *Physical Review Letters* 126, 011102.
- Maiman, T.H. (1960). Stimulated optical radiation in ruby. *Nature* 187, 493–494.
- Morel, L., Yao, Z., Cladé, P. and Guellati-Khélifa, S. (2020). Determination of the fine-structure constant with an accuracy of 81 parts per trillion. *Nature* 588, 61–65.

- Rauch, H., Zeilinger, A., Badurek, G., Wilfing, A., Bauspiess, W. and Bonse, U. (1975). Verification of coherent spinor rotation of fermions. *Physics Letters A* 54(6), 425–427.
- Salpeter, E.E. (1952). Nuclear reactions in stars without hydrogen. *Astrophysical Journal* (cited after the Wikipedia article on the triple-alpha process; volume and pages not verified).
- Tong, D. (2017). Line operators in the Standard Model. arXiv:1705.01853.
- Werner, S.A., Colella, R., Overhauser, A.W. and Eagen, C.F. (1975). Observation of the phase shift of a neutron due to precession in a magnetic field. *Physical Review Letters* 35(16), 1053–1054.
- Witten, E. (1989). Quantum field theory and the Jones polynomial. *Communications in Mathematical Physics* 121(3), 351–399.
- Konstapel, J. (2026). *The Vacuum.Net Theory: Foundations*, Ninth Edition (version 9.3); and *The Vacuum.Net Theory: Physics*, Ninth Edition. Constable Research, Leiden.

Chapter 3 — Forces and Motion

- Gleason, A.M. (1957). Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics (Indiana Univ. Math. J.)* 6(4), 885–893.
- Hamada, N., Sawada, S. and Oshiyama, A. (1992). New one-dimensional conductors: graphitic microtubules. *Physical Review Letters* 68(10), 1579–1581.
- Kane, C.L. and Lubensky, T.C. (2014). Topological boundary modes in isostatic lattices. *Nature Physics* 10, 39–45.
- Kochen, S. and Specker, E.P. (1967). The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics (Indiana Univ. Math. J.)* 17(1), 59–87.
- Laughlin, R.B. (1983). Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations. *Physical Review Letters* 50(18), 1395–1398.
- Lovász, L. (1979). On the Shannon capacity of a graph. *IEEE Transactions on Information Theory* IT-25(1), 1–7.
- Maiman, T.H. (1960). Stimulated optical radiation in ruby. *Nature* 187, 493–494.
- Pólya, G. (1921). Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz. *Mathematische Annalen* 84, 149–160.
- Schnakenberg, J. (1976). Network theory of microscopic and macroscopic behavior of master equation systems. *Reviews of Modern Physics* 48(4), 571.
- Tsui, D.C., Stormer, H.L. and Gossard, A.C. (1982). Two-dimensional magnetotransport in the extreme quantum limit. *Physical Review Letters* 48(22), 1559–1562.
- Wannier, G.H. (1950). Antiferromagnetism. The triangular Ising net. *Physical Review* 79(2), 357–364.
- Watson, G.N. (1939). Three triple integrals. *Quarterly Journal of Mathematics (Oxford)* 10, 266–276.

- Konstapel, J. (2026). *The Vacuum.Net Theory: Foundations*, Ninth Edition (version 9.3); and *The Vacuum.Net Theory: Physics*, Ninth Edition. Constable Research, Leiden.

Chapter 4 — The Cosmos

- Banik, I., Pittordis, C., Sutherland, W., Famaey, B., Ibata, R., Mieske, S. and Zhao, H. (2023). Strong constraints on the gravitational law from Gaia DR3 wide binaries. *Monthly Notices of the Royal Astronomical Society* 527, 4573–4615; arXiv: 2311.03436.
- DESI Collaboration (2025). DESI DR2 Results II: measurements of baryon acoustic oscillations and cosmological constraints. arXiv:2503.14738.
- Fabrycky, D.C., et al. (2014). Architecture of Kepler's multi-transiting systems. II. New investigations with twice as many candidates. *Astrophysical Journal* 790, 146; arXiv:1202.6328.
- Hu, J., Webb, J.K., Ayres, T.R., et al. (2021). Measuring the fine-structure constant on a white dwarf surface. *Monthly Notices of the Royal Astronomical Society*; arXiv: 2007.10905.
- Kepler, J. (1619). *Harmonices Mundi*. Linz.
- Luque, R., Osborn, H.P., Leleu, A., et al. (2023). A resonant sextuplet of sub-Neptunes transiting the bright star HD 110067. *Nature* 623, 932–937.
- Milgrom, M. (1983). A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis. *Astrophysical Journal* 270, 365.
- Milgrom, M. (2014). The MOND paradigm of modified dynamics. *Scholarpedia* 9(6), 31410.
- Planck Collaboration (2020). Planck 2018 results. VI. Cosmological parameters. *Astronomy & Astrophysics* 641, A6.
- Konstapel, J. (2026). *The Vacuum.Net Theory: Physics*, Ninth Edition. Constable Research, Leiden.

Chapter 5 — Engineering of Energy and Signal

- Google Quantum AI and Collaborators (2025). Quantum error correction below the surface code threshold. *Nature*; arXiv:2408.13687.
- Fortescue, C.L. (1918). Method of symmetrical co-ordinates applied to the solution of polyphase networks. *Transactions of the AIEE* 37(2), 1027–1140.
- Google Quantum AI and Collaborators (2025). Observation of constructive interference at the edge of quantum ergodicity. *Nature* 646.
- Hamada, N., Sawada, S. and Oshiyama, A. (1992). New one-dimensional conductors: graphitic microtubules. *Physical Review Letters* 68(10), 1579–1581.
- Jeong, J.W., Choi, Y.E., Kim, W.S., Park, J.H., Kim, S., Shin, S., Lee, K., Chang, J., Kim, S.J. and Kim, K.R. (2019). Tunnelling-based ternary metal–oxide–semiconductor technology. *Nature Electronics* 2(7), 307–312.
- Maiman, T.H. (1960). Stimulated optical radiation in ruby. *Nature* 187, 493–494.

- Konstapel, J. (2026). *The Vacuum.Net Theory: Foundations*, Ninth Edition (version 9.3). Constable Research, Leiden.

Chapter 6 — The Elements, Molecules and Materials

- Bernal, J.D. and Fowler, R.H. (1933). A theory of water and ionic solution, with particular reference to hydrogen and hydroxyl ions. *Journal of Chemical Physics* 1, 515–548.
- Cayley, A. (1875). Ueber die analytischen Figuren, welche in der Mathematik Bäume genannt werden und ihre Anwendung auf die Theorie chemischer Verbindungen. *Berichte der deutschen chemischen Gesellschaft* 8, 1056–1059.
- Chichak, K.S. et al. (2004). Molecular Borromean rings. *Science* 304, 1308–1312.
- Dietrich-Buchecker, C.O. and Sauvage, J.-P. (1989). A synthetic molecular trefoil knot. *Angewandte Chemie International Edition* 28, 189–192.
- Döbereiner, J.W. (1829). Versuch zu einer Gruppirung der elementaren Stoffe nach ihrer Analogie. *Annalen der Physik und Chemie*, 2nd series, 15(2), 301–307.
- Fricke, B., Greiner, W. and Waber, J.T. (1971). The continuation of the periodic table up to Z = 172. *Theoretica Chimica Acta* 21, 235.
- Guinier, A. et al. (1984). Nomenclature of polytype structures. *Acta Crystallographica* A40, 399–404.
- Hamada, N., Sawada, S. and Oshiyama, A. (1992). New one-dimensional conductors: graphitic microtubules. *Physical Review Letters* 68, 1579–1581.
- Mao, C., Sun, W. and Seeman, N.C. (1997). Assembly of Borromean rings from DNA. *Nature* 386, 137–138.
- Pauling, L. (1935). The structure and entropy of ice and of other crystals with some randomness of atomic arrangement. *Journal of the American Chemical Society* 57, 2680–2684.
- Pyykkö, P. (2011). A suggested periodic table up to Z ≤ 172, based on Dirac–Fock calculations on atoms and ions. *Physical Chemistry Chemical Physics* 13, 161–168.
- Saito, R., Fujita, M., Dresselhaus, G. and Dresselhaus, M.S. (1992). Electronic structure of chiral graphene tubules. *Physical Review B* 46, 1804.
- Soai, K. et al. (1995). Asymmetric autocatalysis and amplification of enantiomeric excess of a chiral molecule. *Nature* 378, 767–768.
- Son, Y.-W., Cohen, M.L. and Louie, S.G. (2006). Energy gaps in graphene nanoribbons. *Physical Review Letters* 97, 216803.
- Pandey, D. and Krishna, P. Notations for close-packed structures, §9.2 in *International Tables for Crystallography*, Vol. C, 752–753.

Chapter 7 — Reactions and Heat

- Cross, M.C. and Hohenberg, P.C. (1993). Pattern formation outside of equilibrium. *Reviews of Modern Physics* 65, 851–1112.
- Debye, P. and Hückel, E. (1923). Zur Theorie der Elektrolyte. I. Gefrierpunktserniedrigung und verwandte Erscheinungen. *Physikalische Zeitschrift* 24, 185–206.

- Gibbs, J.W. (1875–1878). On the equilibrium of heterogeneous substances. *Transactions of the Connecticut Academy of Arts and Sciences* 3.
- Li, T.-Y. and Yorke, J.A. (1975). Period three implies chaos. *American Mathematical Monthly* 82, 985–992.
- Lorenz, E.N. (1963). Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences* 20, 130–141.
- Prigogine, I. and Lefever, R. (1968). Symmetry breaking instabilities in dissipative systems. II. *Journal of Chemical Physics* 48, 1695–1700.
- Sharkovskii, O.M. (1964). Co-existence of cycles of a continuous mapping of the line into itself. *Ukrainian Mathematical Journal* 16, 61–71.
- Simoyi, R.H., Wolf, A. and Swinney, H.L. (1982). One-dimensional dynamics in a multicomponent chemical reaction. *Physical Review Letters* 49, 245–248.
- Soai, K. et al. (1995). Asymmetric autocatalysis and amplification of enantiomeric excess of a chiral molecule. *Nature* 378, 767–768.
- Turing, A.M. (1952). The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society B* 237, 37–72.
- Watson, G.N. (1939). Three triple integrals. *Quarterly Journal of Mathematics, Oxford* 10, 266–276.
- Feigenbaum, M.J. (1978). Quantitative universality for a class of nonlinear transformations. *Journal of Statistical Physics* 19, 25–52.

Chapter 8 — The Earth

- Birman, J.S. and Williams, R.F. (1983). Knotted periodic orbits in dynamical systems I: Lorenz’s equations. *Topology* 22, 47–82.
- Delplace, P., Marston, J.B. and Venaille, A. (2017). Topological origin of equatorial waves. *Science* 358, 1075–1077.
- Doodson, A.T. (1921). The harmonic development of the tide-generating potential. *Proceedings of the Royal Society A* 100, 305–329.
- Gutenberg, B. and Richter, C.F. (1944). Frequency of earthquakes in California. *Bulletin of the Seismological Society of America* 34, 185–188.
- Hays, J.D., Imbrie, J. and Shackleton, N.J. (1976). Variations in the Earth’s orbit: pacemaker of the ice ages. *Science* 194, 1121–1132.
- Horton, R.E. (1945). Erosional development of streams and their drainage basins. *Geological Society of America Bulletin* 56(3).
- Hurst, H.E. (1951). Long-term storage capacity of reservoirs. *Transactions of the American Society of Civil Engineers* 116, 770–808.
- Huybers, P. and Wunsch, C. (2005). Obliquity pacing of the late Pleistocene glacial terminations. *Nature* 434, 491–494.
- Huybers, P. (2007). Glacial variability over the last two million years: an extended depth-derived agemodel, continuous obliquity pacing, and the Pleistocene progression. *Quaternary Science Reviews* 26, 37–55.
- Lorenz, E.N. (1963). Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences* 20, 130–141.

- McKenzie, D.P. and Morgan, W.J. (1969). Evolution of triple junctions. *Nature* 224, 125–133.
- Munk, W.H. (1966). Abyssal recipes. *Deep-Sea Research* 13, 707–730.
- Stommel, H. (1961). Thermohaline convection with two stable regimes of flow. *Tellus* 13, 224–230.
- Strahler, A.N. (1957). Quantitative analysis of watershed geomorphology. *Transactions of the American Geophysical Union* 38(6).
- Wilson, J.T. (1966). Did the Atlantic close and then re-open? *Nature* 211, 676–681.

Chapter 9 — Building and Making

- Ardern, E. and Lockett, W.T. (1914). Experiments on the oxidation of sewage without the aid of filters. *Journal of the Society of Chemical Industry* 33, 523–539.
- Calladine, C.R. (1978). Buckminster Fuller’s “tensegrity” structures and Clerk Maxwell’s rules for the construction of stiff frames. *International Journal of Solids and Structures* 14, 161–172.
- Deshpande, V.S., Ashby, M.F. and Fleck, N.A. (2001). Foam topology: bending versus stretching dominated architectures. *Acta Materialia* 49, 1035–1040.
- Gibson, L.J. and Ashby, M.F. (1997). *Cellular Solids: Structure and Properties*, 2nd ed. Cambridge University Press.
- Laman, G. (1970). On graphs and rigidity of plane skeletal structures. *Journal of Engineering Mathematics* 4, 331–340.
- Maxwell, J.C. (1864). On the calculation of the equilibrium and stiffness of frames. *Philosophical Magazine* 27, 294–299.
- Mulder, A., van de Graaf, A.A., Robertson, L.A. and Kuenen, J.G. (1995). Anaerobic ammonium oxidation discovered in a denitrifying fluidized bed reactor. *FEMS Microbiology Ecology* 16, 177–184.
- Simoyi, R.H., Wolf, A. and Swinney, H.L. (1982). One-dimensional dynamics in a multicomponent chemical reaction. *Physical Review Letters* 49, 245–248.

Chapter 10 — The Code and the Cell

- Aartsma-Rus, A., Van Deutekom, J.C.T., Fokkema, I.F., Van Ommen, G.J.B. and Den Dunnen, J.T. (2006). Entries in the Leiden Duchenne muscular dystrophy mutation database: an overview of mutation types and paradoxical cases that confirm the reading-frame rule. *Muscle & Nerve* 34, 135–144.
- Boudko, S.P. and Bächinger, H.P. (2016). Structural insight for chain selection and stagger control in collagen. *Scientific Reports* 6, 37831.
- Caspar, D.L.D. and Klug, A. (1962). Physical principles in the construction of regular viruses. *Cold Spring Harbor Symposia on Quantitative Biology* 27, 1–24.
- Crick, F.H.C., Barnett, L., Brenner, S. and Watts-Tobin, R.J. (1961). General nature of the genetic code for proteins. *Nature* 192, 1227–1232.
- Crick, F.H.C., Griffith, J.S. and Orgel, L.E. (1957). Codes without commas. *Proceedings of the National Academy of Sciences USA* 43, 416–421.

- Elowitz, M.B. and Leibler, S. (2000). A synthetic oscillatory network of transcriptional regulators. *Nature* 403, 335–338.
- Fickett, J.W. (1982). Recognition of protein coding regions in DNA sequences. *Nucleic Acids Research* 10, 5303–5318.
- Gardner, T.S., Cantor, C.R. and Collins, J.J. (2000). Construction of a genetic toggle switch in *Escherichia coli*. *Nature* 403, 339–342.
- Hardy, G.H. (1908). Mendelian proportions in a mixed population. *Science* 28, 49–50.
- Ikeda, T., Fujinaka, H., Goto, K. et al. (2019). Becker muscular dystrophy caused by exon 2-truncating mutation of DMD. *Human Genome Variation* 6, 52.
- Lin, M., Whitmire, S., Chen, J., Farrel, A., Shi, X. and Guo, J. (2017). Effects of short indels on protein structure and function in human genomes. *Scientific Reports* 7, 9313.
- Mandelkow, E.-M., Schultheiss, R., Rapp, R., Müller, M. and Mandelkow, E. (1986). On the surface lattice of microtubules: helix starts, protofilament number, seam, and handedness. *Journal of Cell Biology* 102, 1067–1073.
- Mendel, G. (1866). Versuche über Pflanzenhybriden. *Verhandlungen des naturforschenden Vereines in Brünn* 4, 3–47.
- Mitchison, T. and Kirschner, M. (1984). Dynamic instability of microtubule growth. *Nature* 312, 237–242.
- Monaco, A.P., Bertelson, C.J., Liechti-Gallati, S., Moser, H. and Kunkel, L.M. (1988). An explanation for the phenotypic differences between patients bearing partial deletions of the DMD locus. *Genomics* 2, 90–95.
- Liu, L.F., Liu, C.-C. and Alberts, B.M. (1980). Type II DNA topoisomerases: enzymes that can unknot a topologically knotted DNA molecule via a reversible double-strand break. *Cell* 19, 697–707.
- Micheletti, C., Marenduzzo, D., Orlandini, E. and Sumners, D.W. (2008). Simulations of knotting in confined circular DNA. *Biophysical Journal* 95, 3591–3599.
- Pólya, G. (1921). Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz. *Mathematische Annalen* 84, 149–160.
- Pauling, L., Corey, R.B. and Branson, H.R. (1951). The structure of proteins: two hydrogen-bonded helical configurations of the polypeptide chain. *Proceedings of the National Academy of Sciences USA* 37, 205–211.
- Pokhilko, A., Fernández, A.P., Edwards, K.D., Southern, M.M., Halliday, K.J. and Millar, A.J. (2012). The clock gene circuit in *Arabidopsis* includes a repressilator with additional feedback loops. *Molecular Systems Biology* 8, 574.
- Robertson, W.E., Rehm, F.B.H., Spinck, M. et al. (2025). *Escherichia coli* with a 57-codon genetic code. *Science* 390, eady4368.
- Sanger, F., Air, G.M., Barrell, B.G. et al. (1977). Nucleotide sequence of bacteriophage φX174 DNA. *Nature* 265, 687–695.
- Sinkovits, R.S. and Baker, T.S. (2010). A tale of two symmetrons: rules for construction of icosahedral capsids from trisymmetrons and pentasymmetrons. *Journal of Structural Biology* 170, 109–116.

- Sui, H. and Downing, K.H. (2010). Structural basis of inter-protofilament interaction and lateral deformation of microtubules. *Structure* 18, 1022–1031.
- Tiwari, S., Ramachandran, S., Bhattacharya, A., Bhattacharya, S. and Ramaswamy, R. (1997). Prediction of probable genes by Fourier analysis of genomic sequences. *Bioinformatics (CABIOS)* 13, 263–270.
- Varadi, M., Bertoni, D., Magana, P. et al. (2024). AlphaFold Protein Structure Database in 2024: providing structure coverage for over 214 million protein sequences. *Nucleic Acids Research* 52, D368–D375.
- White, J.H. (1969). Self-linking and the Gauss integral in higher dimensions. *American Journal of Mathematics* 91, 693–728.
- Wang, J.C. (1979). Helical repeat of DNA in solution. *Proceedings of the National Academy of Sciences USA* 76, 200–203.
- Watson, J.D. and Crick, F.H.C. (1953). Molecular structure of nucleic acids: a structure for deoxyribose nucleic acid. *Nature* 171, 737–738.
- Weinberg, W. (1908). Über den Nachweis der Vererbung beim Menschen. *Jahreshefte des Vereins für vaterländische Naturkunde in Württemberg* 64, 368–382.

Chapter 11 — Metabolism and Interaction

- Allesina, S. and Levine, J.M. (2011). A competitive network theory of species diversity. *Proceedings of the National Academy of Sciences USA* 108, 5638–5642.
- Allesina, S. and Tang, S. (2012). Stability criteria for complex ecosystems. *Nature*, doi:10.1038/nature10832 (published online 19 February 2012).
- Eigen, M. (1971). Selforganization of matter and the evolution of biological macromolecules. *Die Naturwissenschaften* 58, 465–523.
- Eigen, M. and Schuster, P. (1979). *The Hypercycle: A Principle of Natural Self-Organization*. Springer, Berlin.
- Fisher, D.C. and Reeves, J. (1995). Optimal strategies for random tournament games. *Linear Algebra and its Applications* 217, 83–85.
- Kerr, B., Riley, M.A., Feldman, M.W. and Bohannan, B.J.M. (2002). Local dispersal promotes biodiversity in a real-life game of rock–paper–scissors. *Nature* 418, 171–174.
- Krebs, C.J., Boonstra, R. and Boutin, S. (2018). Using experimentation to understand the 10-year snowshoe hare cycle in the boreal forest of North America. *Journal of Animal Ecology* 87, 87–100.
- Krebs, H.A. and Johnson, W.A. (1937). Metabolism of ketonic acids in animal tissues. *Biochemical Journal* 31, 645–660.
- Lotka, A.J. (1925). *Elements of Physical Biology*. Williams & Wilkins, Baltimore.
- Luria, S.E. and Delbrück, M. (1943). Mutations of bacteria from virus sensitivity to virus resistance. *Genetics* 28, 491–511.
- May, R.M. (1972). Will a large complex system be stable? *Nature* 238, 413–414.
- Noji, H., Yasuda, R., Yoshida, M. and Kinosita, K. (1997). Direct observation of the rotation of F1-ATPase. *Nature* 386, 299–302.

- Sender, R., Fuchs, S. and Milo, R. (2016). Revised estimates for the number of human and bacteria cells in the body. *PLoS Biology* 14, e1002533.
- Sinervo, B. and Lively, C.M. (1996). The rock–paper–scissors game and the evolution of alternative male strategies. *Nature* 380, 240–243.
- Volterra, V. (1926). Fluctuations in the abundance of a species considered mathematically. *Nature* 118, 558–560.

Chapter 12 — Evolution and Development

- Costantino, R.F., Desharnais, R.A., Cushing, J.M. and Dennis, B. (1997). Chaotic dynamics in an insect population. *Science* 275, 389–391.
- Dennis, B., Desharnais, R.A., Cushing, J.M. and Costantino, R.F. (1997). Transitions in population dynamics: equilibria to periodic cycles to aperiodic cycles. *Journal of Animal Ecology* 66, 704–729.
- Dennis, B., Desharnais, R.A., Cushing, J.M., Henson, S.M. and Costantino, R.F. (2001). Estimating chaos and complex dynamics in an insect population. *Ecological Monographs* 71, 277–303.
- Dodd, A.N., Salathia, N., Hall, A. et al. (2005). Plant circadian clocks increase photosynthesis, growth, survival, and competitive advantage. *Science* 309, 630–633.
- Douady, S. and Couder, Y. (1992). Phyllotaxis as a physical self-organized growth process. *Physical Review Letters* 68, 2098–2101.
- Earn, D.J.D., Rohani, P., Bolker, B.M. and Grenfell, B.T. (2000). A simple model for complex dynamical transitions in epidemics. *Science* 287, 667–670.
- Feigenbaum, M.J. (1978). Quantitative universality for a class of nonlinear transformations. *Journal of Statistical Physics* 19, 25–52.
- Krebs, C.J., Boonstra, R. and Boutin, S. (2018). Using experimentation to understand the 10-year snowshoe hare cycle in the boreal forest of North America. *Journal of Animal Ecology* 87, 87–100.
- Li, T.-Y. and Yorke, J.A. (1975). Period three implies chaos. *American Mathematical Monthly* 82, 985–992.
- Matsuda, M., Hayashi, H., Garcia-Ojalvo, J. et al. (2020). Species-specific segmentation clock periods are due to differential biochemical reaction speeds. *Science* 369, 1450–1455.
- May, R.M. (1976). Simple mathematical models with very complicated dynamics. *Nature* 261, 459–467.
- Maynard Smith, J. and Szathmáry, E. (1995). *The Major Transitions in Evolution*. W.H. Freeman, Oxford.
- Newport, J. and Kirschner, M. (1982). A major developmental transition in early Xenopus embryos: I. Characterization and timing of cellular changes at the midblastula stage. *Cell* 30, 675–686.
- Olsen, L.F. and Schaffer, W.M. (1990). Chaos versus noisy periodicity: alternative hypotheses for childhood epidemics. *Science* 249, 499–504.

- Pokhilko, A., Fernández, A.P., Edwards, K.D., Southern, M.M., Halliday, K.J. and Millar, A.J. (2012). The clock gene circuit in *Arabidopsis* includes a repressilator with additional feedback loops. *Molecular Systems Biology* 8, 574.
- Raup, D.M. and Sepkoski, J.J. (1982). Mass extinctions in the marine fossil record. *Science* 215, 1501–1503.
- Raup, D.M. and Sepkoski, J.J. (1984). Periodicity of extinctions in the geologic past. *Proceedings of the National Academy of Sciences USA* 81, 801–805.
- Sharkovskii, O.M. (1964). Co-existence of cycles of a continuous mapping of the line into itself. *Ukrainian Mathematical Journal* 16, 61–71.
- Sinervo, B. and Lively, C.M. (1996). The rock–paper–scissors game and the evolution of alternative male strategies. *Nature* 380, 240–243.
- Swinton, J., Ochu, E. and the MSI Turing’s Sunflower Consortium (2016). Novel Fibonacci and non-Fibonacci structure in the sunflower: results of a citizen science experiment. *Royal Society Open Science* 3, 160091.
- Turing, A.M. (1952). The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society B* 237, 37–72.

Chapter 13 — Cultivating Life

- Allen, S.K. and Downing, S.L. (1986). Performance of triploid Pacific oysters. *Journal of Experimental Marine Biology and Ecology* 102, 197–208.
- Beverton, R.J.H. and Holt, S.J. (1957). *On the Dynamics of Exploited Fish Populations*. Fishery Investigations Series II, Vol. XIX. Ministry of Agriculture, Fisheries and Food, London.
- Carlowitz, H.C. von (1713). *Sylvicultura oeconomica, oder haußwirthliche Nachricht und naturmäßige Anweisung zur wilden Baum-Zucht*. Braun, Leipzig.
- Elowitz, M.B. and Leibler, S. (2000). A synthetic oscillatory network of transcriptional regulators. *Nature* 403, 335–338.
- Faustmann, M. (1849). Berechnung des Werthes, welchen Waldboden sowie noch nicht haubare Holzbestände für die Waldwirthschaft besitzen. *Allgemeine Forst- und Jagd-Zeitung*, December 1849.
- Gardner, T.S., Cantor, C.R. and Collins, J.J. (2000). Construction of a genetic toggle switch in *Escherichia coli*. *Nature* 403, 339–342.
- Kerr, B., Riley, M.A., Feldman, M.W. and Bohannan, B.J.M. (2002). Local dispersal promotes biodiversity in a real-life game of rock–paper–scissors. *Nature* 418, 171–174.
- Knipling, E.F. (1955). Possibilities of insect control or eradication through the use of sexually sterile males. *Journal of Economic Entomology* 48, 459–462.
- May, R.M. (1976). Simple mathematical models with very complicated dynamics. *Nature* 261, 459–467.
- Ricker, W.E. (1954). Stock and recruitment. *Journal of the Fisheries Research Board of Canada* 11, 559–623.
- Robertson, W.E., Rehm, F.B.H., Spinck, M. et al. (2025). *Escherichia coli* with a 57-codon genetic code. *Science* 390, eady4368.

Chapter 14 — Body and Brain

- Azevedo, F.A.C. et al. (2009). Equal numbers of neuronal and nonneuronal cells make the human brain an isometrically scaled-up primate brain. *Journal of Comparative Neurology* 513(5), 532–541.
- Baker, E.N. et al. (1988). The structure of 2Zn pig insulin crystals at 1.5 Å resolution. *Philosophical Transactions of the Royal Society B* 319, 369–456.
- Belchetz, P.E. et al. (with Knobil, E.) (1978). Hypophysial responses to continuous and intermittent delivery of hypothalamic gonadotropin-releasing hormone. *Science* 202(4368), 631–633.
- Bianconi, E. et al. (2013). An estimation of the number of cells in the human body. *Annals of Human Biology* 40(6), 463–471.
- Buzsáki, G. and Draguhn, A. (2004). Neuronal oscillations in cortical networks. *Science* 304, 1926–1929.
- Butticè, G. et al. (1990). Evolution of collagen IV genes from a 54-base pair exon: a role for introns in gene evolution. *Journal of Molecular Evolution* 30, 479–488.
- Chu, M.-L. et al. (1984). Human proα1(I) collagen gene structure reveals evolutionary conservation of a pattern of introns and exons. *Nature* 310, 337–340.
- Daley, M.A., Bramble, D.M. and Carrier, D.R. (2013). Impact loading and locomotor-respiratory coordination significantly influence breathing dynamics in running humans. *PLoS ONE* 8(8), e70752.
- Dorkenwald, S., Matsliah, A., Sterling, A.R. et al. (2024). [Whole-brain connectome of the adult fly]. *Nature* 634, 124–138.
- Fiete, I.R., Burak, Y. and Brookings, T. (2008). What grid cells convey about rat location. *Journal of Neuroscience* 28(27), 6858–6871.
- Galis, F. (1999). Why do almost all mammals have seven cervical vertebrae? Developmental constraints, Hox genes, and cancer. *Journal of Experimental Zoology*.
- Galis, F., Van Dooren, T.J.M., Feuth, J.D., Metz, J.A.J., Witkam, A., Ruinard, S., Steigenga, M.J. and Wijnaendts, L.C.D. (2006). Extreme selection in humans against homeotic transformations of cervical vertebrae. IIASA Interim Report IR-06-071.
- Gonze, D. and Abou-Jaoudé, W. (2013). The Goodwin model: behind the Hill function. *PLoS ONE* 8(8), e69573.
- Guevara, M.R., Glass, L. and Shrier, A. (1981). Phase locking, period-doubling bifurcations, and irregular dynamics in periodically stimulated cardiac cells. *Science* 214, 1350–1353.
- Hafting, T., Fyhn, M., Molden, S., Moser, M.-B. and Moser, E.I. (2005). Microstructure of a spatial map in the entorhinal cortex. *Nature* 436, 801–806.
- Hodgkin, A.L. and Huxley, A.F. (1952). A quantitative description of membrane current and its application to conduction and excitation in nerve. *Journal of Physiology*.
- Kliewer, S.A. et al. (1992). Retinoid X receptor–COUP-TF interactions modulate retinoic acid signaling. *Proceedings of the National Academy of Sciences USA* 89(4), 1448–1452.

- MICrONS Consortium et al. (2025). [Cubic-millimetre connectome of mouse visual cortex]. *Nature* 640, 435–447.
- Schultz, W., Dayan, P. and Montague, P.R. (1997). A neural substrate of prediction and reward. *Science* 275, 1593–1599.
- Shapson-Coe, A. et al. (2024). A petavoxel fragment of human cerebral cortex reconstructed at nanoscale resolution. *Science* 384(6696), eadk4858.
- Song, S., Sjöström, P.J., Reigl, M., Nelson, S. and Chklovskii, D.B. (2005). Highly nonrandom features of synaptic connectivity in local cortical circuits. *PLoS Biology* 3(3), e68.
- Stensola, H. et al. (2012). The entorhinal grid map is discretized. *Nature* 492, 72–78.
- Wei, X.-X., Prentice, J. and Balasubramanian, V. (2015). A principle of economy predicts the functional architecture of grid cells. *eLife* 4, e08362.
- Zoghbi, M.E., Woodhead, J.L., Moss, R.L. and Craig, R. (2008). Three-dimensional structure of vertebrate cardiac muscle myosin filaments. *Proceedings of the National Academy of Sciences USA* 105(7), 2386–2390.

Chapter 15 — Perception, Behaviour and Mind

- Ballerini, M., Cabibbo, N., Candelier, R., Cavagna, A., Cisbani, E., Giardina, I., Lecomte, V., Orlandi, A., Parisi, G., Procaccini, A., Viale, M. and Zdravkovic, V. (2008). Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study. *Proceedings of the National Academy of Sciences USA*, doi:10.1073/pnas.0711437105.
- Cartwright, D. and Harary, F. (1956). Structural balance: a generalization of Heider’s theory. *Psychological Review* 63(5), 277–293.
- Chalmers, D.J. (1995). Facing up to the problem of consciousness. *Journal of Consciousness Studies*.
- Cogitate Consortium et al. (2025). Adversarial testing of global neuronal workspace and integrated information theories of consciousness. *Nature*, doi:10.1038/s41586-025-08888-1.
- Cowan, N. (2001). The magical number 4 in short-term memory: a reconsideration of mental storage capacity. *Behavioral and Brain Sciences* 24, 87–185.
- Heider, F. (1946). Attitudes and cognitive organization. *Journal of Psychology* 21, 107–112.
- Leskovec, J., Huttenlocher, D. and Kleinberg, J. (2010). Signed networks in social media. arXiv:1003.2424 (CHI 2010).
- Ma, S. et al. (2024). The era of 1-bit LLMs: all large language models are in 1.58 bits. arXiv:2402.17764.
- Miller, G.A. (1956). The magical number seven, plus or minus two: some limits on our capacity for processing information. *Psychological Review* 63(2), 81–97.
- Regenwetter, M., Dana, J. and Davis-Stober, C.P. (2011). Transitivity of preferences. *Psychological Review*.

- Shizuka, D. and McDonald, D.B. (2016). Data from: The network motif architecture of dominance hierarchies. Dryad, doi:10.5061/dryad.f76f2.
- Sinervo, B. and Lively, C.M. (1996). The rock–paper–scissors game and the evolution of alternative male strategies. *Nature* 380, 240–243.
- Thoen, H.H. et al. (2014). A different form of color vision in mantis shrimp. *Science* 343, 411–413.
- Tinbergen, N. (1963). On aims and methods of ethology. *Zeitschrift für Tierpsychologie* 20, 410–433.
- Tversky, A. (1969). Intransitivity of preferences. *Psychological Review*.

Chapter 16 — The Life Course

- Bull, J.R. et al. (2019). Real-world menstrual cycle characteristics of more than 600,000 menstrual cycles. *npj Digital Medicine*.
- Czeisler, C.A. et al. (1999). Stability, precision, and near-24-hour period of the human circadian pacemaker. *Science* 284(5423), 2177 ff.
- Dong, X., Milholland, B. and Vijg, J. (2016). Evidence for a limit to human lifespan. *Nature* (published online 5 October 2016).
- Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality. *Philosophical Transactions of the Royal Society* 115, 513–585.
- Gonze, D. and Abou-Jaoudé, W. (2013). The Goodwin model: behind the Hill function. *PLoS ONE* 8(8), e69573.
- Hayflick, L. and Moorhead, P.S. (1961). The serial cultivation of human diploid cell strains. *Experimental Cell Research* 25(3), 585–621.
- Horvath, S. (2013). DNA methylation age of human tissues and cell types. *Genome Biology* 14(10), R115.
- López-Otín, C., Blasco, M.A., Partridge, L., Serrano, M. and Kroemer, G. (2023). Hallmarks of aging: an expanding universe. *Cell* 186(2), 243–278.
- Nagaoka, S.I., Hassold, T.J. and Hunt, P.A. (2012). Human aneuploidy: mechanisms and new insights into an age-old problem. *Nature Reviews Genetics* 13(7), 493–504.

Chapter 17 — Healing

- Aartsma-Rus, A. et al. (2006). Entries in the Leiden Duchenne muscular dystrophy mutation database: an overview of mutation types and paradoxical cases that confirm the reading-frame rule. *Muscle & Nerve*.
- Baker, E.N. et al. (1988). The structure of 2Zn pig insulin crystals at 1.5 Å resolution. *Philosophical Transactions of the Royal Society B* 319, 369–456.
- Belchetz, P.E. et al. (with Knobil, E.) (1978). Hypophysial responses to continuous and intermittent delivery of hypothalamic gonadotropin-releasing hormone. *Science* 202(4368), 631–633.
- Cade, J.F.J. (1949). Lithium salts in the treatment of psychotic excitement. *Medical Journal of Australia*.
- Caspar, D.L.D. and Klug, A. (1962). Physical principles in the construction of regular viruses. *Cold Spring Harbor Symposia on Quantitative Biology* 27, 1–24.

- Guevara, M.R., Glass, L. and Shrier, A. (1981). Phase locking, period-doubling bifurcations, and irregular dynamics in periodically stimulated cardiac cells. *Science* 214, 1350–1353.
- Karpman, S.B. (1968). Fairy tales and script drama analysis.
- Monaco, A.P. et al. (1988). An explanation for the phenotypic differences between patients bearing partial deletions of the DMD locus. *Genomics* 2, 90–95.

Chapter 18 — Social Structure

- Arcaute, E., Hatna, E., Ferguson, P., Youn, H., Johansson, A. and Batty, M. (2015). Constructing cities, deconstructing scaling laws. *Journal of the Royal Society Interface* 12(102), 20140745.
- Bettencourt, L.M.A., Lobo, J., Helbing, D., Kühnert, C. and West, G.B. (2007). Growth, innovation, scaling, and the pace of life in cities. *Proceedings of the National Academy of Sciences* 104(17), 7301–7306.
- Broido, A.D. and Clauset, A. (2019). Scale-free networks are rare. *Nature Communications* 10, 1017.
- Cartwright, D. and Harary, F. (1956). Structural balance: a generalization of Heider’s theory. *Psychological Review* 63(5), 277–293.
- Christaller, W. (1933). *Die zentralen Orte in Süddeutschland*. Gustav Fischer, Jena.
- Davis, J.A. (1967). Clustering and structural balance in graphs. *Human Relations* 20, 181–187.
- Gabaix, X. (1999). Zipf’s law for cities: an explanation. *Quarterly Journal of Economics* 114(3), 739–767.
- Granovetter, M. (1973). The strength of weak ties. *American Journal of Sociology* 78(6), 1360–1380.
- Harary, F. (1953). On the notion of balance of a signed graph. *Michigan Mathematical Journal* 2(2).
- Heider, F. (1946). Attitudes and cognitive organization. *Journal of Psychology* 21, 107–112.
- Holland, P.W. and Leinhardt, S. (1970). A method for detecting structure in sociometric data. *American Journal of Sociology* 76(3), 492–513.
- Leitão, J.C., Miotto, J.M., Gerlach, M. and Altmann, E.G. (2016). Is this scaling nonlinear? *Royal Society Open Science* 3(7), 150649.
- Leskovec, J., Huttenlocher, D. and Kleinberg, J. (2010). Signed networks in social media. *Proceedings of the SIGCHI Conference on Human Factors in Computing Systems (CHI 2010)*. arXiv:1003.2424.
- Leslie, P.H. (1945). The use of matrices in certain population mathematics. *Biometrika* 33(3), 183–212.
- Lévi-Strauss, C. (1949). *Les structures élémentaires de la parenté*. Presses Universitaires de France, Paris.
- Lösch, A. (1940). *Die räumliche Ordnung der Wirtschaft*. Gustav Fischer, Jena. English translation: *The Economics of Location*, Yale University Press, 1954.

- Pólya, G. (1921). Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz. *Mathematische Annalen* 84, 149–160.
- Simmel, G. (1908). *Soziologie: Untersuchungen über die Formen der Vergesellschaftung*. Duncker & Humblot, Leipzig.
- von Thünen, J.H. (1826). *Der isolierte Staat in Beziehung auf Landwirtschaft und Nationalökonomie*. Perthes, Hamburg.
- Watts, D.J. and Strogatz, S.H. (1998). Collective dynamics of “small-world” networks. *Nature* 393, 440–442.
- Zhou, W.-X., Sornette, D., Hill, R.A. and Dunbar, R.I.M. (2005). Discrete hierarchical organization of social group sizes. *Proceedings of the Royal Society B* 272, 439–444. doi:10.1098/rspb.2004.2970.
- Zipf, G.K. (1949). *Human Behavior and the Principle of Least Effort*. Addison-Wesley, Cambridge, Mass.

Chapter 19 — Markets, Power and Communication

- Arrow, K.J. (1950). A difficulty in the concept of social welfare. *Journal of Political Economy* 58(4), 328–346.
- Black, D. (1948). On the rationale of group decision-making. *Journal of Political Economy* 56(1), 23–34.
- Condorcet, M.J.A.N. de (1785). *Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix*. Imprimerie Royale, Paris.
- Downs, A. (1957). *An Economic Theory of Democracy*. Harper, New York.
- Downs, A. (1972). Up and down with ecology: the “issue-attention cycle”. *The Public Interest* 28, 38–50.
- Ezekiel, M. (1938). The cobweb theorem. *Quarterly Journal of Economics* 52(2), 255–280.
- Fama, E.F. (1970). Efficient capital markets: a review of theory and empirical work. *Journal of Finance* 25(2), 383–417.
- Gibbard, A. (1973). Manipulation of voting schemes: a general result. *Econometrica* 41(4), 587–601.
- Jevons, W.S. (1875). *Money and the Mechanism of Exchange*. Macmillan, London.
- Katz, E. and Lazarsfeld, P.F. (1955). *Personal Influence*. Free Press, Glencoe, Ill.
- Lasswell, H.D. (1948). The structure and function of communication in society. In L. Bryson (ed.), *The Communication of Ideas*, Harper, New York.
- Lippmann, W. (1922). *Public Opinion*. Harcourt, Brace, New York.
- McCombs, M.E. and Shaw, D.L. (1972). The agenda-setting function of mass media. *Public Opinion Quarterly* 36(2), 176–187.
- McKelvey, R.D. (1976). Intransitivities in multidimensional voting models and some implications for agenda control. *Journal of Economic Theory* 12(3), 472–482.
- Newcomb, T.M. (1953). An approach to the study of communicative acts. *Psychological Review* 60(6), 393–404.
- Noelle-Neumann, E. (1974). The spiral of silence: a theory of public opinion. *Journal of Communication* 24(2), 43–51.

- Riker, W.H. (1982). *Liberalism against Populism*. W.H. Freeman, San Francisco.
- Samuelson, P.A. (1965). Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review* 6(2), 41–49.
- Shannon, C.E. (1948). A mathematical theory of communication. *Bell System Technical Journal* 27, 379–423 and 623–656.

Chapter 20 — The Cycles of Societies

- Anderson, R.M. and May, R.M. (1991). *Infectious Diseases of Humans: Dynamics and Control*. Oxford University Press.
- Bettencourt, L.M.A., Lobo, J., Helbing, D., Kühnert, C. and West, G.B. (2007). Growth, innovation, scaling, and the pace of life in cities. *Proceedings of the National Academy of Sciences* 104(17), 7301–7306.
- Burns, A.F. and Mitchell, W.C. (1946). *Measuring Business Cycles*. National Bureau of Economic Research, New York.
- Earn, D.J.D., Rohani, P., Bolker, B.M. and Grenfell, B.T. (2000). A simple model for complex dynamical transitions in epidemics. *Science* 287, 667–670.
- Haensch, S. et al. (2010). Distinct clones of *Yersinia pestis* caused the Black Death. *PLoS Pathogens* 6(10), e1001134.
- Ibn Khaldun (1377). *Al-Muqaddimah*. English translation by F. Rosenthal, *The Muqaddimah: An Introduction to History*, Princeton University Press, 1958.
- Juglar, C. (1862). *Des crises commerciales et de leur retour périodique en France, en Angleterre et aux États-Unis*. Guillaumin, Paris.
- Kermack, W.O. and McKendrick, A.G. (1927). A contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society A* 115, 700–721.
- Kondratiev, N.D. (1926). Die langen Wellen der Konjunktur. *Archiv für Sozialwissenschaft und Sozialpolitik* 56, 573–609.
- Li, T.-Y. and Yorke, J.A. (1975). Period three implies chaos. *American Mathematical Monthly* 82(10), 985–992.
- May, R.M. (1976). Simple mathematical models with very complicated dynamics. *Nature* 261, 459–467.
- Nordhaus, W.D. (1975). The political business cycle. *Review of Economic Studies* 42(2), 169–190.
- Polybius (c. 150–120 BCE). *The Histories*, Book VI. Loeb Classical Library, vol. III.
- Schumpeter, J.A. (1939). *Business Cycles: A Theoretical, Historical and Statistical Analysis of the Capitalist Process*. McGraw-Hill, New York.
- Spyrou, M.A. et al. (2022). The source of the Black Death in fourteenth-century central Eurasia. *Nature* 606, 718–724.
- Strauss, W. and Howe, N. (1997). *The Fourth Turning*. Broadway Books, New York.
- Turchin, P. and Nefedov, S.A. (2009). *Secular Cycles*. Princeton University Press.
- Turchin, P. (2010). Political instability may be a contributor in the coming decade. *Nature* 463, 608.

Chapter 21 — Ordering Society

- Alchourrón, C.E. and Bulygin, E. (1971). *Normative Systems*. Springer, Vienna.
- Bahrick, H.P., Bahrick, L.E., Bahrick, A.S. and Bahrick, P.E. (1993). Maintenance of foreign language vocabulary and the spacing effect. *Psychological Science* 4(5), 316–321.
- Cepeda, N.J., Pashler, H., Vul, E., Wixted, J.T. and Rohrer, D. (2006). Distributed practice in verbal recall tasks: a review and quantitative synthesis. *Psychological Bulletin* 132(3), 354–380.
- Doll, R. and Hill, A.B. (1950). Smoking and carcinoma of the lung. *British Medical Journal* 2(4682), 739–748.
- Dunleavy, P., Margetts, H., Bastow, S. and Tinkler, J. (2006). New public management is dead — long live digital-era governance. *Journal of Public Administration Research and Theory* 16(3), 467–494.
- Ebbinghaus, H. (1885). *Über das Gedächtnis*. Duncker & Humblot, Leipzig.
- Fayol, H. (1916). *Administration industrielle et générale*. Dunod, Paris.
- Graicunas, V.A. (1933). Relationship in organization. *Bulletin of the International Management Institute* 7, 39–42.
- Hart, H.L.A. (1961). *The Concept of Law*. Clarendon Press, Oxford.
- Hill, A.B. (1965). The environment and disease: association or causation? *Proceedings of the Royal Society of Medicine* 58(5), 295–300.
- Hohfeld, W.N. (1913). Some fundamental legal conceptions as applied in judicial reasoning. *Yale Law Journal* 23, 16–59.
- Hohfeld, W.N. (1917). Fundamental legal conceptions as applied in judicial reasoning. *Yale Law Journal* 26, 710–770.
- Hood, C. (1991). A public management for all seasons? *Public Administration* 69(1), 3–19.
- Kermack, W.O. and McKendrick, A.G. (1927). A contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society A* 115, 700–721.
- Łukasiewicz, J. (1920). O logice trójwartościowej. *Ruch Filozoficzny* 5, 170–171.
- Omran, A.R. (1971). The epidemiologic transition: a theory of the epidemiology of population change. *Milbank Memorial Fund Quarterly* 49(4), 509–538.
- Ostrom, E. (1990). *Governing the Commons*. Cambridge University Press.
- Rose, G. (1985). Sick individuals and sick populations. *International Journal of Epidemiology* 14(1), 32–38.
- Shewhart, W.A. (1939). *Statistical Method from the Viewpoint of Quality Control*. Graduate School of the Department of Agriculture, Washington.
- Simon, H.A. (1947). *Administrative Behavior*. Macmillan, New York.
- Taylor, F.W. (1911). *The Principles of Scientific Management*. Harper & Brothers, New York.
- von Wright, G.H. (1951). Deontic logic. *Mind* 60(237), 1–15.
- Weber, M. (1922). *Wirtschaft und Gesellschaft*. Mohr, Tübingen.
- Wilson, W. (1887). The study of administration. *Political Science Quarterly* 2(2), 197–222.

Chapter 22 — Language and Sign

- Bloomfield, L. (1933). *Language*. Henry Holt, New York.
- Chomsky, N. (1957). *Syntactic Structures*. Mouton, The Hague.
- Daniels, P.T. and Bright, W. (eds.) (1996). *The World's Writing Systems*. Oxford University Press.
- Eco, U. (1976). *A Theory of Semiotics*. Indiana University Press, Bloomington.
- Gelb, I.J. (1952). *A Study of Writing*. University of Chicago Press.
- Greenberg, J.H. (1963). "Some universals of grammar with particular reference to the order of meaningful elements". In J.H. Greenberg (ed.), *Universals of Language*, MIT Press, 73–113.
- Grimm, J. (1822). *Deutsche Grammatik*, 2nd ed., vol. 1. Dieterich, Göttingen.
- Parry, M. (1930). "Studies in the epic technique of oral verse-making. I. Homer and Homeric style". *Harvard Studies in Classical Philology* 41, 73–147.
- Peirce, C.S. (1867). "On a new list of categories". *Proceedings of the American Academy of Arts and Sciences* 7, 287–298.
- Rao, R.P.N., Yadav, N., Vahia, M.N., Joglekar, H., Adhikari, R. and Mahadevan, I. (2009). "Entropic evidence for linguistic structure in the Indus script". *Science* 324(5931), 1165.
- Saussure, F. de (1916). *Cours de linguistique générale*. Ed. C. Bally and A. Sechehaye. Payot, Lausanne and Paris.
- Shannon, C.E. (1951). "Prediction and entropy of printed English". *Bell System Technical Journal* 30(1), 50–64.
- Skirgård, H. et al. (2023). "Grambank reveals the importance of genealogical constraints on linguistic diversity and highlights the impact of language loss". *Science Advances* 9(16), eadg6175.
- Sproat, R. (2014). "A statistical comparison of written language and nonlinguistic symbol systems". *Language* 90(2), 457–481.
- Ventris, M. and Chadwick, J. (1953). "Evidence for Greek dialect in the Mycenaean archives". *Journal of Hellenic Studies* 73, 84–103.
- Zipf, G.K. (1949). *Human Behavior and the Principle of Least Effort*. Addison-Wesley, Cambridge MA.

Chapter 23 — The Arts

- Arnheim, R. (1954). *Art and Visual Perception*. University of California Press, Berkeley.
- Cohn, R. (1996). "Maximally smooth cycles, hexatonic systems, and the analysis of late-Romantic triadic progressions". *Music Analysis* 15(1), 9–40.
- Cohn, R. (1997). "Neo-Riemannian operations, parsimonious trichords, and their *Tonnetz* representations". *Journal of Music Theory* 41(1), 1–66.
- Euler, L. (1739). *Tentamen novae theoriae musicae*. St Petersburg.
- Forte, A. (1973). *The Structure of Atonal Music*. Yale University Press, New Haven.

- Jacoby, N. et al. (2024). “Commonality and variation in mental representations of music revealed by a cross-cultural comparison of rhythm priors in 15 countries”. *Nature Human Behaviour* 8. doi:10.1038/s41562-023-01800-9.
- Lewin, D. (1987). *Generalized Musical Intervals and Transformations*. Yale University Press, New Haven.
- Lord, A.B. (1960). *The Singer of Tales*. Harvard University Press, Cambridge MA.
- Lu, P.J. and Steinhardt, P.J. (2007). “Decagonal and quasi-crystalline tilings in medieval Islamic architecture”. *Science* 315(5815), 1106–1110.
- Mehr, S.A. et al. (2019). “Universality and diversity in human song”. *Science* 366(6468), eaax0868.
- Olrik, A. (1909). “Epische Gesetze der Volksdichtung”. *Zeitschrift für deutsches Altertum und deutsche Literatur* 51, 1–12.
- Propp, V. (1928). *Morfologiya skazki*. Academia, Leningrad. English: *Morphology of the Folktale*, 1958.
- Queneau, R. (1963). “Note complémentaire sur la sextine”. *Subsidia Pataphysica* 1.
- Rameau, J.-P. (1722). *Traité de l’harmonie réduite à ses principes naturels*. Ballard, Paris.
- Savage, P.E., Brown, S., Sakai, E. and Currie, T.E. (2015). “Statistical universals reveal the structures and functions of human music”. *PNAS* 112(29), 8987–8992.
- Stedman, F. (1677). *Campanalogia, or the Art of Ringing Improved*. London.
- Toussaint, G. (2005). “The Euclidean algorithm generates traditional musical rhythms”. *Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science*, Banff.
- Weyl, H. (1952). *Symmetry*. Princeton University Press.

Chapter 24 — Memory of Culture

- Braudel, F. (1949). *La Méditerranée et le monde méditerranéen à l’époque de Philippe II*. Armand Colin, Paris.
- Dershowitz, N. and Reingold, E.M. (2018). *Calendrical Calculations: The Ultimate Edition*. Cambridge University Press.
- Dumézil, G. (1958). *L’idéologie tripartie des Indo-Européens*. Latomus, Brussels.
- Durkheim, É. (1912). *Les formes élémentaires de la vie religieuse*. Alcan, Paris.
- Eliade, M. (1949). *Le mythe de l’éternel retour*. Gallimard, Paris.
- Freeth, T., Higgon, D., Dacanalis, A., MacDonald, L., Georgakopoulou, M. and Wojcik, A. (2021). “A model of the Cosmos in the ancient Greek Antikythera Mechanism”. *Scientific Reports* 11, 5821.
- Harris, M.A. and Reingold, E.M. (2004). “Line drawing, leap years, and Euclid”. *ACM Computing Surveys* 36(1), 68–80.
- Kuitems, M. et al. (2022). “Evidence for European presence in the Americas in AD 1021”. *Nature* 601, 388–391.
- Libby, W.F., Anderson, E.C. and Arnold, J.R. (1949). “Age determination by radiocarbon content: world-wide assay of natural radiocarbon”. *Science* 109(2827), 227–228.

- Reimer, P.J. et al. (2020). “The IntCal20 Northern Hemisphere radiocarbon age calibration curve (0–55 cal kBP)”. *Radiocarbon* 62(4), 725–757.
- Scaliger, J.J. (1583). *De emendatione temporum*. Paris.
- Thomsen, C.J. (1836). *Ledetraad til Nordisk Oldkyndighed*. Copenhagen.
- Turchin, P. (2003). *Historical Dynamics: Why States Rise and Fall*. Princeton University Press.
- Turchin, P. and Nefedov, S.A. (2009). *Secular Cycles*. Princeton University Press.
- Tylor, E.B. (1871). *Primitive Culture*. John Murray, London.
- van Gennep, A. (1909). *Les rites de passage*. Émile Nourry, Paris.
- Whitehouse, H. (2004). *Modes of Religiosity*. AltaMira Press, Walnut Creek.

Chapter 25 — Making Meaning

- Alexander, C., Ishikawa, S. and Silverstein, M. (1977). *A Pattern Language*. Oxford University Press, New York.
- Anscombe, G.E.M. (1958). “Modern moral philosophy”. *Philosophy* 33(124), 1–19.
- Arrow, K.J. (1951). *Social Choice and Individual Values*. Wiley, New York.
- Awad, E., Dsouza, S., Kim, R., Schulz, J., Henrich, J., Shariff, A., Bonnefon, J.-F. and Rahwan, I. (2018). “The Moral Machine experiment”. *Nature* 563, 59–64.
- Butlin, P., Long, R. et al. (2023). “Consciousness in artificial intelligence: insights from the science of consciousness”. arXiv:2308.08708.
- Chalmers, D.J. (1995). “Facing up to the problem of consciousness”. *Journal of Consciousness Studies* 2(3), 200–219.
- Chisholm, R.M. (1963). “Contrary-to-duty imperatives and deontic logic”. *Analysis* 24(2), 33–36.
- Christaller, W. (1933). *Die zentralen Orte in Süddeutschland*. Gustav Fischer, Jena.
- Condorcet, Marquis de (1785). *Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix*. Imprimerie Royale, Paris.
- Jacobs, J. (1961). *The Death and Life of Great American Cities*. Random House, New York.
- Kant, I. (1781). *Kritik der reinen Vernunft*. Hartknoch, Riga.
- Kleene, S.C. (1952). *Introduction to Metamathematics*. North-Holland, Amsterdam.
- Lösch, A. (1940). *Die räumliche Ordnung der Wirtschaft*. Gustav Fischer, Jena.
- Łukasiewicz, J. (1920). “O logice trójwartościowej”. *Ruch Filozoficzny* 5, 170–171.
- Maxwell, J.C. (1864). “On the calculation of the equilibrium and stiffness of frames”. *Philosophical Magazine* 27(182), 294–299.
- Norman, D.A. (1988). *The Psychology of Everyday Things*. Basic Books, New York.
- Rawls, J. (1971). *A Theory of Justice*. Harvard University Press, Cambridge MA.
- Rittel, H.W.J. and Webber, M.M. (1973). “Dilemmas in a general theory of planning”. *Policy Sciences* 4(2), 155–169.
- Simon, H.A. (1969). *The Sciences of the Artificial*. MIT Press, Cambridge MA.
- Vitruvius (c. 25 BCE). *De architectura*. English: *Ten Books on Architecture*, trans. M.H. Morgan, Harvard University Press, 1914.

• von Wright, G.H. (1951). “Deontic logic”. *Mind* 60(237), 1–15.
• Wittkower, R. (1949). *Architectural Principles in the Age of Humanism*. Warburg Institute, London.