One Rule at Every Scale: Knots, Ternary Closure and the Lo Shu

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Abstract

A structure that repeats one closure rule at every scale exists and can be measured exactly, but it lives in ternary addressing, not in the new knot invariant Θ. This article tests what knot theory adds to the Vacuum.Net theory, with every number recomputed and every claim checked against a second source.

Four results stand.

1. **Θ compresses infinite families.** For all torus knots T(p,q), θ equals Ohtsuki's closed 2004 formula, verified coefficient by coefficient. For the family 5_2, 7_3, 9_3, … a new closed formula was found, predicted in advance and confirmed.
2. **Θ does not carry the theory's closure rule or its direction.** Three-colourability (the theory's closure) lives in Kauffman's bracket, not in θ; orientation is invisible to θ and to the classical polynomials.
3. **The theory addresses with a different quantity.** Its address is the Hopf charge Q, one integer; Θ describes the shape of a curve. The two meet in torus knots, which are also the shapes physical knotted fields take.
4. **The Lo Shu is a ternary fractal with an exact entropy limit.** Its compound squares of order 3^l obey zero line sums and the closure rule on every ternary digit level, at every scale tested up to 81 × 81. Their singular-value spectrum has a closed form, and the entropy converges to $H\infty = 1.168105$, explaining the value Loly and Cameron found numerically.

One literature formula could not be confirmed: Ohtsuki's cabling formula did not reproduce θ for cables of the trefoil. That and four questions only the theory can answer are left open.

1. Question and method

The question is whether knot theory can reduce the complexity of infinitely many knots to rules, and whether such rules show the self-similarity (fractals) that the Vacuum.Net theory expects at every scale.

The starting point was the essay *The Knot That Rings*, which relates the invariant Θ = (Δ, θ) of Bar-Natan and van der Veen to the theory. The theory is read from its own text, *The Vacuum.Net Theory, Ninth Edition* (Foundations 9.2 / Physics 9.0, 27 October 2026), and its history from *The History of the Derivation as It Happened*. In that history the knot is not an added picture: the trit comes from the three states at a crossing, closure is Fox three-colouring, and orientation is meant to follow from the knotting (programme of 9 July 2026).

The working rule: quantitative claims are recomputed; qualitative claims are checked against a second source. Every finding below is labelled.

- **Source:** what a cited text states.
- **Computed:** an independent calculation, with code available.
- **Interpretation:** a reading that has not been tested.

Where a formula was found from data, it was fixed first and then tested on cases not used to find it.

2. Θ rebuilt and checked

An independent implementation of Θ reproduces the paper and every number in *The Knot That Rings*; four statements in the essay need correction.

Source. Bar-Natan and van der Veen (arXiv 2509.18456 v4, 6 May 2026) define Θ from a long-knot diagram: a matrix A built from the crossings, its inverse G, and three sums F_1, F_2, F_3 (equations 1–6). Δ is the Alexander polynomial; θ is a new two-variable polynomial. The invariant is computable in polynomial time. Up to 15 crossings it leaves a deficit of 6,758 among 313,230 prime knots.

Computed. The implementation works exactly modulo a 61-bit prime, with a second numerical version for large knots.

Check	Result
Trefoil printed in the paper	Reproduced coefficient by coefficient
Mirror image	Gives exactly −θ (Conjecture 21) on six knots
Same knot on different braid indices	T(4,3) on 4 strands = T(3,4) on 3 strands; T(5,2) = T(2,5)
Table of the essay (3_1, 4_1, 5_1, 5_2, 7_1, 8_{19})	Δ, number of terms, Σ|c| and mirror sign all reproduced

Check	Result
Prime knots up to 15 crossings	313,230 confirmed as the sum of OEIS A002863
Separating power	(313,230 − 6,758) / 313,230 = 97.84 % get a unique Θ

Corrections to the essay.

1. "6,758 pairs undistinguished" should read "a deficit of 6,758": the paper counts knots minus distinct values, not pairs.
2. The factor 12 does not come from the order of the symmetry group. For T(2,7), 36 of the 72 terms sit on mirror axes in orbits of 6. The 12 comes from geometry: ring j lies at hexagonal radius 2j, and a hexagonal ring of radius 2j has 12j points.
3. T(2,9) is also three-colourable (determinant 9). The essay lists only the trefoil and T(3,4). The essay's conclusion stands: all values of θ(ω,ω) are divisible by 9 and separate nothing.
4. Two references to the theory are wrong. T20 concerns the Reidemeister moves, not left and right forms; N5 concerns scale (coarse-graining), not open boundary.

3. Infinite families reduced to one rule

For two infinite families, θ of every member follows from one formula with one or two integers; for torus knots the formula was already known.

Torus knots (source and computed). Ohtsuki (*A cabling formula for the 2-loop polynomial of knots*, Publ. RIMS 40, 2004, Theorem 3.1) gives the 2-loop polynomial of every torus knot:

$$\Theta_{T(p,q)}(t_1,t_2,t_3) = -\tfrac{1}{4} \sum_{\{i,j,k\}=\{1,2,3\}} \psi_{p,q}(t_i)\,\psi_{q,p}(t_j)\,\Delta_{T(p,q)}(t_k), \quad t_1t_2t_3 = 1$$

with Δ the Alexander polynomial of T(p,q) and $\psi_{p,q}(t) = \Delta(t)\cdot$[coth-type term in $t^{p/2}$ − q·(same in $t^{pq/2}$)]. Setting $t_1 = T_1$, $t_2 = T_2$, $t_3 = 1/(T_1T_2)$, this equals the paper's θ exactly, with no normalisation factor. That supports Conjecture 25 of Bar-Natan and van der Veen (θ is the 2-loop polynomial) for torus knots.

Test	Knots	Result
Ratio θ / Ohtsuki at 4 random points	T(2,3), T(2,5), T(2,7), T(3,4), T(3,5), T(4,5)	Exactly 1 in every point

Test	Knots	Result
Coefficient by coefficient (exact Fourier recovery)	T(3,13), T(3,14), T(3,16), T(3,17)	All 1,458 / 1,710 / 2,274 / 2,586 coefficients equal

Two special cases follow and were found independently before the literature was checked. For T(2,q), θ is a set of concentric hexagonal rings: ring j at radius 2j carries ±j, with sign + when both exponents are even, for the closed braid σ_1^q; the mirror has opposite signs (verified up to 17 crossings). For T(3,n): number of terms $10n^2 - 18n + 2$ and $\Sigma|c| = n^2(n^2 - 1)$; predicted for n = 13 and 14 before computing, and both came out exactly.

A 2-bridge family (computed, new). The closed braids $\sigma_1^a\sigma_2\sigma_1^{-1}\sigma_2$ with a = 2M + 1 give 5_2, 7_3, 9_3, … (7_3 and 9_3 confirmed in the Knot Atlas by braid word, Alexander polynomial and determinant 13 and 19). They are not torus knots. Their θ has an exact form on concentric hexagonal rings of radius r:

Ring	Where	Coefficient
r = 0	centre	−12
r odd	only the 6 corners	3(r + 1)
r = 2m < 2M	corners / other even–even points / remaining points	3m − 4 / 9m − 2 / −(9m − 2)
r = 2M (outer ring)	corners / other even–even points / remaining points	8M + 1 / 12M + 1 / −(12M + 1)

Hence, for M ≥ 2: number of terms $6M^2 + 12M + 1$ and $\Sigma|c| = 36M^3 + 78M^2 + 42M + 36$. The formula was read off a = 7, 9, 11, matches a = 3, 5, 13, and was then used to predict a = 15 (379 terms, Σ|c| = 16,500) and a = 17 (481 terms, Σ|c| = 23,796). Both came out exactly, coefficient by coefficient. No published formula for this family was found.

Interpretation. The inner rings of this family are identical for every member; each extra pair of twists adds one ring at the rim. An infinite family is therefore a fixed core plus a growing edge, described by one integer. Neither formula is proven for all members; both are tested.

4. What Θ sees and does not see

Θ carries the theory's mirror rule, but not its closure rule and not its direction of travel.

Property	Theory (source: Ninth Edition)	Knot theory	Status
Mirror	Negation 1 ↔ T reverses the address (T12.1); 0 is rest (III.2)	θ changes sign; knots equal to their mirror give θ = 0	Computed on six knots. Agreement, not evidence: knot theory predicts this by itself
Direction	"The first passage fixes the direction" (N3)	θ(−K) = θ(K) (Conjecture 22); Alexander, Jones and HOMFLY-PT are also blind to reversal	Computed: reversal leaves θ unchanged for 5_2, 8_{19}, 8_{17}. 8_{17}, the smallest non-invertible knot, has θ = 0
Closure	Three-colouring (T4, T19; Fox colouring, IV.1)	Number of three-colourings = $3\cdot\lvert V(e^{i\pi/3})\rvert^2$ (Lickorish–Millett, via Przytycki Thm 1.13)	Computed on 12 knots with an independent bracket state sum and brute-force colouring: exact. θ at ω separates nothing
Third crossing state	Over, under, or not crossing (derivation step 3)	Virtual knot theory: classical +, classical −, virtual ("not really there", Kauffman 1999)	If "not crossing" is virtual, θ loses its hexagonal symmetry (Θ paper §6; computed on their example KS: both reflections fail)

Kauffman's bracket and the trit (source and interpretation). The bracket replaces each crossing by its two smoothings, both states without a crossing: ⟨over⟩ = $A\langle 0\rangle + A^{-1}\langle\infty\rangle$ and ⟨under⟩ the reverse; the mirror is $A \leftrightarrow A^{-1}$. Over and under are thus combinations of the third state. The weight of a state is A raised to a sum of ±1 entries: the digit sum of a trit word, not its place value as in the theory's address (T12). The bracket therefore matches the theory's crossing alphabet more closely than Θ, and it carries the closure rule.

Consequence. An address that serves the theory needs three parts: closure (Δ or the bracket at $e^{i\pi/3}$), handedness (θ), and an invariant that sees direction. Ordinary polynomial invariants supply the first two only.

5. Two kinds of knot

Physics and the theory use the word "knot" for a field whose lines link, counted by one integer; knot theory uses it for the shape of one closed curve. Torus knots are where the two coincide.

Source. Cho, Oh and Zhang (*Knots in Physics*, arXiv 1803.08952, 2018) define a physical knot by its Hopf index $Q \in \pi_3(S^2) = \mathbb{Z}$, the number of times two field lines wind around each other. For their knot soliton Q = m·n: m flux quanta through the ring and n along it. The theory uses exactly this quantity: "the Hopf charge $Q \in \pi_3(S^2) = \mathbb{Z}$ is conserved" (T22), written as one balanced-ternary address (T12).

Source. Arrayás and Trueba (2011) construct light fields in vacuum whose field lines are torus knots (n,m), with Hopf index n·m. Sutcliffe (2007) computed which shapes are stable for Hopf charges up to 16 in the Skyrme–Faddeev model: many torus knots appear, often as local minima, while the global minimum is sometimes a link. Rañada's knotted solutions of Maxwell's equations, with integer helicities, form a dense set: locally every radiation field is a superposition of knots (via Cho et al., §II).

Computed. The one formula of §3 that covers a whole two-parameter class is the one for torus knots, the shape physical knotted fields take.

Interpretation. In torus knots the theory's address (Q = p·q) and the shape (Θ by Ohtsuki's formula) are both readable from the same two winding numbers. That makes torus knots the natural meeting point of the theory and Θ. It is a correspondence, not a demonstrated link: Q does not determine the knot type, and Θ does not determine Q.

Counting (computed). There are infinitely many knot types; the number of prime knots grows by a factor of about 6 per extra crossing between 17 and 20 crossings (1,847,319,428 at 20 crossings, OEIS A002863). The observable universe holds about 10^{80} nuclei and 10^{89}

photons (recomputed from standard cosmological parameters). Three symbols suffice to give each its own address: 168 trits for 10^{80}, 187 for 10^{89}.

6. Nilpotency as a constraint

Nilpotency acts in physics as a demonstrable constraint on which states may exist and on what is conserved; whether it is a property of the universe or of our descriptions cannot be decided here.

Source. Rowlands calls the state of a particle a "square root of 0". A nilpotent particle is therefore "automatically Pauli exclusive": two identical ones combine to zero. The universe is a "zero totality" in which particle and vacuum together give the zero state, $\psi_v = -\psi_f$. He calls nilpotency "a physical principle, rather than a purely mathematical operation". The Ninth Edition places Rowlands' nilpotent condition under its closure axiom.

Source. Wheeler's principle "the boundary of a boundary is zero" ($\partial\partial = 0$; Misner, Thorne and Wheeler, *Gravitation*, chapters 15 and 17) yields the conservation of energy and momentum in general relativity through the Bianchi identities.

Interpretation. $\partial\partial = 0$ says that a boundary never has a boundary of its own: it is closed, a loop. That is N1 ("every distinction is a loop") in geometric language. The knot invariants of §2–§4 rest on it too: the Alexander polynomial comes from homology, which exists only because $\partial^2 = 0$.

Computed. The Lo Shu of §7 is not a nilpotent matrix: after subtracting its centre, $M^3 = -24 \cdot M$. Its constraint is of a different kind: every line sums to zero, and every moment up to order 2 vanishes.

7. Looking for the fractal

The knot formulas grow at the rim and are not fractal; the Lo Shu and its compound squares are a ternary fractal in which one zero-sum closure rule holds on every digit level at every scale, with an exactly computable entropy limit.

7.1 Knots grow at the rim

Interpretation of §3. For T(2,q) and for 5_2, 7_3, 9_3, … a larger member adds a ring at the outside. The whole does not reappear, scaled down, inside itself. That is growth by accretion, not self-similarity.

Source. In knot theory self-similarity lives in *cabling*: a knot is replaced by a strand that winds around it, and this can be repeated. Ohtsuki's cabling formula (published version Theorem 4.1) states that Θ of the (p,q)-cable is the torus term $\Theta_{T(p,q)}$, plus Θ_K evaluated at

t_i^p, plus a coupling term built from Δ_K, its derivative and $\varphi_{q,p}$. The substitution $t \to t^p$ scales the companion's picture by the factor p: exactly a fractal rule.

Computed: not confirmed. Cables were built as braids (p parallel strands plus k twists; the slope $q = p \cdot w + k$ was confirmed for every cable through $\Delta_{cable}(t) = \Delta_{T(p,q)}(t) \cdot \Delta_K(t^p)$). For seven cables of the trefoil (13 to 31 crossings) and for cables of cables (53 and 213 crossings), θ computed from the paper's definition did not equal the formula. Twenty-four variants of sign, subscript and derivative conventions left a residual of at least 36 %; a correction of the form $a \cdot \Delta_1\Delta_2\Delta_3$ did not explain it either. Since the trefoil is a torus knot for which θ equals Ohtsuki's Θ exactly, the discrepancy lies either in a convention not yet identified or in a real difference between θ and Θ beyond torus knots. This is left open; nothing below depends on it.

7.2 The Lo Shu as a ternary fractal

Source. The Lo Shu, 4 9 2 / 3 5 7 / 8 1 6, is the only normal 3×3 magic square up to rotation and reflection; every line sums to 15. Composite squares of order 9 whose nine 3×3 sub-squares are magic go back to Yang Hui. Loly and Cameron (arXiv 2008.11020) build squares of order 3^l recursively by Kronecker products, call the construction "fractal", report rank 2l + 1, and find numerically that the entropy of the normalised singular values approaches "about 1.168". Loly (*Mathematical Gazette*, 2004) shows that the moment of inertia of a magic square about its centre, $N^2(N^4 - 1)/12$, does not depend on the arrangement: 60 for the Lo Shu.

Computed: the 3 × 3 square.

- Subtracting the centre gives −1 4 −3 / −2 0 2 / 3 −4 1: every row, column and diagonal sums to 0, and the centre is 0.
- Each entry −4 … 4 is a two-trit number 3a + b with a, b ∈ {−1, 0, 1}. The high-trit square 0 1 −1 / −1 0 1 / 1 −1 0 and the low-trit square −1 1 0 / 1 0 −1 / 0 −1 1 have "all three different" on every row, column and diagonal: the theory's closure rule T4. Every pair (high, low) occurs exactly once. This decomposition is not stated in the sources consulted.
- All moments up to order 2 vanish (total, both dipoles, all quadrupoles); the first non-zero moments are of order 3 ($x \cdot y^2 = -4$, $x^2 \cdot y = -8$). The moment of inertia 60 is confirmed.
- There are exactly 8 normal 3×3 magic squares, confirming uniqueness.

Computed: the compound squares. Cell value $9 \cdot (L_{outer} - 1) + L_{inner}$, repeated, gives squares of order 9, 27, 81, …

Order	Magic	Every trit	Rank	Entropy H

Order	Magic	Every trit level: line sum 0 and T4 per window of 3	Rank	Entropy H
9	yes	yes (4 levels)	5	1.1300
27	yes	yes (6 levels)	7	1.1625
81	yes	yes (8 levels)	9	1.1673

Every single trit level has rank 2 and entropy exactly ln 2 at every scale.

Computed: the exact spectrum (new). For order n = 3^l the non-zero singular values are the magic sum and one pair per scale:

$$s_0 = \frac{n(n^2+1)}{2}, \quad s_k = \sqrt{3} \cdot 2 \cdot 3^{l-1} \cdot 9^k \cdot (2,\ 1), \quad k = 0,\ \ldots,\ l-1$$

This reproduces every numerically computed spectrum (orders 3 to 81) and every entropy above. As $l \to \infty$ the centre carries the share $p_0 = 2/(2+\sqrt{3}) \approx 0.5359$ of the spectral weight, the pairs share the rest geometrically with ratio 1/9, and each pair splits 2 : 1. The limit is

$$H_\infty = -p_0 \ln p_0 - m \ln m + m\,(H_{\text{geo}}(1/9) + H(2/3,\ 1/3)) = 1.168105, \quad m = 1 - p_0$$

with $H_{\text{geo}}(1/9) = 0.3924$ the entropy of the geometric distribution and $H(2/3, 1/3) = 0.6365$. Order 243 gives 1.1680 and order 729 gives 1.1681. This explains the value Loly and Cameron found numerically.

Interpretation. The rank 2l + 1 splits as 1 (the centre: rest) plus 2 per scale. Every scale adds the same pair, in the same ratio 2 : 1, a factor $9 = 3^2$ smaller, with the same zero sum and the same closure. That is self-similarity in a measurable quantity, and it sits in the ternary digits, not in θ.

8. What this means for the Vacuum.Net theory

This whole section is interpretation; it says where the tested results touch the theory, not that they confirm it.

1. **The theory's closure rule has a home in knot theory.** Fox three-colouring is the theory's closure (T4, T19), and it is carried by the Alexander determinant and, more finely, by the Jones polynomial at $e^{i\pi/3}$. It is not carried by θ.
2. **The theory's sign rule is matched by θ.** Mirror reverses the address and the self-mirror case is rest; θ does the same. This is consistency, not support, because knot theory predicts it on its own.
3. **Direction is a real gap.** N3 makes the direction of travel part of a configuration; the invariants tested here do not see it. If the theory derives orientation from the knotting, as its programme of 9 July 2026 says, it should produce an invariant that tells 8_{17} from its reverse. That would be a contribution of the theory to knot theory, and it is testable.
4. **The address is coarser than the shape.** One integer Q cannot separate 313,230 knot types; Θ with two polynomials still leaves a deficit of 6,758. The theory's address and Θ are complementary measures.
5. **The fractal is in the ternary addressing.** The Lo Shu family shows N5 (the same rules at every scale), T4 (closure per window of three) and "0 = rest" in one object whose spectrum and entropy are exact. The knot invariant θ shows growth at the rim instead.
6. **Nilpotency and zero sums point the same way.** $\partial\partial = 0$ says every boundary is a loop (N1); Rowlands' zero totality and the Lo Shu's vanishing lower moments are zero-sum constraints of the same family. Whether the universe obeys them, or our descriptions do, is not decided by this material.

9. Open questions

Four questions can only be answered from within the theory; four can be settled by further computation.

For the theory.

1. Is the direction of travel (N3) part of the address? If so, none of the tested polynomials is sufficient.
2. Does "not crossing" mean no crossing, or a virtual crossing? The answer decides whether the hexagonal symmetry of θ applies.
3. Does "knot" in the theory mean the Hopf charge Q or the shape of the curve?
4. Is Kauffman's bracket, which carries the closure rule, the theory's natural instrument?

For computation.

- Resolve the discrepancy between θ and Ohtsuki's cabling formula, for example with an independent value of the 2-loop polynomial of a non-torus knot.
- Prove the ring formula for the family 5_2, 7_3, 9_3, … for all members, or find it in the literature.
- Extend the Lo Shu result: is every ternary digit-level matrix of every compound square a Kronecker product of an all-ones matrix with a Lo Shu trit square (which would prove the spectrum formula)?
- Compare Θ of the stable and the unstable knotted solitons that Sutcliffe found for the same Hopf charge.

Sources and code

All sources below were opened, except Lickorish & Millett, which is cited through Przytycki; the code (Python: exact Θ modulo a prime, numerical Θ, braid-to-long-knot conversion, cable construction, Lo Shu analysis) is available from the author.

Knots and invariants

- Bar-Natan & van der Veen, A Fast, Strong, Topologically Meaningful and Fun Knot Invariant, arXiv 2509.18456 v4
- Ohtsuki, A cabling formula for the 2-loop polynomial of knots, arXiv math/0310216; published in Publ. RIMS 40 (2004)
- Kauffman, Virtual Knot Theory
- Przytycki, 3-coloring and other elementary invariants of knots; Lickorish & Millett (1986)
- Knot Atlas 7_3, Knot Atlas 9_3, OEIS A002863

Physics

- Cho, Oh & Zhang, Knots in Physics, arXiv 1803.08952 (2018)
- Arrayás & Trueba, Torus-knotted electromagnetic fields
- Sutcliffe, Knots in the Skyrme–Faddeev model
- Rowlands, Are there alternatives to our present theories of physical reality?
- Does General Relativity Highlight Necessary Connections in Nature? (on Wheeler and MTW ch. 15)

Magic squares

- Loly & Cameron, Frierson's 1907 Parameterization of Compound Magic Squares Extended to Orders 3^l, arXiv 2008.11020
- Loly, The Invariance of the Moment of Inertia of Magic Squares

- Navas, Khajuraho's magic square is an hypercube, arXiv 2608.06740
- Wikipedia: Lo Shu Square, Magic square, Constructions of magic squares

The theory

- The Vacuum.Net Theory, Ninth Edition
- The History of the Derivation as It Happened
- Exploring Knot Theory: Kauffman's Vision and Applications (constable.blog, 13 January 2026)