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# CAN PHYSICS BE DERIVED FROM A SINGLE EQUATION?

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## ABSTRACT

Physics, at the fundamental level, is concerned only with a single system, that of the fermion or fundamental particle, and it is possible to conceive the actions of the entire universe as those of a single fermion. We have had a quantum mechanical equation for the fermion – the Dirac equation for nearly a hundred years – but no one has imagined that that equation alone could even lead to all the developments encoded in the Standard Model of particle physics. This is partly because the equation is not normally expressed in its most significantly meaningful form, but it is also because 'derivation' is generally taken to mean a deductive mathematical consequence given certain conditions rather than an unfolding of the innate structure that is built into the equation as a result of more fundamental principles. The equation is not necessarily the *source* of these principles, rather the codification of them. In fact, given these additional considerations, it is possible to see the equation in its most physically meaningful form as the source of all current aspects of the Standard Model and even some things beyond it. In addition, the full statement of the equation is not necessary for these derivations, only the definition of the fermion creation operator that the equation requires. So, the question that we will be answering is the more restricted one of whether physics can be defined by a single *operator*, rather than a single equation.

## THE ALGEBRA OF THE DIRAC EQUATION

At the fundamental level, physics is the description of the behaviour of just one type of object. This is the fermion or fundamental particle, and there is a way of looking at the actions of the entire universe as though they were those of a single fermion. The relativistic quantum mechanical equation for the fermion has been known for nearly a hundred years, having been published by Dirac in 1928 [1]. The Standard Model, which is the basis for the theory of fundamental particles, has 12 fermions, made up of 6 quarks (each in 3 'colours') and 6 leptons, and 12 corresponding antifermions. Seemingly, only 4 types of interaction occur between these elementary objects – the three gauge forces, electromagnetic, weak and strong, with gravity as the fourth. The interactions are mediated by spin 1 gauge bosons: $\gamma$ or photon for electric, $W^+$, $W^-$ and $Z^0$ for weak and 8 gluons for strong, and the so far hypothetical spin 2 graviton, if gravity is a similarly-constructed gauge force. The bosons are, in principle, aspects of the gauge interactions, rather than independent particles, and can be seen effectively as fermion-antifermion combinations. One other boson, the spin 0 Higgs boson, is a vacuum particle, which gives mass to the fermions and the weak bosons. Physics on higher scales is ultimately a consequence of the physics of fermions, and gravity is most conveniently treated in the same way, rather than scaled down from a macroscopic perspective, using principles which may sometimes contradict those of the Standard Model.

The Dirac equation is the only viable candidate for the equation we want and, like the other known quantum mechanical equations (Schrödinger and Klein-Gordon), it is an expression of conservation of energy and momentum, $E$ and $\mathbf{p}$, but, unlike them, it is not obviously derived from the 'quantization' of an analogous classical equation using wave-particle duality and the de Broglie relations, $E \rightarrow i\,\partial / \partial t$ and $\mathbf{p} \rightarrow -i\,\nabla$. In fact, its original derivation by Dirac was a remarkable feat of *ad hoc* guesswork and required a mathematics not normally found in physics at the time. It also led to some strange predictions, though these later proved to be true! However, at first sight, though it is the only known equation for physics's most fundamental state, it does not in the least look like the sort of equation from which the rest of physics could be derived.

Using the convention (as we will throughout the paper) that the fundamental constants $\hbar$ and $c$ are both equated to 1, the Dirac equation can be written:

$$i\frac{\partial}{\partial t}\psi(x,t) = [\beta m - i\boldsymbol{\alpha}\nabla]\psi(x,t)$$

The main terms are exactly those normally used in quantum physics, but the $\boldsymbol{\alpha}$ and $\beta$ terms seem to be purely mathematical and without obvious physical meaning. They are anticommuting operators (ones in which, typically, AB = –BA) whose purpose is to reduce an equation in squared terms (Klein-Gordon) to one with the same number of linear terms which will eliminating cross terms when the equation is squared again. Dirac was familiar with anticommuting matrices and chose 4 × 4 matrices, with entries ranging between 1, –1, 0, $i$ and $-i$. The $\boldsymbol{\alpha}$ symbol was, in fact, a 3-component vector, so was represented by 3 matrices. However, the matrices were not unique and could be exchanged for ones with quite different entries. Though this is not normally discussed, we can show that $\beta$ and the three components of $\boldsymbol{\alpha}$ are just four members of a 64-component group of 4 × 4 matrices.

The reason for using 4 × 4 matrices seems to be that these map on to the 4-dimensional relativistic structure of the space and time terms, and a more convenient version of the equation, which makes this structure explicit, is the covariant form, where we rename $\beta$ as $\gamma^0$ and take $\beta\boldsymbol{\alpha}$ as the new 3-component $\boldsymbol{\gamma}$ when we premultiply all the terms by $\gamma^0$ to obtain:

$$\left(\gamma^0\frac{\partial}{\partial t} + \boldsymbol{\gamma}.\nabla + im\right)\psi = \left(\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z} + im\right)\psi = 0$$

The four $\gamma$ terns are mutually anticommutative 4 × 4 matrices, squaring respectively to $I$, $-I$, $-I$ and $-I$, where $I$ is the 4 × 4 identity matrix. The algebra using four terms, however, is incomplete, and a fifth anticommuting matrix, $\gamma^5$, another square root of $I$, defined as $i\gamma^0\gamma^1\gamma^2\gamma^3$, is needed to complete it.

Because of the way the equation developed, the terms $\boldsymbol{\alpha}$ and $\beta$ or $\boldsymbol{\gamma}$ and $\gamma^0$ have been used in their matrix form for nearly a hundred years as a mathematical 'convenience' though they are not actually convenient. As we will show, however, they are, in fact, very significant additions to quantum physics, for, almost accidentally, they reveal the source for almost the entire Standard Model of particle physics as well as physics in general. Despite being used mostly as purely mathematical operators, they can also be shown to have significant physical meaning. In fact, from the beginning it was clear that new physics was contained in the $\boldsymbol{\alpha}$ and $\beta$ terms. Using 4 × 4 matrices required the simultaneous existence of four solutions for a single particle, which could be interpreted as representing particle and antiparticle, each of which is in two possible spin states. Not only did this predict that antiparticles must exist. It was also a significant departure from conventional physics, implying that a particle could somehow include its own antiparticle in its description.

However, to progress further we need a more convenient representation of the mathematics [2-18]. Many early authorities (including Dirac) and quite a few contemporary ones have realised that the system must be equivalent to a Clifford algebra. As first proposed in 1878, these are very strictly defined to be composed of any number of commutative (i.e. independent) sets of quaternions, which may or may not be complexified [19]. Now quaternions can be very simply defined as dimensional square roots of –1. A set of three such units, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, defined by anticommutative multiplication, form a closed system, whose existence is the obvious explanation of the special fundamental significance of 3-dimensionality. Here

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k};\ \boldsymbol{jk} - \boldsymbol{kj};\ \boldsymbol{ki} = -\boldsymbol{ik};$$

and

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

Clifford algebras and their components form groups with orders based on factors of 2. Quaternions are of order 8, complexified quaternions (vectors) of order 16, double quaternions of order 32, complexified double quaternions of order 64, etc.

So, how many quaternion sets are required for the Dirac algebra? If we take the $\gamma$ algebra, and create + and – versions of the tensor product of all the basic units we find that the algebra is represented by a group of order 64 terms:

$$\begin{array}{l}
1 \\
\gamma^{o}, \gamma^{1}, \gamma^{2}, \gamma^{3}, \gamma^{5}, \\
\gamma^{o}\gamma^{1}, \gamma^{o}\gamma^{2}, \gamma^{o}\gamma^{3}, \gamma^{o}\gamma^{5}, \gamma^{1}\gamma^{2}, \gamma^{1}\gamma^{3}, \gamma^{1}\gamma^{5}, \gamma^{2}\gamma^{3}, \gamma^{2}\gamma^{5}, \gamma^{3}\gamma^{5}, \\
\gamma^{o}\gamma^{1}\gamma^{2}, \gamma^{o}\gamma^{1}\gamma^{3}, \gamma^{o}\gamma^{1}\gamma^{5}, \gamma^{o}\gamma^{2}\gamma^{3}, \gamma^{o}\gamma^{2}\gamma^{5}, \gamma^{o}\gamma^{3}\gamma^{5}, \gamma^{1}\gamma^{2}\gamma^{3}, \gamma^{1}\gamma^{2}\gamma^{5}, \gamma^{1}\gamma^{3}\gamma^{5}, \gamma^{2}\gamma^{3}\gamma^{5}, \\
\gamma^{o}\gamma^{1}\gamma^{2}\gamma^{3}, \gamma^{o}\gamma^{1}\gamma^{2}\gamma^{5}, \gamma^{o}\gamma^{1}\gamma^{3}\gamma^{5}, \gamma^{o}\gamma^{2}\gamma^{3}\gamma^{3}, \gamma^{1}\gamma^{2}\gamma^{3}\gamma^{5}, \\
\gamma^{o}\gamma^{1}\gamma^{2}\gamma^{3}\gamma^{5} = -i.
\end{array}$$

64 units requires 2 sets of quaternions, complexified. Complexified quaternions are also isomorphic to multivariate vectors or Pauli matrices. Pauli introduced these as an ad hoc addition to the Schrödinger equation to represent the experimentally discovered property of spin, and it is significant that the matrix representations of the three components of $\boldsymbol{\gamma}$ incorporate the Pauli matrices $\sigma_1$, $\sigma_2$, $\sigma_3$.

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix};\ \gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix};\ \gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Alternative representations of the Dirac algebra include quaternion × vector, or double vector. One way of deriving the 64 matrices is by taking the product of two commuting sets of Pauli matrices. This has the same structure as the product of two commuting sets of vector units. The last algebra is significant as being equivalent to a **double space.**

Once the Clifford algebra is recognised, then the matrix structures become an option. We can use algebraic symbols and we can, at any time, convert them to their 4 × 4 matrix equivalents, but they can also be used as algebraic operators, creating greater flexibility in manipulating the equation.

Working with Sydney Rowlands, I identified at least 8 completely equivalent ways of generating this Clifford algebra: 2 sets of quaternions complexified; 1 set of quaternions and 1 of vector units; 2 sets of vector unit; the products of $\gamma^0$, $\gamma^1$, $\gamma^2$, $\gamma^3$, $\gamma^5$; the products of $\gamma^0$, $\gamma^1$, $\gamma^2$, $\gamma^3$, $i$; the products of $\beta$, $\alpha_1$, $\alpha_2$, $a_3$, $i$; 2 sets of Pauli matrices, converted to 4 × 4 using the 4 × 4 identity matrix; and the full set of 4 × 4 matrices worked out from these [20].

The full set of units of order 64, exactly isomorphic to the $\gamma$ algebra, can be set out in a way that emphasizes the 5 required generators of the group. In the table, 60 terms apart from the complex numbers, form 12 pentads, any of which could serve instead of the five $\gamma$ terms (though in some case after multiplication by $i$):

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $1$ | $i$ | | | | $-1$ | $-i$ | | | |
| $\boldsymbol{i}\mathbf{i}$ | $\boldsymbol{i}\mathbf{j}$ | $\boldsymbol{i}\mathbf{k}$ | $i\boldsymbol{k}$ | $\boldsymbol{j}$ | $-\boldsymbol{i}\mathbf{i}$ | $-\boldsymbol{i}\mathbf{j}$ | $-\boldsymbol{i}\mathbf{k}$ | $-i\boldsymbol{k}$ | $-\boldsymbol{j}$ |
| $\boldsymbol{j}\mathbf{i}$ | $\boldsymbol{j}\mathbf{j}$ | $\boldsymbol{j}\mathbf{k}$ | $i\boldsymbol{i}$ | $\boldsymbol{k}$ | $-\boldsymbol{j}\mathbf{i}$ | $-\boldsymbol{j}\mathbf{j}$ | $-\boldsymbol{j}\mathbf{k}$ | $-i\boldsymbol{j}$ | $-\boldsymbol{i}$ |
| $\boldsymbol{k}\mathbf{i}$ | $\boldsymbol{k}\mathbf{j}$ | $\boldsymbol{k}\mathbf{k}$ | $i\boldsymbol{j}$ | $\boldsymbol{i}$ | $-\boldsymbol{k}\mathbf{i}$ | $-\boldsymbol{k}\mathbf{j}$ | $-\boldsymbol{k}\mathbf{k}$ | $-i\boldsymbol{j}$ | $-\boldsymbol{i}$ |
| $i\mathbf{i}\boldsymbol{i}$ | $i\mathbf{i}\boldsymbol{j}$ | $i\mathbf{i}\boldsymbol{k}$ | $i\mathbf{k}$ | $\mathbf{j}$ | $-i\mathbf{i}\boldsymbol{i}$ | $-i\mathbf{i}\boldsymbol{j}$ | $-i\mathbf{i}\boldsymbol{k}$ | $-i\mathbf{k}$ | $-\mathbf{j}$ |
| $i\mathbf{j}\boldsymbol{i}$ | $i\mathbf{j}\boldsymbol{j}$ | $i\mathbf{j}\boldsymbol{k}$ | $i\mathbf{i}$ | $\mathbf{k}$ | $-i\mathbf{j}\boldsymbol{i}$ | $-i\mathbf{j}\boldsymbol{j}$ | $-i\mathbf{j}\boldsymbol{k}$ | $-i\mathbf{i}$ | $-\mathbf{k}$ |
| $i\mathbf{k}\boldsymbol{i}$ | $i\mathbf{k}\boldsymbol{j}$ | $i\mathbf{k}\boldsymbol{k}$ | $i\mathbf{j}$ | $\mathbf{i}$ | $-i\mathbf{k}\boldsymbol{i}$ | $-i\mathbf{k}\boldsymbol{j}$ | $-i\mathbf{k}\boldsymbol{k}$ | $-i\mathbf{j}$ | $-\mathbf{i}$ |

Here the quaternions, ***i***, ***j***, ***k***, and vectors, **i**, **j**, **k**, are commutative or independent of each other. The actual units in each pentad have clear physical significance, as we will see later. An alternative formulation would use two commutative vector sets, **i**, **j**, **k** and **I**, **J**, **K**:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $1$ | $i$ | | | | $-1$ | $-i$ | | | |
| $\mathbf{iI}$ | $\mathbf{iJ}$ | $\mathbf{iK}$ | $i\mathbf{k}$ | $\mathbf{j}$ | $-\mathbf{iI}$ | $-\mathbf{iJ}$ | $-\mathbf{iK}$ | $-i\mathbf{k}$ | $-\mathbf{j}$ |
| $\mathbf{jI}$ | $\mathbf{jJ}$ | $\mathbf{jK}$ | $i\mathbf{i}$ | $\mathbf{k}$ | $-\mathbf{jI}$ | $-\mathbf{jJ}$ | $-\mathbf{jK}$ | $-i\mathbf{j}$ | $-\mathbf{i}$ |
| $\mathbf{kI}$ | $\mathbf{kJ}$ | $\mathbf{kk}$ | $i\mathbf{j}$ | $\mathbf{i}$ | $-\mathbf{kI}$ | $-\mathbf{kJ}$ | $-\mathbf{kK}$ | $-i\mathbf{j}$ | $-\mathbf{i}$ |
| $i\mathbf{Ii}$ | $i\mathbf{Ij}$ | $i\mathbf{Ik}$ | $i\mathbf{K}$ | $\mathbf{J}$ | $-i\mathbf{Ii}$ | $-i\mathbf{Ij}$ | $-i\mathbf{Ik}$ | $-i\mathbf{K}$ | $-\mathbf{J}$ |
| $i\mathbf{Ji}$ | $i\mathbf{Jj}$ | $i\mathbf{Jk}$ | $i\mathbf{I}$ | $\mathbf{K}$ | $-i\mathbf{Ji}$ | $-i\mathbf{Jj}$ | $-i\mathbf{Jk}$ | $-i\mathbf{I}$ | $-\mathbf{K}$ |
| $i\mathbf{Ki}$ | $i\mathbf{Kj}$ | $i\mathbf{Kk}$ | $i\mathbf{J}$ | $\mathbf{I}$ | $-i\mathbf{Ki}$ | $-i\mathbf{Kj}$ | $-i\mathbf{Kk}$ | $-i\mathbf{J}$ | $-\mathbf{I}$ |

Because, we only need a pentad of 5 to generate the algebra, we don't normally see all 64 terms. But, essentially dual spaces are all that we need to do relativistic quantum physics. As we will see, there is **no other source of physical information**. One **space** is the familiar space of observation and measurement. The **other** contains everything else. I call it vacuum space. Together they create the singularity that we call the fermion, existing on the boundary of two spaces.

## RESTRUCTURING THE EQUATION

The Dirac equation tends to be used in one of the two standard versions given above, but, in fact, these are neither the most useful nor the most physically informative versions. If we take the version using $\gamma$ matrices, we can immediately make it much more powerful by pre-multiplying all the terms by $\gamma^5$. Pre-multiplying all the $\gamma^n$ terms ($n = 0, 1, 2, 3$) by $\gamma^5$ creates new matrices, but does not change their functions. They are simply alternative versions of the operators that they were before multiplication. However, multiplication of $im$ by $\gamma^5$ creates $i\gamma^5 m$, which is completely different from $im$, and means that all five terms now have a $\gamma$ operator.

$$\left(\gamma^0 \frac{\partial}{\partial t} + \boldsymbol{\gamma}.\nabla + i\gamma^5 m\right)\psi = \left(\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} + i\gamma^5 m\right)\psi = 0$$

Just as we obtained the covariant form of the Dirac equation by premultiplying by $\gamma^0$, so, equivalently, we are now premultiplying by the equivalent of $\gamma^5$ to obtain the *covariant nilpotent* form. This changes the $\gamma^0$ and $\boldsymbol{\gamma}$ terms in the equation, but only to alternative forms of the same operators and adds the new $\gamma^5$ coefficient to $m$. And, when we convert the operators to their Clifford algebra equivalents, meaning immediately begins to appear. The 5 generators of the group can be matched to the 5 gamma matrices in a number of ways, for example:

$$\gamma^o = i\boldsymbol{k} \qquad \gamma^1 = \mathbf{i}\boldsymbol{i} \qquad \gamma^2 = \mathbf{j}\boldsymbol{i} \qquad \gamma^3 = \mathbf{k}\boldsymbol{i} \qquad \gamma^5 = i\boldsymbol{j}$$

or

$$\gamma^o = \mathbf{k} \qquad \gamma^1 = i\mathbf{i}\mathbf{i} \qquad \gamma^2 = i\mathbf{j}\mathbf{i} \qquad \gamma^3 = i\mathbf{k}\mathbf{i} \qquad \gamma^5 = \mathbf{j}$$

There are many ways of doing this but the overall structure is always the same. The key thing is that the symmetry of one 3-D quantity (vector space) is preserved, while that of the other is broken.

The dual space structure gives us an insight into such things as the meaning of a point-particle, spin ½, the Berry phase, *zitterbewegung*, vacuum, CPT, etc. The fact that it can be generated from 5 asymmetric generators gives us the first inkling of the meaning of symmetry-breaking in the Standard Model. It produces a much more compact and powerful formalism for relativistic quantum mechanics and quantum field theory which makes much of the existing apparatus redundant.

Though people have argued that we have a formalism that works, so we don't need to change it, my answer is that it doesn't work. It doesn't give us the Standard Model. If there is mathematical structure that has no physical explanation, then the formalism is not complete. In addition, the existing formalism lacks symmetry and has a massive amount of redundancy and fragmentation. A whole apparatus of mathematical objects has been created for the current approach – spinors, primitive idempotents, minimum left ideals – which is not strictly necessary. They cause fragmentation of the equation, mixing up energy, momentum and mass terms. They also take up too much logical space, requiring 16 × 4 pieces of information for one operation. And they lack symmetry. There are 5 terms in the equation, but only 4 have a $\gamma$ matrix. Yet there is a fifth matrix ($\gamma^5$) in the algebra.

## WHERE DOES THE ALGEBRA COME FROM?

As we have seen, the Clifford algebra has intrinsic physical meaning, and so we should expect to find even more physical meaning in areas we have not yet fully investigated. If we do a dimensional analysis on the basic units of the algebra,

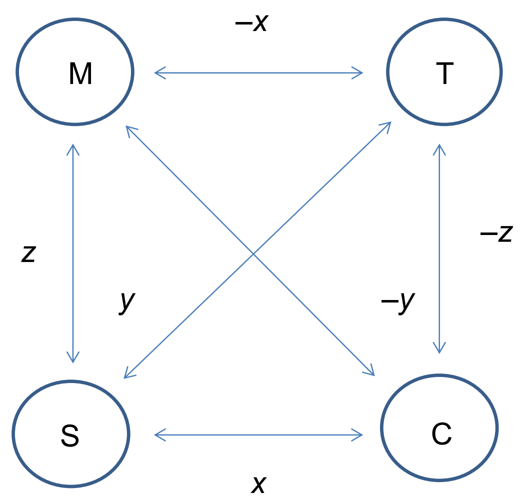
| | | | |
|---|---|---|---|
| $i$ | **i j k** | 1 | ***i j k*** |
| complex | vector | scalar | quaternion |

where the quaternion is a separate 3-D system to the vector, we see that mass is scalar, time is complex and space vector, but we have not yet assigned any physical meaning to the quaternion terms. There is, however, an obvious candidate for this. If we extend dimensional analysis beyond classical dynamics and gravity, the first thing we encounter is electromagnetism, founded on the concept of *charge*. And the strong and weak interactions suggest fundamental concepts of a similar nature, which, under ideal conditions would be exactly the same. Also, identical charges produce forces in the opposite direction to identical masses.

So, we can propose that charge is, in principle, globally, a quaternion of the form ***i***, ***j***, ***k*** (our convention defining ***i*** as strong, ***j*** as electric and ***k*** as weak) with the symmetry being broken when we create the local (fermionic) state with 5 generators instead of 8 units. When this occurs, ***i*** takes on vector characteristics, ***j*** scalar and ***k*** pseudoscalar. These become reflected in the interactions with which they are associated. A combined 3-D ‘charge’ is the one required for ‘charge conjugation’ (which reverses all 3 forces), and for CPT symmetry, and, if taken to be a global concept, is the one thing needed to make the four parameters absolutely symmetrical. It is then that the local fermionic state requires symmetry-breaking of the charges. Globally, we have:

| | | | |
|---|---|---|---|
| **mass** | conserved | real | commutative |
| **time** | nonconserved | imaginary | commutative |
| **charge** | conserved | imaginary | anticommutative |
| **space** | nonconserved | real | anticommutative |

There are many representations of this global symmetry – tetrahedral, cubical, coloured circles, Venn diagram, or even diagrams inspired by category theory:




This gives us the table for a D2 (or Klein-4) group [21-22, 2-18 *passim*]:

| * | mass | time | charge | space |
|---|---|---|---|---|
| mass | mass | time | charge | space |
| time | time | mass | space | charge |
| charge | mass | space | mass | time |
| space | space | charge | time | mass |

CPT symmetry, in the form MCPT, where M (mass) acts as identity, is a Klein-4 group, which is totally derived from the larger mass-time-charge-space group. The same structure is observed in the 4 examples of cellular automata which are believed to be universally Turing-compatible and which have analogues to T, C and P transformations (Mainzer and Chua [23]), in addition to the $H_4$ algebra, made up from identity and three double quaternions.

| * | 1 | ***iI*** | ***jJ*** | ***kK*** |
|---|---|---|---|---|
| 1 | 1 | ***iI*** | ***jJ*** | ***kK*** |
| ***iI*** | ***iI*** | 1 | ***kK*** | ***jJ*** |
| ***jJ*** | ***jJ*** | ***kK*** | 1 | ***iI*** |
| ***kK*** | ***kK*** | ***jJ*** | ***iI*** | 1 |

Double quaternions are clearly a Klein-4 group or $H_4$ algebra. The same effect is produced by negative values of double vector units. Remarkably, the same group connects the four bases which make up DNA, adenine, thymine, guanine and cytosine, which have 3 dualities (purine / pyrimidine, strong hydrogen bonds / weak hydrogen bonds, keto / amine), and which produce 64 units of the genetic code. The Klein-4 group symmetries appear to be absolutely exact for physical quantities in the global state, with the only exception being for the local states (fermions) produced by the compression of 8 terms in the group generators down to 5.

Before any consideration of the Dirac algebra, I saw the complex vector quaternion algebra as the fundamental algebra of Nature because of its emergence in the Klein-4 group of the fundamental parameters, which I had long seen as the only independent information in Nature. It then became very significant that it emerged also as the algebra behind the equation of the most fundamental physical state. In fact, once we have identified the true nature of the algebraic coefficients in the equation, the coefficients actually take over all the physical meaning. So the equation effectively becomes the coefficients, with the differentials of time and space reflecting the nonconserved or variable nature of these quantities, which emerges from the complex components in these terms. Clifford, interestingly, saw quaternions as fixed or conserved quantities in the algebra compared to vectors, whose additional $i$ factor makes them variable.

## SYMMETRY BREAKING

We can see also the origin of a broken symmetry in the algebra as applied to the equation. The compactification of all the information about two spaces into a set of just 5 units generating the group also becomes the origin of much of the symmetry-breaking in physics. As in the Higgs mechanism, the 'lowest' state

$$\mathbf{K} \qquad i\mathbf{iI} \qquad i\mathbf{jI} \qquad i\mathbf{kI} \qquad \mathbf{J}$$

is not the most symmetrical

$$\mathbf{i} \qquad \mathbf{j} \qquad \mathbf{k} \qquad \mathbf{I} \qquad \mathbf{J} \qquad \mathbf{K}$$

Symmetry-breaking is essential to the construction of locality. The numerous ways of writing out the generators all basically require distributing the units of one 3-D structure among the rest, e.g., using notation in which the two 3-D structures $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are commutative to each other:

$$\begin{array}{cccc} i & \mathbf{i}\ \mathbf{j}\ \mathbf{k} & 1 & \boldsymbol{i}\ \boldsymbol{j}\ \boldsymbol{k} \\ i\boldsymbol{k} & \mathbf{i}\boldsymbol{i}\ \mathbf{j}\boldsymbol{i}\ \mathbf{k}\boldsymbol{i} & 1\boldsymbol{j} & \end{array}$$

Here one of the two spaces retains its symmetry (our space, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$). The other, associated with vacuum ($i\boldsymbol{i}$, $i\boldsymbol{j}$, $i\boldsymbol{k}$), breaks it. We notice also that the symmetry-breaking in this instance is between the three charges or sources of the three interactions, in that the three sources are associated with different mathematical objects, which affects the way they behave. This is the source of all the symmetry breaking between the gauge interactions and the respective pseudoscalar, vector and scalar coefficients ultimately become the causes of the $SU(2)$, $SU(3)$ and $U(1)$ symmetries which apply to the respective weak, strong and electric interactions.

## THE NILPOTENT EQUATION

The fully symmetrical form of the equation using the fifth gamma or equivalent allows us to defragment – that is, to use the quaternion units $\boldsymbol{k}$, $\boldsymbol{i}$ and $\boldsymbol{j}$, to separate energy, momentum and mass terms from each other. But, the real significance of the extra symmetry from the fifth gamma or equivalent apparent comes when we try inserting a plane wave solution for $\psi$:

$$\psi = A\, e^{-i(Et - \mathbf{p.r})}$$

Then

$$(\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{i}\mathbf{k}p_x + \boldsymbol{ij}m)\, A\, e^{-i(Et - \mathbf{p.r})} = 0$$
$$(\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)\, A\, e^{-i(Et - \mathbf{p.r})} = 0$$

where **p** is multivariate.

However, this equation is only valid if $A$ is a multiple of $(\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)$. That is, $A$ is a *nilpotent* or square root of zero. It is also completely explicit. There is no mysterious 'black box' wavefunction. This must also be true of $\psi$, and must have been true *even before we made the substitution of algebraic operators for matrices*. Of course, from the conventional Dirac equation we know that there are really *four* solutions within the 4-component spinor $\psi$. But with the new symmetrical form of the equation, and with the knowledge (from Hestenes) that a multivariate **p** or **∇** already incorporates fermionic spin, these are easy to identify:

| | |
|---|---|
| fermion / antifermion | $\pm E$ |
| spin up / down | $\pm \mathbf{p}$ |

These options apply to both amplitude and phase. So we can represent the Dirac 4-spinor by a column vector with these 4 components:

$$\psi_1 = (\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)\, e^{-i(Et - \mathbf{p.r})}$$
$$\psi_2 = (\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)\, e^{-i(Et + \mathbf{p.r})}$$
$$\psi_3 = (-\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)\, e^{i(Et - \mathbf{p.r})}$$
$$\psi_4 = (-\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)\, e^{i(Et + \mathbf{p.r})}$$

each of which is operated on by

$$(\boldsymbol{ik}\partial / \partial t + \boldsymbol{i}\nabla + \boldsymbol{j}m)$$

But this is not the only transformation we can do to the equation. Four different phases is a calculator's nightmare, but, by removing the 4 × 4 matrices, we have effectively reduced the differential operator to a single term. This gives us the logical space to add an extra twist of symmetry to the equation, by making the operator a 4-component spinor, identical in representation to the amplitude. To do this, we transfer the variation in the signs of $E$ and **p** from the phase term to the differential operator.

$$(-\boldsymbol{ik}\partial / \partial t - \boldsymbol{i}\nabla + \boldsymbol{j}m)\, (\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)\, e^{-i(Et - \mathbf{p.r})}$$
$$(-\boldsymbol{ik}\partial / \partial t + \boldsymbol{i}\nabla + \boldsymbol{j}m)\, (\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)\, e^{-i(Et - \mathbf{p.r})}$$
$$(\boldsymbol{ik}\partial / \partial t - \boldsymbol{i}\nabla + \boldsymbol{j}m)\, (-\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)\, e^{-i(Et - \mathbf{p.r})}$$
$$(\boldsymbol{ik}\partial / \partial t + \boldsymbol{i}\nabla + \boldsymbol{j}m)\, (-\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)\, e^{-i(Et - \mathbf{p.r})}$$

Here, we use **∇** and **p** instead of **σ.∇** and **σ.p** as the vectors are multivariate and incorporate spin. We see here that the Feynman representation of negative energy states being associated with negative time is automatically applied. Also, with only a single phase, the expression

$$(\pm\, \boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{ij}m)$$

or, more conveniently,

$$(\pm\, \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

can be taken to refer either to the differential term in operator form, or to the amplitude produced by applying this to the phase. In any case, the term $(\pm ikE \pm i\mathbf{p} + jm)$ remains a nilpotent, and the expression

$$(\pm ikE \pm i\mathbf{p} + jm)\,(\pm ikE \pm i\mathbf{p} + jm) = 0$$

(in which the first term is conveniently a row vector and the second a column vector) becomes a perfect expression of the Pauli exclusion principle. It also demonstrates that the Dirac equation is not peculiar. We can derive it in the same way as the other quantum mechanical equations from a classical expression.

We take the Einstein relation (with $c = 1$)

$$E^2 - p^2 - m^2 = 0$$

and factorise to give

$$(\pm ikE \pm i\mathbf{p} + jm)\,(\pm ikE \pm i\mathbf{p} + jm) = 0$$

Again, the four terms in first bracket could be considered as arranged as a row vector and the four terms in the second bracket as a column vector. Applying a canonical quantization to the first bracket, we obtain, for a free particle, the Dirac equation

$$(\mp ik\partial / \partial t \mp i\nabla + jm)\,(\pm ikE \pm i\mathbf{p} + jm)\, e^{-i(Et - \mathbf{p.r})} = 0$$

We can also use the convention that $E$ and $\mathbf{p}$ represent operators as well as amplitudes to express it as

$$(\pm ikE \pm i\mathbf{p} + jm)\,(\pm ikE \pm i\mathbf{p} + jm)\, e^{-i(Et - \mathbf{p.r})} = 0$$

The significant fact now is that the amplitude is always nilpotent. The nilpotent expression

$$(\pm ikE \pm i\mathbf{i}p_x \pm i\mathbf{j}p_y \pm i\mathbf{k}p_z + jm)\,(\pm ikE \pm i\mathbf{i}p_x \pm i\mathbf{j}p_y \pm i\mathbf{k}p_z + jm) = 0$$

or

$$(\pm ikE \pm i\mathbf{p} + jm)\,(\pm ikE \pm i\mathbf{p} + jm) = 0$$

can be used to create the Klein-Gordon equation that applies to bosons as well as fermions, by treating the $E$ and $\mathbf{p}$ terms in *both* brackets as operators, acting on a particle with wavefunction $A\, e^{-i(Et - \mathbf{p.r})}$, leading to

$$(\mp ik\partial / \partial t \mp i\nabla + jm)\,(\mp ik\partial / \partial t \mp i\nabla + jm)\, e^{-i(Et - \mathbf{p.r})} = 0$$

or

$$\left(\nabla^2 - \frac{\partial^2}{\partial t} - m^2\right) e^{-i(Et - \mathbf{p.r})} = 0$$

Pauli exclusion, of course, is not unique to free fermions, and this brings us to a step which, in retrospect, might appear revolutionary, a possible transition to universality. We assume that *all fermionic amplitudes in all states* are nilpotent. We postulate that the most general form for a wavefunction is nilpotent, and that we should seek specifically nilpotent solutions for all problems. A zero condition is the most certain way to universality. The principle is revolutionary because it means **we no longer need an equation of any kind**. Relativistic quantum mechanics is completely specified by an operator of the form

$$(\pm ikE \pm i\mathbf{p} + jm)$$

which completely determines the phase that will provide a nilpotent amplitude. We can see the operator, once defined, can only be completed by finding one particular phase. The operator acts like a tuned antenna, and finding the 'solution' generally means finding the phase.

Using this convention, we can replace $E$ and $\mathbf{p}$ with covariant derivatives (e.g. $\partial / \partial t + e\phi + ...$, $-\nabla - e\mathbf{A} + ...$), or incorporate any number of field (or curvature) terms after the differential operators. The expression

$$(\pm\, \boldsymbol{i}\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,(\pm\, \boldsymbol{i}\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \rightarrow 0$$

will still apply. The phase term will no longer be a simple exponential, but we can assume that the amplitude will continue to be nilpotent.

A particle with a nilpotent wavefunction will be automatically Pauli exclusive, as the combination state with an identical particle $\psi_1\psi_1$ will be zero. Such wavefunctions are also Pauli exclusive in the conventional sense of being automatically antisymmetric ($\psi_1\psi_2 - \psi_2\psi_1$):

$$(\pm\, \boldsymbol{i}\boldsymbol{k}E_1 \pm \boldsymbol{i}\mathbf{p}_1 + \boldsymbol{j}m_1)\,(\pm\, \boldsymbol{i}\boldsymbol{k}E_2 \pm \boldsymbol{i}\mathbf{p}_2 + \boldsymbol{j}m_2) - (\pm\, \boldsymbol{i}\boldsymbol{k}E_2 \pm \boldsymbol{i}\mathbf{p}_2 + \boldsymbol{j}m_2)\,(\pm\, \boldsymbol{i}\boldsymbol{k}E_1 \pm \boldsymbol{i}\mathbf{p}_1 + \boldsymbol{j}m_1)$$
$$= 4\mathbf{p}_1\mathbf{p}_2 - 4\mathbf{p}_2\mathbf{p}_1 = 8\, i\, \mathbf{p}_1 \times \mathbf{p}_2 = -\, 8\, i\, \mathbf{p}_2 \times \mathbf{p}_1$$

We see immediately that

$$(\psi_1\psi_2 - \psi_2\psi_1) = -(\psi_2\psi_1 - \psi_1\psi_2)$$

The result is clearly antisymmetric. But it also tells us something new, for it requires a nilpotent wavefunction to have a **p** vector in spin space at a different orientation to any other. The instantaneous nonlocal correlation of the wavefunctions of all nilpotents could then require the intersection of the planes corresponding to all the different **p** vector directions, creating their Hilbert space. We can even consider these intersections as actually creating the *meaning* of Euclidean space, with an intrinsic spherical symmetry generated by the fermions themselves.

The formalism drives us towards an understanding of the universe that cannot be separated from the way we define a fermion. And now, at last we begin to understand the question: what is vacuum? Suppose we want to create a fermion *ab initio*, that is, from entirely nothing, and that we want it to be created with certain interaction potentials embedded in its operator. Let us give this fermion a 'wavefunction' $\psi_f$. Then the 'hole' that its creation leaves in 'nothing' may be described as $-\psi_f$. The superposition of the states of fermion + 'hole' or $\psi_f - \psi_f$ equals 0, as does the product state $-\psi_f\psi_f$ if the wavefunction is nilpotent. The nilpotent structure and Pauli exclusion now become comprehensible if the entire universe adds up to nothing, and 'vacuum' becomes the 'hole' in nothing left by creating the fermion.

However, vacuum itself is *not* nothing. To create the fermionic state $\psi_f$ with the interaction potentials that we assigned to it means to create an 'environment' with makes this possible (i.e. a system of other fermions which create those potentials), but whose total effect adds up to $-\psi_f$. So, vacuum, for any fermionic state, becomes *the rest of the universe* for that state. This is what the fermionic operator is actually acting on. And that 'rest of the universe' must be so constructed as to appear in total as the mirror image of the fermion state. The same condition must additionally be true for each of the other fermions which collectively create the vacuum state for the first. So, we could express the Pauli exclusion principle in the statement that no two fermions share the same vacuum (or the same phase factor).

We also have a strict and absolute definition for the difference between local and nonlocal. For an object such as

$$(\pm \, i\boldsymbol{k}E \pm \mathbf{i}\mathbf{p} + \boldsymbol{j}m)$$

everything inside the bracket is local. Everything outside (superposition, combination) is nonlocal. But the nonlocal can cause local changes and vice versa. local changes *inside* nilpotent brackets (the creation of local potentials added to the energy and momentum operators) can produce the same results as nonlocal changes *outside* them. Interactions have both local and nonlocal descriptions because both particles and their environments are equally involved and each type of description determines the behaviour of its counterpart.

## FURTHER PHYSICAL MEANING IN THE ALGEBRAIC OPERATORS

Locality is about defining a fermionic state and it is entirely specified by the Clifford algebra operators, with the $E$, $p$ and $m$ acting merely as scalar factors. As we have seen, the Clifford algebra has intrinsic physical meaning, and so we should expect to find even more physical meaning in areas we have not yet fully investigated.

The three quaternion units in the nilpotent operator have multiple, but connected, meanings. One of these is as operators for fundamental symmetry transformations, by pre- and post-multiplication of the nilpotent operator.

$$
\begin{array}{ll}
P & \boldsymbol{i}\,(i\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{i} = (i\boldsymbol{k}E - \mathbf{i}\mathbf{p} + \boldsymbol{j}m) \\
T & \boldsymbol{k}\,(i\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{k} = (-i\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m) \\
C & -\boldsymbol{j}\,(i\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,\boldsymbol{j} = (-i\boldsymbol{k}E - \mathbf{i}\mathbf{p} + \boldsymbol{j}m)
\end{array}
$$

It is easy to show that $CPT \equiv$ identity, etc. The terms in the nilpotent 4-spinor, other than the lead term which determines the nature of the 'real' particle state, are effectively, the $P$-, $T$- and $C$-transformed versions of this state, the states into which it could transform without changing the magnitude of its energy or momentum. We can see them as vacuum 'reflections' of the real particle state, arising from vacuum operations that can be mathematically defined.

Although a fermion cannot form a combination state with itself, we can imagine it forming a combination state with each of these vacuum 'reflections', and, if the 'reflection' exists or materialises as a 'real' state, then the combined state can form one of the three classes of bosons or boson-like objects. The three possible transformations also lead to the production of three types of bosonic state, which, when summed up over 4 terms (just one is shown for convenience), yield products which are scalars (i.e. not Dirac spinors):

Spin 1 boson:

$$(i\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,(-\,i\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m) \qquad T$$

Spin 0 boson:

$$(i\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,(-\,i\boldsymbol{k}E - \mathbf{i}\mathbf{p} + \boldsymbol{j}m) \qquad C$$

Fermion-fermion – Bose-Einstein condensate / Berry phase, etc.:

$$(i\boldsymbol{k}E + \mathbf{i}\mathbf{p} + \boldsymbol{j}m)\,(i\boldsymbol{k}E - \mathbf{i}\mathbf{p} + \boldsymbol{j}m) \qquad P$$

A spin 1 boson can be massless as (taking just the first term of the 4 in each amplitude for convenience)

$$(i\boldsymbol{k}E + \mathbf{i}\mathbf{p})\,(-\,i\boldsymbol{k}E + \mathbf{i}\mathbf{p}) \neq 0$$

A spin 0 boson cannot be massless as

$$(\boldsymbol{i}kE + \boldsymbol{i}\mathbf{p})\ (-\ \boldsymbol{i}kE - \boldsymbol{i}\mathbf{p}) = 0$$

So there are no Goldstone bosons and Higgs has mass. At the same time, no fermion can be massless because, if a fermion is reduced to ($\pm\ \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p}$) then it cannot be distinguished from other fermions with the same $E / p$ ratio, and so can't be specified as unique.

We have seen that it is possible to define the entire structure of quantum mechanics by defining the creation operator for a single fermion ($\pm\ \boldsymbol{ik}E \pm \mathbf{ip} + \boldsymbol{j}m$) as a nilpotent. There are four creation (or annihilation) operators here (or two of each):

| | |
|---|---|
| $(\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)$ | fermion spin up |
| $(\boldsymbol{ik}E - \mathbf{ip} + \boldsymbol{j}m)$ | fermion spin down |
| $(-\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)$ | antifermion spin down |
| $(-\boldsymbol{ik}E - \mathbf{ip} + \boldsymbol{j}m)$ | antifermion spin up |

But only the first term defines the real fermionic state. The others are vacuum 'reflections', or the states into which it could transform.

To demonstrate the scalar nature of the result of one of the boson combinations, over the 4 terms in the spinor, let us take the fermion-fermion combination:

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (\boldsymbol{ik}E - \mathbf{ip} + \boldsymbol{j}m) = (\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)\ (-2\boldsymbol{i}\mathbf{p})$$
$$(\boldsymbol{ik}E - \mathbf{ip} + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m) = (\boldsymbol{ik}E - \mathbf{ip} + \boldsymbol{j}m)\ (2\mathbf{ip})$$
$$(-\ \boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (-\ \boldsymbol{ik}E - \mathbf{ip} + \boldsymbol{j}m) = (-\ \boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)\ (-2\boldsymbol{i}\mathbf{p})$$
$$(-\ \boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (-\ \boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m) = (-\ \boldsymbol{ik}E - \mathbf{ip} + \boldsymbol{j}m)\ (2\boldsymbol{i}\mathbf{p})$$

Clearly, the terms on the right hand side in $\boldsymbol{ik}E$ and $\boldsymbol{j}m$ total to 0, leaving only $4 \times (-\mathbf{ip})(2\boldsymbol{i}\mathbf{p}) = 8p^2$, which is a scalar (normalizable to 1).

There is another way to look at this. If we take $(\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)$ and postmultiply it by

$$\boldsymbol{k}(\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)$$

or $$\boldsymbol{i}(\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)$$

or $$\boldsymbol{j}(\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)$$

the result is $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, multiplied by a scalar. This can be done an indefinite number of times. So these three *idempotent* terms behave as a vacuum operators. We can describe the partitions as strong, weak and electric 'vacua', and assign to them particular roles within existing physics.

We can see how the 3 bosonic states are related to vacua produced by the 3 quaternionic operators. Here we take $\boldsymbol{k}$ as example and take just the first term of the four for convenience:

$$(\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)\ \boldsymbol{k}\ (\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)\ \boldsymbol{k}\ (\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)\ \boldsymbol{k}\ (\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)\ \dots$$

This can also be written as

$$(\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)\ (-\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)\ (-\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)\ \dots$$

Both of these expressions are *identical* to $(\boldsymbol{ik}E + \mathbf{ip} + \boldsymbol{j}m)$ after normalization, but the second also represents spin 1 vacuum bosons. In principle, this means that we can do relativistic quantum mechanics for a fermion in any state, subject to any number of interactions, simply by defining an operator of the form

$$(\pm\, ikE \pm \mathbf{ip} + jm)$$

This will then uniquely determine the phase factor which makes the amplitude nilpotent. There is no need to define any equation at all.

$$(\text{operator acting on phase factor})^2 = \text{amplitude}^2 = 0$$

The adoption of nilpotency as a condition for a fermion wavefunction also applies in a more collective sense, for a sum (superposition) of two nilpotent and antisymmetric wavefunctions $\psi_1$ and $\psi_2$ will also be nilpotent, for

$$(\psi_1 + \psi_2)^2 = \psi_1{}^2 + \psi_2{}^2 + \psi_1\psi_2 + \psi_2\psi_1 = 0$$

and this will extend to a sum of any number of nilpotents, even including coefficients. The same will apply to any product (combination) of independent nilpotent wavefuntions, for if we take, say, $\psi_1\psi_2$, then $\psi_1\psi_2\psi_1\psi_2 = -\,\psi_1\psi_2\;\psi_2\psi_1 = 0$, and so on. Also, any large-scale system that conserves energy can be expressed in a nilpotent way because it implies $E^2 - p^2 - m^2 = 0$, even if we use a nonrelativistic approximation.

Adopting an appropriate sign convention, we can now, for example, specify the four amplitude states as derived from four creation operators:

| | |
|---|---|
| $(ikE + \mathbf{ip} + jm)$ | fermion spin up |
| $(ikE - \mathbf{ip} + jm)$ | fermion spin down |
| $(-ikE + \mathbf{ip} + jm)$ | antifermion spin down |
| $(-ikE - \mathbf{ip} + jm)$ | antifermion spin up |

With this convention, $\mathbf{ip}$ / $ikE$ represents the same helicity or handedness as $(-\mathbf{ip})$ / $(-ikE)$, but the opposite helicity to $(\mathbf{ip})$ / $(-ikE)$ or $(-\mathbf{ip})$ / $(ikE)$. The phase factor is now simply an expression of all the possible variations in space and time which are encoded in the creation operator. This is uniquely defined once the operator is specified. A fermion is thus specified as a set of space and time variations. The mass term is purely passive, convenient, rather than necessary information. We can no longer define an isolated system. Defining a fermion 'disturbs the universe'. The energy conservation only works over the entire universe, since the $E$ and $\mathbf{p}$ terms contain the entire range of the fermion's interactions. The laws of thermodynamics become a necessary consequence, and the fermion becomes an *intrinsically dissipative* system.

With a further refinement we can reduce the amount of information needed to specify relativistic quantum mechanics to just a two-term operator, while reducing the 'wavefunction' to an amplitude without a phase factor. Here, we use a discrete or anticommutative differentiation process, with a correspondingly discrete wavefunction. We start by defining a discrete differentiation (mainly following Lou Kauffman) [24, 4] of the function $F$, which preserves the Leibniz chain rule, by taking:

$$\frac{\partial F}{\partial t} = [F, H] = [F, E]$$

and

$$\frac{\partial \Psi}{\partial X_i} = [F,\ P_i]$$

with $H = E$ and $P_i$ representing energy and momentum coordinates. Taking the case where velocity operators are not in evidence, we use $\partial F / \partial t$ rather than $dF / dt$. The mass term (which, of course, has only a passive role in quantum mechanics) disappears in the operator, though it is retained in the amplitude. Suppose we define a nilpotent amplitude

$$\psi = \boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}P_1 + \boldsymbol{i}\mathbf{j}P_2 + \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m$$

and an operator

$$D = \boldsymbol{ik}\frac{\partial}{\partial t} + \boldsymbol{i}\mathbf{i}\frac{\partial}{\partial X_1} + \boldsymbol{i}\mathbf{j}\frac{\partial}{\partial X_2} + \boldsymbol{i}\mathbf{k}\frac{\partial}{\partial X_3}$$

with

$$\frac{\partial \Psi}{\partial t} = [\Psi, H] = [\Psi, E]$$

and

$$\frac{\partial \Psi}{\partial X_i} = [\Psi, P_i]$$

After some basic algebraic manipulation, we obtain

$$-D\Psi = \Psi(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}P_1 + \boldsymbol{i}\mathbf{j}P_2 + \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m) + (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}P_1 + \boldsymbol{i}\mathbf{j}P_2 + \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m)\Psi \\ - 2\,i(E^2 - P_1^2 - P_2^2 - P_3^2 - m^2)$$

When $\psi$ is nilpotent, then

$$D = \left(\boldsymbol{ik}\frac{\partial}{\partial t} + \boldsymbol{i}\boldsymbol{\nabla}\right)\Psi = 0$$

Generalising to 4 states, with $D$ and $\psi$ as 4-spinors, then

$$D = \left(\pm\boldsymbol{ik}\frac{\partial}{\partial t} \pm \boldsymbol{i}\boldsymbol{\nabla}\right)(\pm\boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{i}P_1 \pm \boldsymbol{i}\mathbf{j}P_2 \pm \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m) = 0$$

The reason why this derivation is important is that it does not require the $\pm\, i$ (or $i\hbar$) term usually applied to the differential operators in canonical quantization, though, because our expression incorporates all four sign variations, we can include this if we want to without changing anything significant in the equation. This means that we can make a smooth transition between classical and quantum conditions, which points to an important connection between mass and the appearance of $\pm\, i$ terms in quantum mechanical operators and amplitudes.

Mass in this context will appear in the amplitude term (which this time does not require a phase intermediary) but not in the specifying operator. Here, we may mention a second fundamental result from the early days of the Dirac equation. Schrödinger early on found a solution of the free particle equation for a velocity operator which contained a term which had no equivalent classical counterpart. He interpreted this as a *zitterbewegung*, or 'jittery motion' between the components of the spinor. This he took as equivalent to mass and the 'rest' or invariant mass of a fermion is still considered in this way, and related to the Higgs mechanism. In effect, it is as much a dynamic mass as any mass derived from an energy term, and all mass is in this sense dynamic. It might be called the structural mass, as it doesn't strictly come from 'rest'. In principle, Schrödinger's result also tells us that there is no such thing as the velocity $v$ of a material particle. It is always oscillating between $+c$ or $-c$, and $v$ is only the averaged out apparent result.

Now, relativistic quantum mechanics was always assumed to require idempotent, rather than nilpotent wavefunctions, essentially because spinors are built up from primitive idempotents. We can, in fact, make *exactly the same equation* look either idempotent or nilpotent simply by redistributing a single algebraic unit between the sections of the equation defined as operator and wavefunction. So, we can write the basic equation as either

$$\underbrace{[(\mp \boldsymbol{k}\partial / \partial t \mp \boldsymbol{ii}\nabla + \boldsymbol{j}m)\,\boldsymbol{j}]}_{operator}\;\underbrace{[\boldsymbol{j}(\pm\, \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\; e^{-i(Et - \mathbf{p.r})}]}_{idempotent\ \ wavefunction} = 0$$

or

$$\underbrace{[(\mp \boldsymbol{k}\partial / \partial t \mp \boldsymbol{ii}\nabla + \boldsymbol{j}m)\,\boldsymbol{jj}]}_{operator}\;\underbrace{[(\pm\, \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\; e^{-i(Et - \mathbf{p.r})}]}_{nilpotent\ \ wavefunction} = 0$$

There is no change in either physics or mathematics. The wavefunction, in fact, has both idempotent and nilpotent properties. Even the wavefunction in the original, 'conventional' version of the equation was a nilpotent, because we brought out the nilpotent characteristics merely by multiplying the operator from the left, and not by changing the wavefunction. The idempotent aspect ensures that the 4-component wavefunction acts as a spinor, as in the conventional version of relativistic quantum mechanics.

The nilpotent formalism is optimised for calculating efficiency, even by comparison with the nonrelativistic versions of quantum mechanics. The reduction to a single phase factor means that all aspects of a problem can be dealt with in a single calculation, but a more important one is that the extra constraint of nilpotency immediately forces on us an interpretation which connects every fermion holistically and nonlocally with the rest of the universe, in a way that makes much of the formal apparatus of relativistic quantum mechanics redundant. It seems to be impossible to make quantum physics any more minimal.

## SOLUTIONS FOR A POINT-LIKE PARTICLE: THE COULOMB SOLUTION

Anything which claims to derive fundamental physics from a single equation needs to show how all the fundamental forces – the three gauge forces, electric, strong and weak, plus gravity – can be derived, both in structure and in relevant solutions, along with specifying how all the known fundamental physical particles to which they apply originate in the structure of the equation. The key aspect of the nilpotent structure is that the nonlocal processes determining each interaction can be derived from the algebraic representation, and these presuppose an equivalent local structure for potentials within $E$, and from these to exact analytic solutions.

It is a significant fact, which will be now demonstrated, that there are only three Dirac solutions for a point-like fundamental particle. This has not been done previously, and requires a nilpotent wavefunction for the demonstration. This takes us to the heart of the Standard Model and its symmetry-breaking between the interactions. The simplest solution refers to spherical symmetry round a point source, and to consider the effect of one spherically symmetric point-like particle on another we need a version of Dirac's conversion to polar coordinates for the momentum operator:

$$\left( \pm\, \boldsymbol{k}E \pm \boldsymbol{i}\left( \frac{\partial}{\partial \boldsymbol{r}} + \frac{1}{r} \pm i\,\frac{j + ½}{r} \right) + \boldsymbol{ij}m \right)$$

By inspection, we find that this has no nilpotent solutions unless the $\boldsymbol{k}E$ term acquires a component varying with $1 / r$, that is a Coulomb term. All interactions between fermions include such a term. Nilpotency demands it. So, if all point particles are spherically symmetric sources, then the minimum nilpotent operator will be of the form

$$\left( \pm \boldsymbol{k}\left( E + \frac{A}{r} \right) \pm \boldsymbol{i}\left( \frac{\partial}{\partial \boldsymbol{r}} + \frac{1}{r} \pm i\frac{j+½}{r} \right) + \boldsymbol{ij}m \right)$$

To establish that this is a nilpotent, we must now find the phase to which this must apply to create a nilpotent amplitude. This is a convenient example for showing how an operator fixes the phase factor and quite quickly produces the characteristic solution for the Coulomb force (hydrogen atom, etc.). Deriving the solution for this case provides a model for all other cases.

The ease of calculation is due to the fact that the structure provides dual information about both fermion and vacuum. We apply the specified operator to the phase

$$e^{-ar} r^{\gamma} \sum_{\nu=0} a_{\nu} r^{\nu}$$

to find the amplitude (derived, as in the conventional solution, by inspired guesswork or trial and error), and equate the squared amplitude to zero.

$$4\left( E + \frac{A}{r} \right)^2 = -2\left( -a + \frac{\gamma}{r} + \frac{\nu}{r} + \ldots \frac{1}{r} + i\frac{j+½}{r} \right)^2 - 2\left( -a + \frac{\gamma}{r} + \frac{\nu}{r} + \ldots \frac{1}{r} - i\frac{j+½}{r} \right)^2 + 4m^2$$

Successively equating constant terms and terms in $1 / r^2$ with $\nu = 0$, we find

$$a = \sqrt{m^2 - E^2}$$

and

$$\gamma = -1 + \sqrt{(j+½)^2 - A^2}$$

Then, assuming the power series terminates at $n'$, and equating coefficients of $1 / r$ for $\nu = n'$:

$$2EA = -2\sqrt{m^2 - E^2}\,(\gamma + 1 + n')$$

and

$$\frac{E}{m} = \frac{1}{\sqrt{1 + \dfrac{A^2}{(\gamma + 1 + n')^2}}} = \frac{1}{\sqrt{1 + \dfrac{A^2}{\left( \sqrt{(j+½)^2 - A^2} + n' \right)^2}}}$$

This is the solution for any Coulomb-type force, such as electrostatics and gravity. When $A = Ze^2$ we have the 'hydrogen atom' solution in just 6 lines, incomparably shorter and simpler than any standard derivation.

## THE STRONG INTERACTION SOLUTION

The pure Coulomb solution is well known, but it is possible to find analytic solutions when the Coulomb term is combined with potentials varying in other ways with $r$. In fact, there are only two other solutions for point particles, neither previously achieved analytically. When the extra term $\propto r$, the solution has the characteristics of the strong interaction – infrared slavery, asymptotic freedom, etc. When the extra term is any other polynomial or collection of polynomials, the solution is a harmonic oscillator. The equations for these terms can be derived by looking at the nonlocal aspects of the Dirac operator.

The strong interaction has never been explained analytically, though there is a general understanding, on empirical grounds, that it involves a linear, as well as a Coulomb, potential. The interaction begins in a combination state, which reflects the *vector* nature of the **p** term in the nilpotent wavefunction. Effectively, the vector aspect requires a source term and corresponding vacuum with three components. A combination is outside the bracket and so is nonlocal. Nilpotency, of course, prevents us writing a combination of the form

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = 0$$

which would automatically zero. However, if we separate out three components of **p** in the form

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m)$$

then we have a nonzero combination. Spin is defined in only 1 direction at a time, so, at any given instant, the wavefunction will reduce to

$$\begin{aligned}
(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m) &\to (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m) \\
(\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_y + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m) &\to (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m) \\
(\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_z + \boldsymbol{j}m) &\to (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_z + \boldsymbol{j}m)
\end{aligned}$$

The change of sign in the second term is significant. It means we have to have a version with the opposite sign of $\boldsymbol{i}\mathbf{i}p_x$ to maintain the symmetry. Six separate phases are identifiable, 3 symmetric and 3 antisymmetric:

$$\begin{aligned}
(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m) &\to (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m) \\
(\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_y + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m) &\to (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m) \\
(\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_z + \boldsymbol{j}m) &\to (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_z + \boldsymbol{j}m) \\
(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m) &\to (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m) \\
(\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_y + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m) &\to (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m) \\
(\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{j}m)\ (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_z + \boldsymbol{j}m) &\to (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_z + \boldsymbol{j}m)
\end{aligned}$$

The duality of the ± **p** or helicity states in these structures is a clear indication that the composite state described does not have zero mass. The mediators of the transitions between the six component states will be six bosons of the form:

$$(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_x)\ (-\ \boldsymbol{ik}E - \boldsymbol{i}\mathbf{j}p_y)$$

and two combinations of the three bosons of the form:

$$(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_x)\ (-\ \boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_x)$$

These structures are, of course, identical to an equivalent set in which both brackets undergo a complete sign reversal. The six 'phases' represented here must all be valid at the same time, as required by the rotation symmetry of space or of gauge invariance. The symmetry is recognisably that of the group $SU(3)$, with the same structure as the three symmetric and three antisymmetric 'colour' combinations in the standard representation:

$$\psi \sim (\mathrm{BGR} - \mathrm{BRG} + \mathrm{GRB} - \mathrm{RGB} + \mathrm{RBG} - \mathrm{GBR})$$

The structure solves several problems at once. One is the mass gap – the proton must have a dynamic mass which is not mostly from the masses of the 'quarks' or colour units, because it has to have both helicities simultaneously. (The solution of this problem has been put forward for a prize by the Clay Institute Prize.) The mass is also a Higgs-type mass even though it comes from the gluon exchange – the mechanisms are not mutually exclusive. So, contrary to many 'authoritative' pronouncements on this issue, the mass of the proton is entirely derived, directly or indirectly, from the Higgs mechanism. There is no mysterious alternative method operating. In the same way, we can understand the nonchirality of the strong interaction; the $SU(3)$ structure; the need for a linear as well as Coulomb potential; the zero electric dipole moment of the neutron; and possibly the spin contribution from the valence quarks (1/3 of the total).

We have represented the strong force as a nonlocal interaction between the parts of the wavefunction or 'quarks' incorporating $p_x$, $p_y$, and $p_z$. This interaction is independent of the physical separation of the components (and they must be spatially or temporally separated to have different local and nonlocal manifestations). The local effect of this is a transfer of **p** between the three brackets of each wavefunction. As it is local, with **p** changing within each bracket, this is not instantaneous, unlike the actual combinations and superpositions, and so there will be a nonzero local force or rate of change of momentum. However, because it does not depend on separation, this force will be constant. The local manifestation of a constant force is a potential that varies linearly with distance, and, in principle, it will be the same whether we are referring to a quark-antiquark combination, with a temporal cycle of the components $p_x$, $p_y$, and $p_z$, and a separation of quark and antiquark, or to a combination of three quarks, with a separation of each from the centre of the system.

Using this argument based on fundamental principles, we create a differential operator incorporating Coulomb and linear potentials from a source with spherical symmetry, which is, in physical terms, either the centre of a 3-quark system or one component of a quark-antiquark pairing.

$$\left( \pm \boldsymbol{k}\left( E + \frac{A}{r} + Br \right) \mp \boldsymbol{i}\left( \frac{\partial}{\partial r} + \frac{1}{r} \pm i\,\frac{j+½}{r} \right) + \boldsymbol{ij}m \right)$$

As with the Coulomb solution, we solve by finding the phase which leads to a nilpotent solution. Here, by analogy, we can take it to be something of the form:

$$\phi = \exp\left(-ar - br^2\right) r^{\gamma} \sum_{\nu=0} a_{\nu} r^{\nu}$$

Applying the operator to the phase, and then the nilpotent condition, we obtain:

$$E^2 + 2AB + \frac{A^2}{r^2} + B^2 r^2 + \frac{2AE}{r} + 2BEr = m^2$$
$$-\left( a^2 + \frac{(\gamma + \nu + ... + 1)^2}{r^2} - \frac{(j+½)^2}{r^2} + 4b^2 r^2 + 4abr - 4b(\gamma + \nu + ... + 1) - \frac{2a}{r}(\gamma + \nu + ... + 1) \right)$$

with the positive and negative $i(j + ½)$ terms cancelling out over the four solutions, as previously. Then, assuming a termination in the power series (as with the Coulomb solution), we can equate successive coefficients of $r^2$, $r$ and $1 / r$, to give

$$B^2 = -4b^2$$

$$2BE = -4ab$$
$$2AE = 2a(\gamma + \nu + 1)$$

These equations immediately lead to:

$$b = \pm \frac{iB}{2}$$
$$a = \mp iE$$
$$\gamma + \nu + 1 = \mp iA$$

The ground state case (where $\nu = 0$) then requires a phase factor of the form:

$$\phi = \exp\left(\pm\, iEr \mp iBr^2 / 2\right) r^{\mp iA - 1}$$

The imaginary exponential terms in $\phi$ can be seen as representing asymptotic freedom, the exp ($\mp iEr$) being typical for a free fermion. The complex $r^\gamma$ term can be structured as a component phase, $\chi(r) = \exp(\pm\, iA \ln(r))$, which varies less rapidly with $r$ than the rest of $\phi$.

We can therefore write $\phi$ as

$$\phi = \frac{\exp(kr + \chi(r))}{r}$$

where

$$k = \pm iE \mp iBr / 2$$

The first term in $k$ dominates at high energies, where $r$ is small, approximating to a free fermion solution, which can be interpreted as asymptotic freedom, while the second term, with its confining potential $Br$, dominates, at low energies, when $r$ is large, and this can be interpreted as infrared slavery.

## THE WEAK INTERACTION SOLUTION

Just as the special characteristic of the strong interaction appear to originate in the (horizontal) combination state required to represent the 3-dimensional **p** vector, so those of the weak interaction appear to originate in the (vertical) superposition state of the 4-component Dirac spinor, and we can model all weak interactions on this basis. There does not seem to be any coherent idea of how the weak interaction should be expressed in terms of a distance-related potential, although its symmetry is known to be that of the *SU*(2) group. We, however, have presented evidence for a dipole component, in its superposition of two oscillating states, which, in more complicated situations, might be expected to become multipole. This time we will approach the problem more generally to show that the nilpotent solutions are exclusive. We will begin with a nilpotent operator with a form such as

$$\left( \boldsymbol{k}\left( E - \frac{A}{r} - Cr^n \right) + \boldsymbol{i}\left( \frac{\partial}{\partial r} + \frac{1}{r} \pm i\frac{j + ½}{r} \right) + \boldsymbol{ij}m \right)$$

where $n$ is an integer greater than 1 or less than –1 (so including the dipole-monopole and dipole-dipole cases of $n = -2$ and $n = -3$).

As usual, we look for a phase factor which will make the amplitude nilpotent. So, extending our work on the Coulomb solution, we may suppose that the phase factor is of the form:

$$\phi = \exp\left(-ar - br^{n+1}\right) r^{\gamma} \sum_{\nu=0} a_{\nu} r^{\nu}$$

Applying the operator and squaring to zero, with a termination in the series, we obtain

$$4\left(E - \frac{A}{r} - Cr^{n}\right)^{2} = -2\left(-a + (n+1)br^{n} + \frac{\gamma}{r} + \frac{\nu}{r} + \frac{1}{r} + i\frac{j+½}{r}\right)^{2} - 2\left(-a + (n+1)br^{n} + \frac{\gamma}{r} + \frac{\nu}{r} + \frac{1}{r} - i\frac{j+½}{r}\right)^{2}$$

Equating constant terms, we find

$$a = \sqrt{m^{2} - E^{2}}$$

Equating terms in $r^{2n}$, with $\nu = 0$:

$$C^{2} = -(n+1)^{2} b^{2}$$

$$b = \pm \frac{ic}{n+1}$$

Equating coefficients of $r$, where $n = 0$:

$$AC = -(n+1)\, b\, (1+\gamma)$$
$$(1+\gamma) = \pm iA$$

Equating coefficients of $1 / r^{2}$ and coefficients of $1 / r$, for a power series terminating in $\nu = \nu'$, we obtain

$$A^{2} = -(1 + \gamma + n')^{2} + (j + ½)^{2}$$

and

$$-EA = a\,(1 + \gamma + n').$$

Combining the last three equations produces:

$$\left(\frac{m^{2} - E^{2}}{E^{2}}\right)(1 + \gamma + n')^{2} = -(1 + \gamma + n')^{2} + (j + ½)^{2}$$

$$E = -\frac{m}{j+½}\left(\pm iA + n'\right)$$

This equation has the form of a harmonic oscillator, with evenly spaced energy levels deriving from integral values of $n'$. Though it does not immediately suggest the value for the term $iA$, we can make the additional assumption, based on our interpretation of *zitterbewegung*, that $A$, the phase term required for spherical symmetry, has some connection with the random directionality of the fermion spin. We, therefore, assign to it a half-unit value ($\pm ½\, i$), or ($\pm ½\, i\hbar c$), using explicit values for the constants, and obtain the complete formula for the fermionic simple harmonic oscillator:

$$E = -\frac{m}{j+½}\left(½ + n'\right)$$

The dimensions of $A$ are those of charge ($q$) squared or interaction energy × range, and an $A$ numerically equal to $\pm ½\, \hbar c$ would be exactly that required by the uncertainty principle, allowing the value of the range of an interaction mediated by the $Z$ boson to be calculated as $\hbar / 2M_{Z}c = 2.166 \times 10^{-18}$ m, as observed.

The ½ $\hbar c$ term is also significant in the expressions for zero-point energy and *zitterbewegung*, which connect with both spin and the uncertainty principle. Interpreting the *zitterbewegung* as a dipolar switching between fermion and vacuum antifermion states, we can describe this in terms of a weak dipole moment $(\hbar c\ /\ 2)^{3/2}\ /\ M_Z c^2$, of magnitude $8.965 \times 10^{-18}$ $e$ m ($1.44 \times 10^{-36}$ Cm). Because of the specific appearance of the ½ $\hbar c$ term for spin ($s$) in $\mu = gqs\ /\ 2m$, an identical expression can additionally be used to define a weak magnetic moment, of order $4.64 \times 10^{-5}$ × the magnetic moment of the electron. The existence of such a dipole moment would make the spin ½ term an expression of the dipolarity of the weak vacuum, and a physical representation of the weak interaction as a link between the fermion and vacuum, or between real space and vacuum space.

The value of $A$ we have assumed seems to be the one required to link a number of different physical manifestations of the weak interaction as related to fermionic structure. But, in any case, the solution we have derived indicates that an additional potential of the form $Cr^n$, where $n$ is an integer greater than 1 or less than −1, has the effect of creating a harmonic oscillator solution for the nilpotent operator, irrespective of the value of $n$. In fact, we can show that any polynomial sum of potentials of this form will produce the same result, and consequently virtually any function of $r$ other than the pure Coulomb of inverse linear, and the combination of inverse and direct linear.

Potentials of this kind will emerge from any system in which there is complexity, aggregation, or a multiplicity of sources, even if the individual sources have Coulomb or linear potentials. In the case of dipolar weak sources, the minimum additional term will be of the form $Cr^{-3}$, and so will provide the correct characteristics for the weak interaction from the kind of potential that weak sources must necessarily produce. In addition, because this solution is exclusive for distance related potentials of the form $Cr^n$, except where $r = 1$ or −1, we have also, in effect, shown that a fermion interaction specified in relation to a spherically symmetric point source has only three physical manifestations, and that these are the ones associated with the electric (or other pure Coulomb), strong and weak interactions.

## OTHER DEVELOPMENTS

The nilpotent method allows an immediate transformation to quantum field theory, without the need for the cumbersome apparatus of quantum field integrals, etc. For complete specification, it requires the interaction between the fermion and the rest of the universe. The single phase factor makes calculation relatively easy. It also corresponds with Feynman's interpretation of negative energy states requiring reversed time. Symmetry between $E$, $\mathbf{p}$ and $m$ is determined by coefficients $\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$, thus making clear the significance of the second vector space.

The nilpotent structure of the lead term combined with the idempotent structure of the vacuum operator ensures that all real fermions have exactly supersymmetric boson partners (with the same $E$, $\mathbf{p}$, $m$) and vice versa. The fermion is its own vacuum boson. Ultimately, this means that we don't need renormalization for the self-interaction (as we can also show through perturbation theory) or extra supersymmetric particles; and there is no hierarchy problem. The fermion and boson loops cancel automatically. Supersymmetric particles emerge from the calculations, but they are vacuum states, just like the negative energy solutions in Dirac's representation of the electron. In addition, the nilpotent formalism ensure that there is no infrared divergence in propagators, and no singularity separating the particle and antiparticle solutions because the two come as a single unit incorporating real space and vacuum space on an equal footing [13].

The nilpotent wavefunctions and the $SU(3)$, $SU(2)$ and $U(1)$ symmetries which emerge lead to a new understanding of the quark theory, especially in relation to the electric interaction, where the fractional charges are emergent (as in the quantum Hall effect) rather than fundamental. This completely changes the running coupling constant equations in a way that leads to a Grand Unification of all three gauge forces at the Planck mass, the energy associated with 'quantum gravity'. This also changes the relationship between the gauge forces and gravity, which now takes on a character as the vacuum equivalent of these forces (Gravity-Gauge Theory Correspondence), leading to an inertial reaction as the local observable manifestation (with quantum and wave properties). In addition, gravity as a vacuum force (corresponding to the continuous Higgs field) leads to a prediction of the 'dark energy' at 2/3 of the total universe value, compared to the current experimental value of 68 % [2, 5]

Outside of quantum mechanics and fundamental physics, there are other indications that the Klein-parameter 4 group and the 64-part Clifford algebra applied to the Dirac equation are generic consequences of a more fundamental universal rewrite system based on establishing a zero totality for the universe at all levels [25-26]. The Klein-4 group is based on three dualities which ensure that the system of the in this sense totals zero and creates a degree of closure. There are 256 (4 × 64) cellular automata, of which only four (110, 124, 137 and 193), which are known to be Turing compatible and capable of universal computation, and, like the four fundamental parameters, these automata transform into each other by analogues of $C$, $P$ and $T$ symmetries. Even more striking is the genetic code, which has 64 components, made up of triplet combinations of the four fundamental bases, and these bases, like the parameters, have three dualities, so ensuring closure. The three dualities were postulated with V. Hill in 2007 [2], with two being identified (purine / pyrimidine and A-T / G-C); the second has now been more definitely characterized (as weak / strong hydrogen bonding), and a third established (G-T keto / A-C amino) [27]. There are almost exact parallels with the parameter group. The codons can also be arranged in four groups exhibiting known experimental properties, based on $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ for the first base, $(1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ for the second, and $(1, -1, i, -i)$ for the third. If we arrange these on the faces of a regular icosidodecahron, the pentagonal faces are surrounded by algebraic units from regular nilpotent pentads [28].

## THE SPECTRUM OF PARTICLES

So far, we have demonstrated that the properties of the three nongravitational interactions of the Standard Model can be derived from the Clifford algebra structure of the nilpotent fermionic state. Yet this uses only 5 out of the 64 components of the entire algebra, in a single pentad. If the mathematical operators have real physical meaning, there are simultaneous possibilities for at least 6 pentad structures to exist at the same time representing 6 leptons and 6 quark types, and another 12 for their antitypes, the colours coming from options on the vector terms $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ (specified in the table by the OR symbol $\vee$). Transitions between these would require transfers of energy. If this is a quantum mechanical system, then all the possibilities must exist at once, though probabilities for different structures may differ, depending on the energies needed to produce them. The structure we have in the algebra seems to be exactly that needed to produce the fermion spectrum [29].

| *generation* | | *isospin* | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | electron neutrino | up | $\boldsymbol{i}\mathbf{i}$ | $\boldsymbol{i}\mathbf{j}$ | $\boldsymbol{i}\mathbf{k}$ | $i\boldsymbol{k}$ | $\boldsymbol{j}$ |
| | electron | down | $i\mathbf{i}\boldsymbol{i}$ | $i\mathbf{i}\boldsymbol{j}$ | $i\mathbf{i}\boldsymbol{k}$ | $i\mathbf{k}$ | $\mathbf{j}$ |
| 2 | muon neutrino | up | $\boldsymbol{j}\mathbf{i}$ | $\boldsymbol{j}\mathbf{j}$ | $\boldsymbol{j}\mathbf{k}$ | $i\boldsymbol{i}$ | $\boldsymbol{k}$ |
| | muon | down | $i\mathbf{j}\boldsymbol{i}$ | $i\mathbf{j}\boldsymbol{j}$ | $i\mathbf{j}\boldsymbol{k}$ | $i\mathbf{i}$ | $\mathbf{k}$ |
| 3 | tau neutrino | up | $\boldsymbol{k}\mathbf{i}$ | $\boldsymbol{k}\mathbf{j}$ | $\boldsymbol{k}\mathbf{k}$ | $i\boldsymbol{j}$ | $\boldsymbol{i}$ |
| | tau | down | $i\mathbf{k}\boldsymbol{i}$ | $i\mathbf{k}\boldsymbol{j}$ | $i\mathbf{k}\boldsymbol{k}$ | $i\mathbf{j}$ | $\mathbf{i}$ |
| 1 | antielectron-neutrino | up | $-\boldsymbol{i}\mathbf{i}$ | $-\boldsymbol{i}\mathbf{j}$ | $-\boldsymbol{i}\mathbf{k}$ | $-i\boldsymbol{k}$ | $-\boldsymbol{j}$ |
| | antielectron | down | $-i\mathbf{i}\boldsymbol{i}$ | $-i\mathbf{i}\boldsymbol{j}$ | $-i\mathbf{i}\boldsymbol{k}$ | $-i\mathbf{k}$ | $-\mathbf{j}$ |
| 2 | antimuon-neutrino | up | $-\boldsymbol{j}\mathbf{i}$ | $-\boldsymbol{j}\mathbf{j}$ | $-\boldsymbol{j}\mathbf{k}$ | $-i\boldsymbol{i}$ | $-\boldsymbol{k}$ |
| | antimuon | down | $-i\mathbf{j}\boldsymbol{i}$ | $-i\mathbf{j}\boldsymbol{j}$ | $-i\mathbf{j}\boldsymbol{k}$ | $-i\mathbf{i}$ | $-\mathbf{k}$ |
| 3 | antitau-neutrino | up | $-\boldsymbol{k}\mathbf{i}$ | $-\boldsymbol{k}\mathbf{j}$ | $-\boldsymbol{k}\mathbf{k}$ | $-i\boldsymbol{j}$ | $-\boldsymbol{i}$ |
| | antitau | down | $-i\mathbf{k}\boldsymbol{i}$ | $-i\mathbf{k}\boldsymbol{j}$ | $-i\mathbf{k}\boldsymbol{k}$ | $-i\mathbf{j}$ | $-\mathbf{i}$ |

| | | | | | |
|---|---|---|---|---|---|
| 1 | up quark | up | $\boldsymbol{i}\mathbf{i} \vee \boldsymbol{i}\mathbf{j} \vee \boldsymbol{i}\mathbf{k}$ | $i\boldsymbol{k}$ | $\boldsymbol{j}$ |
| | down quark | down | $i\mathbf{i}\boldsymbol{i} \vee i\mathbf{i}\boldsymbol{j} \vee i\mathbf{i}\boldsymbol{k}$ | $i\mathbf{k}$ | $\mathbf{j}$ |
| 2 | charm quark | up | $\boldsymbol{j}\mathbf{i} \vee \boldsymbol{j}\mathbf{j} \vee \boldsymbol{j}\mathbf{k}$ | $i\boldsymbol{i}$ | $\boldsymbol{k}$ |
| 3 | strange quark | down | $i\mathbf{j}\boldsymbol{i} \vee i\mathbf{j}\boldsymbol{j} \vee i\mathbf{j}\boldsymbol{k}$ | $i\mathbf{i}$ | $\mathbf{k}$ |
| 3 | top quark | up | $\boldsymbol{k}\mathbf{i} \vee \boldsymbol{k}\mathbf{j} \vee \boldsymbol{k}\mathbf{k}$ | $i\boldsymbol{j}$ | $\boldsymbol{i}$ |
| | bottom quark | down | $i\mathbf{k}\boldsymbol{i} \vee i\mathbf{k}\boldsymbol{j} \vee i\mathbf{k}\boldsymbol{k}$ | $i\mathbf{j}$ | $\mathbf{i}$ |
| 1 | antiup-quark | up | $-\boldsymbol{i}\mathbf{i} \vee -\boldsymbol{i}\mathbf{j} \vee -\boldsymbol{i}\mathbf{k}$ | $-i\boldsymbol{k}$ | $-\boldsymbol{j}$ |
| | antidown-quark | down | $-i\mathbf{i}\boldsymbol{i} \vee -i\mathbf{i}\boldsymbol{j} \vee -i\mathbf{i}\boldsymbol{k}$ | $-i\mathbf{k}$ | $-\mathbf{j}$ |
| 2 | anticharm-quark | up | $-\boldsymbol{j}\mathbf{i} \vee -\boldsymbol{j}\mathbf{j} \vee -\boldsymbol{j}\mathbf{k}$ | $-i\boldsymbol{i}$ | $-\boldsymbol{k}$ |
| | antistrange-quark | down | $-i\mathbf{j}\boldsymbol{i} \vee -i\mathbf{j}\boldsymbol{j} \vee -i\mathbf{j}\boldsymbol{k}$ | $-i\mathbf{i}$ | $-\mathbf{k}$ |
| 3 | antitop-quark | up | $-\boldsymbol{k}\mathbf{i} \vee -\boldsymbol{k}\mathbf{j} \vee -\boldsymbol{k}\mathbf{k}$ | $-i\boldsymbol{j}$ | $-\boldsymbol{i}$ |
| | antibottom-quark | down | $-i\mathbf{k}\boldsymbol{i} \vee -i\mathbf{k}\boldsymbol{j} \vee -i\mathbf{k}\boldsymbol{k}$ | $-i\mathbf{j}$ | $-\mathbf{i}$ |

Apart from the pentads, another possibility is to investigate the algebra of individual symbol combinations. This leads to representations of 48 possible fermion states (leptons + coloured quarks) as created through algebraic *transitions*, with those states with no quaternion component standing for the 16 algebraic structures required for particle-free space.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{i}\mathbf{i}$ | $\boldsymbol{i}\mathbf{j}$ | $\boldsymbol{i}\mathbf{k}$ | $\boldsymbol{i}$ | u / $\nu_e$ | $-\boldsymbol{i}\mathbf{i}$ | $-\boldsymbol{i}\mathbf{j}$ | $-\boldsymbol{i}\mathbf{k}$ | $-\boldsymbol{i}$ |
| $i\boldsymbol{i}\mathbf{i}$ | $i\boldsymbol{i}\mathbf{j}$ | $i\boldsymbol{i}\mathbf{k}$ | $i\boldsymbol{i}$ | d / e | $-i\boldsymbol{i}\mathbf{i}$ | $-i\boldsymbol{i}\mathbf{j}$ | $-i\boldsymbol{i}\mathbf{k}$ | $-i\boldsymbol{i}$ |
| $\boldsymbol{j}\mathbf{i}$ | $\boldsymbol{j}\mathbf{j}$ | $\boldsymbol{j}\mathbf{k}$ | $\boldsymbol{j}$ | c / $\nu_\mu$ | $-\boldsymbol{j}\mathbf{i}$ | $-\boldsymbol{j}\mathbf{j}$ | $-\boldsymbol{j}\mathbf{k}$ | $-\boldsymbol{j}$ |
| $i\boldsymbol{j}\mathbf{i}$ | $i\boldsymbol{j}\mathbf{j}$ | $i\boldsymbol{j}\mathbf{k}$ | $i\boldsymbol{j}$ | s / $\mu$ | $-i\boldsymbol{j}\mathbf{i}$ | $-i\boldsymbol{j}\mathbf{j}$ | $-i\boldsymbol{j}\mathbf{k}$ | $-i\boldsymbol{j}$ |
| $\boldsymbol{k}\mathbf{i}$ | $\boldsymbol{k}\mathbf{j}$ | $\boldsymbol{k}\mathbf{k}$ | $\boldsymbol{k}$ | $\tau$ / $\nu_\tau$ | $-\boldsymbol{k}\mathbf{i}$ | $-\boldsymbol{k}\mathbf{j}$ | $-\boldsymbol{k}\mathbf{k}$ | $-\boldsymbol{k}$ |
| $i\boldsymbol{k}\mathbf{i}$ | $i\boldsymbol{k}\mathbf{j}$ | $i\boldsymbol{k}\mathbf{k}$ | $i\boldsymbol{k}$ | b / t | $-i\boldsymbol{k}\mathbf{i}$ | $-i\boldsymbol{k}\mathbf{j}$ | $-i\boldsymbol{k}\mathbf{k}$ | $-i\boldsymbol{k}$ |

Here, 48 / 64 algebra units have ***quaternions*** = charges. Quarks / leptons are distinguished by these being attached to vectors / scalars; the 3 colours require the vector unit **i**, **j**, **k**, Isospin up / down are determined by the respective factors 1 / *i*. The 3 generations are characterised by the 3 quaternions, ***i***, ***j***, ***k***; Fermions and antifermions are separated by the factors + / –. It is somewhat extraordinary that the algebra contains two different ways of generating the particle spectrum, via the pentads and via the units. However, a 64 × 64 component table of the algebra, worked out with Sydney Rowlands, appears to provide a ready explanation [30]. Here, each row and column are headed by one of the 64 units of the algebra whereas the nonzero contents of the rows and columns are the versions of the pentads.

## THE KLEIN-4 PARAMETER GROUP

When Dirac set out to produce a relativistic equation for the fermion, he realised he needed to square root a quadratic expression to create a linear equation with the same number of terms. This can be done by using anticommutating operators, taken from abstract mathematics. Dirac chose to use 4 × 4 matrices but this is only one option, and, even with matrices, there is no unique formulation. It turns out that the 4 original Dirac operators, and every subsequent modelling of them, are components of a group with 64 elements and 5 generators – a double space creating a zero totality. Ultimately, these become the only components we need to construct physics as we know it. The operators actually used in the equation and those that could be used in place of them have different functions in leading respectively to the interactions of the Standard Model and the broken symmetry, and the particles which interact.

The algebra was forced upon us by the mathematical requirement to make the equation linear but the result is that it has the potential for explaining a huge amount of physical structure. It has also turned out to be relevant in other areas, particularly in the genetic code, which has 64 elements, constructed in exactly the same way (collaboration with Vanessa Hill) [2], and in any context where two spaces are relevant, or object and environment (e.g. John Torday on the cell), which seems to be a universal condition (collaboration with Peter Marcer [31]). The nilpotent structure which emerges from the generators, and incorporates all aspects of the double space, is relevant in any context where conservation of energy is relevant. Ultimately, Dirac's $\boldsymbol{\alpha}$ and $\beta$, though introduced as something of a convenient 'fix' and still largely treated that way, have, a century later, ended up by opening the whole world of fundamental physics and a great deal more besides.

## REFERENCES


[1] Dirac PAM. 1928 The Quantum Theory of the Electron. Proc Roy Soc A. 117, pp. 610-24

[2] Rowlands P. Zero to Infinity The Foundations of Physics. Singapore: World Scientific; 2007.

[3] Rowlands P. 2009. Are There Alternatives to Our Present Theories of Physical Reality? arXiv:0912.3433

[4] Rowlands P. 2013. Symmetry in physics from the foundations. In: Symmetry. Volume 24. pp 41-56.

[5] Rowlands P. The Foundations of Physical Law. Singapore: World Scientific; 2014.

[6] Rowlands P. How Schrödinger's Cat Escaped the Box. Singapore: World Scientific; 2015.

[7] Rowlands P. 1994. An algebra combining vectors and quaternions: A comment on James D. Edmonds' paper. In: Speculat. Sci Tech. Volume 17. pp 279-282.

[8] Rowlands P. 1996. Some interpretations of the Dirac algebra. In: Speculat Sci Tech. Volume 19. pp 243-51.

[9] Rowlands P. The nilpotent Dirac equation and its applications in particle physics. arXiv:quant-ph/0301071.

[10] Rowlands P. 2004. Symmetry breaking and the nilpotent Dirac equation. In: AIP Conference Proceedings. Volume 718. pp. 102-115.

[11] Rowlands P. 2005. Removing redundancy in relativistic quantum mechanics. arXiv.org:physics/0507188.

[12] Rowlands P. 2008. What is vacuum? arXiv:0810.0224.

[13] Rowlands P. 2010. Physical interpretations of nilpotent quantum mechanics. arXiv: 1004.1523.

[14] Rowlands P. 2017. How symmetries become broken. In: Symmetry. Volume 28. Number 3. pp 244-254.

[15] Rowlands P. 2010. Dual Vector Spaces and Physical Singularities. In: AIP Conference Proceedings. Volume 1316. pp 102-111.

[16] Rowlands P. Space and antispace. In: Amoroso RL, Kauffman LH, Rowlands P (eds). The Physics of Reality Space, Time, Matter, Cosmos. Singapore: World Scientific; 2013. pp 29-37.

[17] Rowlands P. A dual space as the basis of quantum mechanics and other aspects of physics. In: Amson JC, Kauffman LH (eds). Scientific Essays in Honor of H Pierre Noyes on the Occasion of His 90th Birthday. Singapore: World Scientific; 2014. pp 318-338.

[18] Rowlands P. Dual vector spaces and physical singularities. In: Amoroso RL Kauffman LH and Rowlands P (eds). Unified Field Mechanics Natural Science Beyond the Veil of Spacetime. Singapore: World Scientific; 2015. pp 92-101.

[19] Clifford, WK. 1878. Applications of Grassmann's Extensive Algebra. American Journal of Mathematics Pure and Applied, 1, 350-358.

[20] Rowlands P, Rowlands S. Representations of the Nilpotent Dirac Matrices. In: Amoroso RL Kauffman LH, Rowlands P and Albertini G (eds). Unified Field Mechanics. II: Formulations and Empirical Tests, World Scientific, pp. 26-33.

[21] Rowlands P. 1983. The fundamental parameters of physics. In: Speculat Sci Tech. Volume 6. pp 69-80.
[22] Rowlands P. 2001. A foundational approach to physics. arXiv:physics/0106054.
[23] Mainzer K, Chua L. The Universe as Automaton From Simplicity and Symmetry to Complexity. Springer; 2012.
[24] Kauffman LH. 2004. Non-commutative worlds. New Journal of Physics. 6, 246.
[25] Rowlands P, Diaz B. 2002. A universal alphabet and rewrite system. arXiv:cs.OH/0209026.
[26] Diaz B, Rowlands P. 2005. A computational path to the nilpotent Dirac equation. In: International Journal of Computing Anticipatory Systems. Volume 16. pp. 203-18.
[27] Pethoukov S. 2024. presentation, meeting on Symmetry, Pisa, July.
[28] Hill V, Rowlands P. 2015. A Mathematical Representation of the Genetic Code. In: Amoroso RL Kauffman LH and Rowlands P (eds). Unified Field Mechanics Natural Science Beyond the Veil of Spacetime. Singapore: World Scientific; 2015. pp 553-339.
[29] Rowlands P. 2019. Constructing the Standard Model fermions. In: Journal of Physics Conference Series. Volume 1251. 012004.
[30] Rowlands S, Rowlands P. 2024. The Universal Rewrite System as a Variable Regular Expression. Researchgate, May.
[31] Marcer P, Rowlands P. 2017. Nilpotent quantum mechanics: analogs and applications. In: Frontiers in Physics. Volume 5. Article 28. pp 1-8.