

The Comma of Pythagoras, Carried to Infinity

The Proved Foundation of the Maze

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1. What this article claims

The MAZE is a store in which every finite thing has one address. The address is a route of trits: turns of +1, 0 or −1. The claim that this store is a foundation of mathematics sounds large. It is not a programme. It is a collection of theorems, proved between 300 BC and 1970, restated here in maze terms.

Six results carry the claim. Every integer has exactly one route (Stifel 1544, Knuth). Every finite text has exactly one number, hence one address (Gödel 1931). Arithmetic itself arises from three turns and one carry rule, so all of standard mathematics is interpretable on the routes (Gödel 1931, Matiyasevich 1970). Every provable statement has a route, and systematic walking finds it (Gödel 1929). No machine decides every address; the undecidable addresses form a column that never closes (Church 1936, Turing 1936). And the oldest proof of all: the ternary tower and the binary tower never meet. Their measured miss is the comma of Pythagoras. Carried to infinity, the comma fills the circle evenly, its gaps come in exactly three sizes, and its best closures are forced at depths 12, 41 and 53. Nothing here is conjecture. Every step is written out below, with the hand calculations in the text.

One thing is not yet proved, and the article names it at the end.

2. The maze in five definitions

Trit. A turn: +1, 0 or −1. Written +, 0, −.

Route. A finite sequence of trits, read from the deepest turn to the shallowest.

Value. A route $t(d-1) \dots t(1) t(0)$ has the value $\sum t(i) \cdot 3^i$. The weights are 1, 3, 9, 27, 81, ...

Depth. The length d of the route. Depth d reaches every integer n with $|n| \leq (3^d - 1)/2$. Depth 5 reaches −121 ... +121.

Address. Depth plus value. The empty route is rest: address 0.

The image throughout is a net. A route is a thread. An address is a knot where threads meet. A gap is a mesh. The question of foundations is the question whether every thread, every knot and every mesh is accounted for.

3. Theorem 1 — every integer has exactly one route

Theorem (T7). For every integer n there is exactly one finite route with value n (up to leading zeros).

Proof of existence. Divide n by 3. If the remainder is 0 or 1, write that trit. If the remainder is 2, write $-$ and carry: replace n by $(n+1)/3$. Repeat. The absolute value shrinks at every step, so the process stops.

Hand calculation for 43: $43 = 3 \cdot 14 + 1 \rightarrow$ trit $+$, continue with 14. $14 = 3 \cdot 5 - 1 \rightarrow$ trit $-$, continue with 5. $5 = 3 \cdot 2 - 1 \rightarrow$ trit $-$, continue with 2. $2 = 3 \cdot 1 - 1 \rightarrow$ trit $-$, continue with 1. $1 \rightarrow$ trit $+$. Stop.

Route, deepest first: $+ - - - +$. Check: $81 - 27 - 9 - 3 + 1 = 43$.

Proof of uniqueness. Suppose two different routes have the same value. Subtract them. The difference is zero. Look at the deepest turn where the two routes differ. The difference at that position is ± 1 or ± 2 times a power of 3. All shallower positions contribute multiples of the next power of 3. A sum of a non-multiple of 3 and multiples of 3 is never zero. Contradiction. ■

This theorem was used in 1544. Stifel showed that weights of 1, 3, 9 and 27 pounds weigh every whole load up to 40 pounds, if weights may sit in either pan. Either pan: that is the $+$ and the $-$. The modern statement is in Knuth. Three signs also carry the most value per sign among whole bases: the cost measure $r/\ln r$ is minimal at $e = 2.718$, and 3 lies nearest. The maze is not one choice among bases. It is the economical one.

4. Theorem 2 — everything expressible has an address

Theorem. Every finite text, in any fixed finite alphabet, has exactly one address. Hence every formula, every proof, every theory, and every future theorem already has its address.

Proof. Fix an alphabet of 27 symbols. $27 = 3^3$, so each symbol is exactly one triple of trits. A text of L symbols is a route of $3L$ trits: one address at depth $3L$. Different texts differ in at least one symbol, hence in at least one trit, hence land on different addresses. Every address at a depth divisible by 3 reads back as exactly one text. ■

Gödel proved the same statement in 1931 with a different code: he numbered symbols with primes and texts with prime powers. His code and the trit code are both computable, in both directions. The translation between them is therefore computable: one line, composing one decoder with the other encoder. Everything Gödel proved about his numbers transfers to the trit addresses without change. This is the translation theorem, and it is the whole bridge: the mathematics of 1931 is already the mathematics of the maze.

Two consequences deserve plain statement. First: a theorem that will be proved in 2126 is a finite text. Its address exists now, empty. The future of mathematics is not outside the maze; it is the set of vacant addresses. Second: what does not fit is exactly the uncountable. A single real number, taken whole, is an infinite route. The maze holds its cut at every depth, with an error of at most half a unit. Cutting is rounding. That is a rule of the maze, not a defect, and Section 8 shows it is also the oldest engineering practice in music.

5. Theorem 3 — arithmetic from three turns and one carry

Theorem. Addition of routes is given by one table and one carry. From it follow successor, order and multiplication. The axioms of Peano arithmetic hold on the routes. Hence every recursively axiomatized theory — set theory, analysis, type theory — is interpretable in the maze.

The table. Add trits position by position. $(+) + (+) = 2 = 3 - 1$: write $-$, carry $+$ one position deeper. $(-) + (-) = -2 = -3 + 1$: write $+$, carry $-$. All other pairs sum to $-1, 0$ or $+1$: write it, no carry.

Hand calculation, successor of 4. Four is $++(3+1)$. Add $+$: Lowest position: $+++ \rightarrow$ write $-$, carry $+$. Next position: $+++ \rightarrow$ write $-$, carry $+$. New deepest position: $+$. Result: $+-$, value $9 - 3 - 1 = 5$.

Hand calculation, successor of 43. Route $+- - - +$. Add $+$: Lowest position: $+++ \rightarrow$ write $-$, carry $+$. Next position: $-++ \rightarrow$ write 0 , no carry. Rest unchanged. Result: $+- - 0 -$, value $81 - 27 - 9 + 0 - 1 = 44$.

From the table to Peano. Zero is the empty route. The successor is "add $+$ ": total and injective by Theorem 1. Induction on numbers is induction on depth: whatever holds at rest and survives one turn holds everywhere. Addition and multiplication are the table and its iteration. These are the Peano axioms. Gödel's 1931 arithmetization then does the rest: provability in any recursively axiomatized theory is an arithmetical relation, and arithmetic lives on the routes.

The strongest form is the DPRM theorem (Davis, Putnam, Robinson, Matiyasevich, 1970). Every listable set of numbers — including the set of theorem-numbers of any formal theory — is the solution set of one polynomial equation with integer coefficients. Not "codeable in" arithmetic: equal to a piece of it. The theorems of mathematics are a Diophantine figure drawn in the integers, and by Theorem 1 the integers are the routes.

6. Theorem 4 — every provable statement has a route, and walking finds it

Theorem. Every statement that is provable at all has a proof-text, hence a proof-address. Walking the addresses in order of depth, and checking each one, reaches that address after finitely many steps.

Proof. A proof is a finite text, so it has an address (Theorem 2). Checking whether a given text is a correct proof of a given statement is a finite, mechanical act: compare each line with the axioms and the inference rules. Walk all addresses by increasing depth; at each, run the check. Every address lies at a finite depth, so every proof is reached. ■

Gödel's completeness theorem (1929) adds the guarantee underneath: whatever is logically valid is provable. So for valid statements the route is never missing. The maze proves on demand in exactly this sense: the route exists, and systematic walking finds it. What the walk cannot promise is a timetable — and for statements that have no proof, it never stops. That boundary is the next theorem, and it is a theorem, not a weakness of this construction.

7. Theorem 5 — the column

Theorem (Church 1936, Turing 1936). No procedure decides, for every address, whether it holds a theorem. **Theorem (Gödel 1931).** Every consistent theory that contains arithmetic leaves true statements unprovable.

The undecidable addresses do not form a local hole that deepens and closes. They return at every depth. In the net image: not a torn mesh, but a column running through the whole fabric — a mesh that recurs on every level and never closes. The maze does not hide this column. It locates it.

The mechanism is the diagonal lemma: for every property there is a sentence that states that property of its own number. In maze terms: a route that contains its own address. A route that refers to itself is a loop, and loops are the periodic routes, which have the closed form $v_p/(3^p - 1)$ for period p . The Gödel sentence is not a distant room in the maze. It is a loop in the corridor.

Note what this section does and does not say. It does not say the maze is incomplete where other foundations are complete. Incompleteness holds for every system containing arithmetic, in any coding — the translation theorem of Section 4 carries it over unchanged. What is specific to the maze is the geometry: the column is visible as a column.

8. Theorem 6 — the comma, carried to infinity

The column of Section 7 is logical, found in 1931. There is an older column, purely arithmetical, found by Pythagoras and proved by Euclid. It is the comma.

8.1 The comma measured. Stack twelve fifths: $(3/2)^{12}$. Stack seven octaves: 2^7 . Their ratio is $3^{12}/2^{19} = 531.441 / 524.288 \approx 1.013643$. In cents (hundredths of a tempered semitone): 23,46. Note the exponents: 12 and 19. The comma is the measured distance between the ternary tower and the binary tower, read at depth 19.

8.2 The towers never meet. Theorem. $3^m = 2^n$ has no solution except $m = n = 0$. **Proof.** Every power of 3 is odd. Every positive power of 2 is even. ■ Equivalently: $\log_2 3$ is irrational; if it were p/q , then $3^q = 2^p$. Two lines, and they close the question for all depths at once. The spiral of fifths is a true spiral. It never returns.

8.3 Carried to infinity, the comma fills everything. Theorem (Weyl 1916). Because $\log_2 3$ is irrational, the points of the spiral — $k \cdot \log_2(3/2)$ modulo 1, for $k = 1, 2, 3, \dots$ — are equidistributed on the circle. They come arbitrarily close to every point and land on none twice. "The comma carried to infinity" is thus a theorem: an everywhere-dense, never-closing thread through the octave. The same signature as the logical column: returning at every depth, never shut.

8.4 The gaps come in three sizes. Theorem (three-distance theorem; conjectured by Steinhaus, proved by Sós 1958, Świerczkowski 1959). Place the first n points of the spiral on the circle. The open gaps between them take at most three distinct lengths, and where three occur, the largest is the sum of the other two.

Hand calculation at $n = 12$. The twelve Pythagorean fifths cut the octave into twelve gaps of exactly two sizes: the limma, $256/243 = 90,225$ cents — seven of them; the apotome, $2187/2048 = 113,685$ cents — five of them. Check: $7 \times 90,225 + 5 \times 113,685 = 631,575 + 568,425 = 1200,000$. And the difference of the two gap sizes: $113,685 - 90,225 = 23,46$ cents. The comma itself. At general n a third size appears, equal to the sum of the other two. The structure of the gaps is ternary: two generators and their sum. The mesh left open by the binary–ternary miss is itself a three-valued pattern.

8.5 The forced ladder: 12, 41, 53. Where does the spiral almost close? The continued fraction of $\log_2 3 = 1,5849625\dots$ answers it, step by step: $1,5849625 \rightarrow$ whole part 1, remainder 0,5849625; $1/0,5849625 = 1,70951 \rightarrow$ 1, remainder 0,70951; $1/0,70951 = 1,40942 \rightarrow$ 1, remainder 0,40942;

$1/0,40942 = 2,44248 \rightarrow 2$, remainder 0,44248; $1/0,44248 = 2,25998 \rightarrow 2$, ... So $\log_2 3 = [1; 1, 1, 2, 2, 3, 1, 5, \dots]$. The convergents: 1, 2, $3/2$, $8/5$, **19/12**, **65/41**, **84/53**, ...

Each convergent p/q says: $3^q \approx 2^p$, as closely as any fraction with denominator $\leq q$ can — proved best possible (Lagrange). Read as music: q fifths nearly equal $p - q$ octaves. The ladder: 12 fifths $\approx$ 7 octaves, miss $3^{12}/2^{19} = 23,46$ cents — 1,955 cents per step; 41 fifths $\approx$ 24 octaves, miss $2^{65}/3^{41} = 19,84$ cents — 0,484 cents per step; 53 fifths $\approx$ 31 octaves, miss $3^{53}/2^{84} = 3,62$ cents (Mercator's comma) — 0,068 cents per step.

Twelve tones, 41 tones, 53 tones: not conventions, but the forced best closures of the comma. The depth pair (12, 19) is the first rung; (41, 65) is the next. The ladder of near-closures is computed once, by the continued fraction, and holds forever.

8.6 Cutting is rounding, executed in 1584. The working solution was not to close the spiral — impossible by 8.2 — but to cut it: spread the 23,46 cents evenly over twelve steps. Each tempered fifth becomes 700 cents instead of 701,955: an error of 1,955 cents per step, far below half of the smallest gap. Zhu Zaiyu computed the twelfth root of 2 to nine digits in 1584; Stevin did the same in Leiden around 1585. Equal temperament is the rule "cutting is rounding, error at most half a unit", applied to the oldest known incommensurability — four centuries before the rule was written down as a rule. The maze did not invent its rounding principle. It inherited it from the one craft that had lived with the comma for two thousand years.

9. What is proved, and the one thing that is not

Proved, and collected above: every integer one route (T7); every text one address, future theorems included (Gödel numbering plus the translation theorem); arithmetic, and through it all recursively axiomatized mathematics, generated by three turns and one carry (Peano on routes; DPRM); every provable statement reachable by walking (completeness plus enumeration); the undecidable addresses a structural column, with the self-referential loop in closed periodic form (Gödel, Church, Turing); and the comma — no closure ever, even filling, three gap sizes, the forced ladder 12–41–53, rounding as the working answer (Euclid, Weyl, Sós, Lagrange).

Not yet proved: that the carry rule R, which generates arithmetic in Section 5, is also a complete derivation rule for logic. Section 6 guarantees that every proof-route exists and that a systematic walk finds it. Whether R itself walks it is a construction task, not a theorem on the shelf. It is the open build item of the maze, and it is the only one.

10. Conclusion

The foundation of the maze was not waiting to be proved. It was proved, piecewise, by Euclid, Stifel, Weyl, Gödel, Church, Turing, Sós and Matiyasevich, and it only needed to be read as one building. The building has three features. Everything finite has its address, including everything not yet thought. All of mathematics runs on three turns and a carry. And the holes are not failures of the map: they are two columns — one logical, one arithmetical — and the older of the two was heard before it was proved, as a comma, by a man with a monochord.

11. Annotated references

Michael Stifel, Arithmetica Integra (Nuremberg, 1544). Why read it? The oldest use of Theorem 1: weights 1, 3, 9, 27 in either pan weigh every load to 40. The maze's addressing, four centuries early.

Donald Knuth, The Art of Computer Programming, Vol. 2, §4.1 (3rd ed., 1997). Why read it? The standard modern statement of balanced ternary: uniqueness, the carry table, and the remark that it is "perhaps the prettiest number system of all".

Euclid, Elements, Book IX. Why read it? Propositions on odd and even and on prime decomposition that settle $3^m \neq 2^n$ in two lines. The oldest theorem in this article, and the one all of Section 8 stands on.

Kurt Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls" (1929). Why read it? The guarantee of Section 6: whatever is valid has a proof, so the route is never missing.

Kurt Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I" (1931). Why read it? Two things at once: every text one number (Section 4), and the first column (Section 7). The numbering differs from the trit code; the translation is one computable line.

Alonzo Church, "An Unsolvable Problem of Elementary Number Theory" (1936); Alan Turing, "On Computable Numbers" (1936). Why read them? The proof that no machine decides every address. The column is structural, in every coding.

Yuri Matiyasevich, Hilbert's Tenth Problem (MIT Press, 1993). Why read it? The DPRM theorem with full proofs: every listable set, theorem sets included, is a polynomial figure in the integers. The strongest form of "everything is in the numbers".

Hermann Weyl, "Über die Gleichverteilung von Zahlen mod. Eins" (1916). Why read it? The equidistribution theorem of Section 8.3: the comma, carried to infinity, fills the circle evenly.

Vera Sós, "On the distribution mod 1 of the sequence $n\alpha$ " (1958); S. Świerczkowski (1959). Why read them? The three-distance theorem: the gaps of the spiral come in at most three sizes, the largest the sum of the other two. The ternary structure of the mesh, proved.

G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, chapters X–XI. Why read it? Continued fractions and Lagrange's best-approximation theorem: why 12, 41 and 53 are forced, not chosen.

J. Murray Barbour, Tuning and Temperament: A Historical Survey (1951). Why read it? The historical execution of cutting-is-rounding: Zhu Zaiyu (1584), Stevin (c. 1585), and the long practice before them.

J. Konstapel, "Everything Has Its Number in the Maze"; "Reengineering the Universe V3" (constable.blog, 2026). Why read them? The maze definitions, the address convention and the loaded store that this article grounds. Ordinary source references for Sections 2 and 4.