The Sound of the Trit Tile

A number system that never settles, a proof that it does not, and what music already knows about it

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Abstract

Take a point of the triangular lattice. Subtract a digit −1, 0 or +1 and divide by 1 − ω, where ω is a cube root of one. Repeat. Some points reach 0. The rest end in one of two points that turn forever. The Vacuum.Net theory asked what share of points turns. We show that the share has no limit. It swings between about 29.6 % and 32.0 %. The swing repeats exactly each time the size grows by a factor 3. We prove this with a computer-assisted argument. We measure the shape of the swing and split it into overtones. The repeat ratio 3:1 is the tritave of the Bohlen–Pierce scale. The swing has the structure of a Shepard–Risset tone. Its overtones give a timbre that can be played. The essay argues that this is a meeting point of number theory and music that has not been described before. It treats the musical side at length.

1. Why this question

Mathematics often moves forward until an operation cannot be undone. Then it steps back and adds a new object. A companion essay lists fourteen such barriers.[1] It predicts that the next answers will often be rhythms rather than single numbers. This essay gives a concrete case. The question is small. The answer is not a number but a rhythm. The rhythm can be heard.

The essay has two halves. Sections 2 to 6 are mathematics: the question, the measurement, the proof and the spectrum. Sections 7 to 12 are music: ratios and the comma, the tritave, endless tones, spectral composition, chord geometry and what is new. Section 13 states what is proved and what is not.

2. The Vacuum.Net theory in brief

The Vacuum.Net theory starts from movement, not from numbers.[2] There is one strand. It moves and returns. Every distinction is a loop of this strand. The second axiom, N2, is called closure. A loop persists only if it closes. What does not close passes on as tension.

Each step of the strand is a trit. It is forward (+1), standing still (0) or backward (−1). Trits combine into ladders. Two ladders are basic: doubling (×2) and tripling (×3). Powers of 2 and powers of 3 never meet exactly. The theory calls the near meetings closure depths. They are 1, 2, 5, 12, 41, 53, 306, 665 and so on. At depth 12 the miss is the Pythagorean comma: $3^{12}/2^{19} \approx 1.0136$.[3] Musicians know these numbers. Section 7 returns to them.

A book derives mathematics from these rules in 22 rungs.[4] Rung 10 puts the trits in the plane. Its numbers are the Eisenstein integers $a + b\omega$, with $\omega = (-1 + i\sqrt{3})/2$. They form a triangular lattice. The base is $\beta = 1 - \omega$. Dividing by β shrinks distances by $\sqrt{3}$ and turns the plane by 30 degrees. The book left

[1] [3] Konstapel, J. (2026). Barriers and Back-Steps, version 1.0 (21 November 2026), and The Next Mathematics, constable.blog (21 November 2026).

[2] [1] Konstapel, J. (2026). Vacuum.Net Theory, Ninth Edition: Foundations 9.2. Constable Research, Leiden.

[3] [1] Konstapel, J. (2026). Vacuum.Net Theory, Ninth Edition: Foundations 9.2. Constable Research, Leiden.

[4] [2] Konstapel, J. (2026). Mathematical Theory Generated with the Vacuum.Net Theory, version 7.0 (19 November 2026). Constable Research, Leiden.

one question open. It is called IX.3. It asks whether the share of points that keep turning settles to a fixed value.

3. The number system

Every Eisenstein integer z leaves a remainder −1, 0 or +1 after division by β. Call it the digit d. The step is T(z) = (z − d)/β. An example: 2 has digit −1. So T(2) = 3/β = 2 + ω. Then 2 + ω has digit 0, and T(2 + ω) = 1 + ω. And 1 + ω maps to itself. It turns forever.

Every orbit ends in one of three places: 0, the point 1 + ω, or the point −(1 + ω). A point that reaches 0 has a finite expansion $z = d_0 + d_1\beta + \ldots + d_k\beta^k$. We call it finite. The others we call turning. By symmetry $z \mapsto -z$, exactly half of the turning points go to each of the two fixed points.

This system is known. It is a symmetric canonical number system in the sense of Akiyama and Scheicher.[5] Their Theorem 2.2 says when every point is finite. For the base polynomial $x^2 + Ax + B$ it requires $|A| < 1 + B/2$ with $|B| \geq 2$, or $A = 1 + B/2$ with $|B| > 2$. Our base satisfies $x^2 - 3x + 3 = 0$. So $A = -3$ and $B = 3$. Neither condition holds. That turning points exist is therefore known. The wider literature on shift radix systems studies which orbits are finite or periodic.[6][7] We found no study of the share of finite points, and no report that this share fails to settle.

4. The measurement

We counted with two independent programs.[8] The first starts from each point and applies T until the orbit ends. The second works forwards. It lists all finite expansions with up to 16 digits: $3^{16} = 43\,046\,721$ of them. All are distinct. It then counts how many lie inside a disc. Both use exact integer arithmetic. The lists with 15 and 16 digits give the same counts up to norm 1.1 million. So the counts are complete there.

Both programs agree to four decimals. The table shows the share of turning points in discs of growing norm. The norm of $a + b\omega$ is $a^2 - ab + b^2$, the squared distance from 0.

Norm N	Share turning	Note
72 900	0.3076	low phase
126 266	0.3102	high phase
218 700	0.3076	N × 3: same value
378 799	0.3102	N × 3: same value
656 100	0.3076	N × 3: same value
1 136 398	0.3102	N × 3: same value

Table 1. Share of points that never reach 0, in discs of norm N. Multiplying the norm by 3 returns the same value.

[5] [4] Akiyama, S. and Scheicher, K. Symmetric shift radix systems and finite expansions. Preprint, University of Tsukuba (www.math.tsukuba.ac.jp/~akiyama/papers/centralsrs6.pdf).

[6] [5] Akiyama, S., Borbély, T., Brunotte, H., Pethő, A. and Thuswaldner, J. M. (2005). Generalized radix representations and dynamical systems I. Acta Mathematica Hungarica 108, 207–238.

[7] [6] Kirschenhofer, P., Pethő, A., Surer, P. and Thuswaldner, J. (2010). Finite and periodic orbits of shift radix systems. Journal de Théorie des Nombres de Bordeaux 22, 421–448.

[8] [18] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.8–2.2, Python scripts (controle/*.py) and audio files (1_tile_timbre, 2_bohlen_pierce, 3_risset_tritave, v1.0). Constable Research, Leiden (unpublished).

The share does not drift toward a limit. It moves up and down. Each factor 3 in the norm brings it back. On a fine grid it ranges from 29.6 % to 32.0 %. Hexagons instead of discs give other values but the same swing. Figure 1 shows one full period, measured three times.

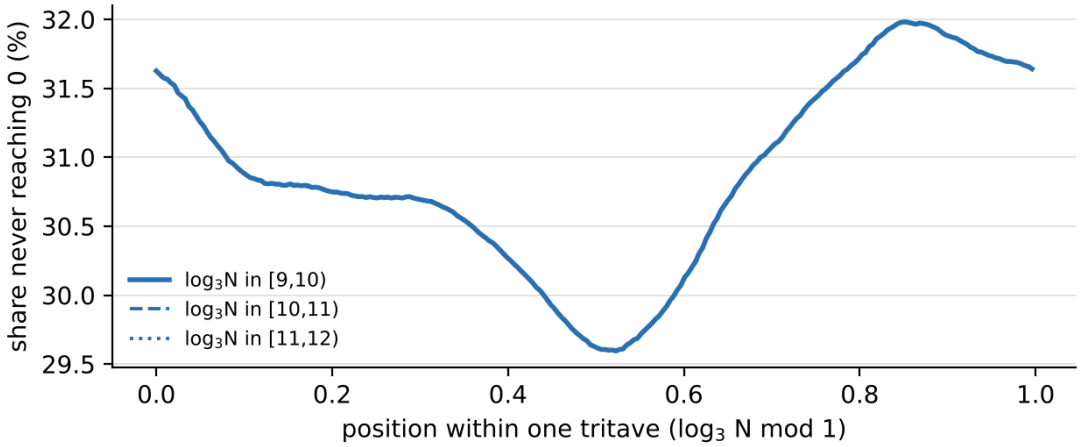


Figure 1. The share of turning points over one period, for three consecutive periods. The three curves lie on top of each other.

5. The proof

We now prove that the share has no limit. The argument has four steps. It uses one external theorem and one computer certificate.

5.1 The tile

Let F be the set of all infinite sums $d_1\beta^{-1} + d_2\beta^{-2} + \ldots$ with digits −1, 0, +1. This is a compact fractal set: the tile. It obeys $\beta F = (F - 1) \cup F \cup (F + 1)$. Let U be the set of finite points. Let W be the union of the translated tiles u + F over all u in U.

5.2 W is self-similar

For u in U and a digit d, the point βu + d is again finite, because T maps it to u. Every finite point arises in this way. With the tile equation this gives βW = W. So W looks the same after scaling by √3 and turning by 30 degrees. Let G(R) be the fraction of a disc of radius R that W covers. Discs do not care about turning. Hence G(√3·R) = G(R) exactly.

5.3 Counting equals area

Lagarias and Wang proved that tiles of this kind tile the plane by a lattice.[9] For an irreducible base polynomial and a complete set of digit residues, the lattice is the smallest invariant one containing the digit differences.[10] Here that is the Eisenstein lattice itself. So the tiles u + F, for u in the lattice, do not overlap except on a set of zero area. Each has the area of one lattice cell, √3/2.

Every point of F lies within $\rho = 1/(\sqrt{3} - 1) \approx 1.366$ of 0, because $|\beta| = \sqrt{3}$. Let A(r) be the number of finite points in a disc of radius r. Then the area of W inside a disc of radius R lies between $(\sqrt{3}/2)\cdot A(R - \rho)$ and $(\sqrt{3}/2)\cdot A(R + \rho)$. As R grows, the share of finite points therefore differs from G(R) by at most a constant times 1/R.

[9] [7] Lagarias, J. C. and Wang, Y. (1997). Integral self-affine tiles in R^n, part II: lattice tilings. Journal of Fourier Analysis and Applications 3, 83–102.

[10] [8] Steiner, W. and Thuswaldner, J. M. (2012). Rational self-affine tiles. arXiv:1203.0758.

5.4 The certificate

It remains to show that G is not constant. We bound it at two radii with exact counts. Every finite point within radius R + ρ needs at most 15 digits. Reason: one step reduces the distance from |z| to at most (|z| + 1)/√3. After 13 steps the point lies within √3 of 0. There only 0 and the six units remain. Four units reach 0 in at most two steps. The other two are the turning points. Our list with 16 digits is therefore complete.

Radius R	Lower bound of G(R)	Upper bound of G(R)
389.73 (high phase of turning)	0.67540	0.68513
321.57 (low phase of turning)	0.69797	0.70998

Table 2. Certified bounds on the covered fraction G at two radii. The intervals do not overlap: the margin is 0.0128. Two further pairs of radii give margins 0.0130 and 0.0116.

The upper bound at the first radius is below the lower bound at the second. So G is not constant. G repeats under scaling by √3. The share of finite points differs from G by O(1/R). Therefore the share takes values at most 0.6851 and at least 0.6980 infinitely often.

Theorem. For the base 1 − ω with digits −1, 0, +1, the share of Eisenstein integers in a disc that never reach 0 has no limit. Its lower limit is at most 0.302. Its upper limit is at least 0.315.

Status. This is a computer-assisted proof. It relies on the Lagarias–Wang tiling theorem and on exact integer counts. The numbers in Table 2 can be recomputed with the scripts. An independent check by a second party is welcome.

6. The spectrum

The swing repeats exactly each time the norm grows by a factor 3. So it can be split into a mean and overtones, like a periodic sound. We sampled one period at 243 points, three times. Then we took the Fourier transform.

Component	Amplitude	Phase
Mean (fundamental level)	30.87 %	—
Overtone 1	0.90 %	23.9°
Overtone 2	0.38 %	131.0°
Overtone 3	0.12 %	−4.7°
Overtone 4	0.03 %	148.6°
Overtone 5	0.05 %	−46.8°
Overtones 6–8	< 0.03 % each	—

Table 3. Decomposition of the swing. Values agree over three periods to 0.00001 in amplitude.

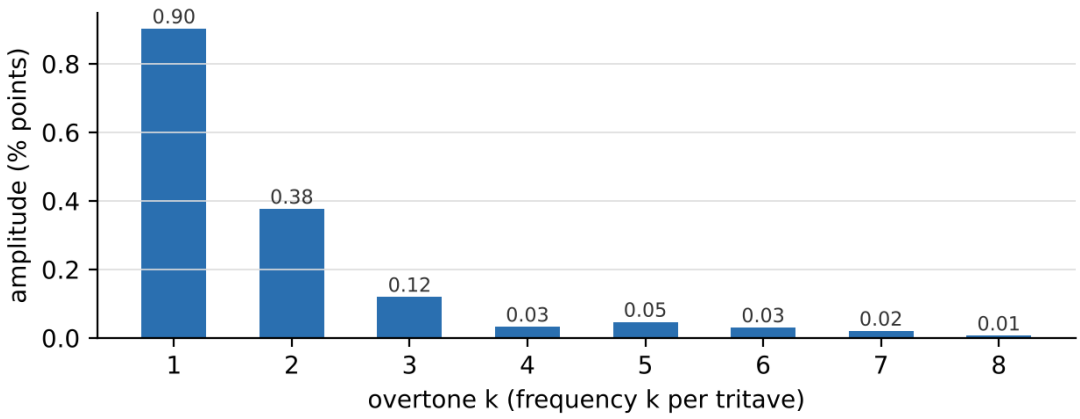


Figure 2. Amplitudes of the first eight overtones.

The mean, 30.87 %, also has a meaning of its own. Weighted by $1/|z|^2$, the finite points have a logarithmic density. We computed it directly: 0.3089 turning. So a mean exists, even though a plain limit does not.

Lapidus and van Frankenhuijsen built a theory for exactly this kind of oscillation.[11] In their language, a fractal has complex dimensions. The imaginary parts set the frequencies of its oscillations. They speak of hearing the shape of a fractal.[12] Here the complex dimensions lie at $1 + 2\pi ik/\ln 3$ in the norm, for k = 1, 2, 3 and so on. Table 3 gives their strengths. In plain words: the trit tile has a fundamental and overtones.

7. Music I: the two ladders and the comma

Music is the oldest applied mathematics of ratios. Two strings sound well together when their lengths have a simple ratio. The octave is 2:1. The fifth is 3:2. These are the two ladders of the Vacuum.Net theory: doubling and tripling.

A circle of twelve fifths almost returns to the start. It overshoots seven octaves by the Pythagorean comma, about 23.5 cents. The theory proves that the miss never vanishes.[13] No power of 3 equals a power of 2. In the language of the net this is a closure that fails. The loop does not close. What does not close passes on as tension.

Musicians met this barrier for two thousand years. Their answers follow the same pattern as the barriers of mathematics. They stepped back and added a new object, and paid with a lost property.

Barrier in music	Resolution	Property lost	Link to the net
Twelve fifths do not return to the octave (the comma)	Temperament: spread the comma	Pure fifths	Closure fails (T16–T18)
Spreading the comma evenly needs 2^(1/12)	Equal temperament (Zhu Zaiyu 1584; Stevin)	Rational ratios	Closure depth 12
Twelve steps give poor thirds	Finer divisions: 41 and 53 steps (Jing Fang; Mercator; Holder)	Simplicity of the keyboard	Closure depths 41 and 53

[11] [9] Lapidus, M. L. and van Frankenhuijsen, M. (2013). Fractal Geometry, Complex Dimensions and Zeta Functions, 2nd ed. Springer.

[12] [10] Lapidus, M. L. (2015). The sound of fractal strings and the Riemann hypothesis. arXiv:1505.01548.

[13] [1] Konstapel, J. (2026). Vacuum.Net Theory, Ninth Edition: Foundations 9.2. Constable Research, Leiden.

Barrier in music	Resolution	Property lost	Link to the net
The octave itself as frame	Tritave 3:1 (Bohlen–Pierce, 1970s)	Octave equivalence	The ×3 ladder alone
Pitch as a fixed height	Circular pitch (Shepard 1964; Risset)	A top and a bottom	Log-periodicity
The note as the unit	The spectrum as material (Grisey, Murail, 1970s)	The note	Spectrum of the tile
Chords as lists	Chords as points of an orbifold (Tymoczko 2006)	Flat chord space	The node is an orbifold

Table 4. Barriers in the history of tuning and composition, read with the pattern of Reference 3. The last column is a reading.

The closure depths of the theory are the well-known good equal divisions. Twelve-tone equal temperament was computed by Zhu Zaiyu in 1584 and, independently, by Simon Stevin.[14] The 53-step division goes back to the Chinese theorist Jing Fang. Mercator computed its comma, and Holder showed in 1694 that it also gives good thirds.[15] The numbers 12, 41 and 53 appear in the net as closure depths, and in music as tunings. This is not new: it is the continued fraction of $\log_2 3$. It shows that the net and music share their first barrier.

8. Music II: the tritave

Most music repeats at the octave. A note and its double are heard as 'the same note, higher'. The Bohlen–Pierce scale breaks this habit. It repeats at 3:1. This interval is called the tritave. It equals an octave plus a fifth. The scale divides it into 13 equal steps, each a ratio 3^(1/13).[16] It was described independently by Heinz Bohlen, Kees van Prooijen and John Pierce. Instruments now exist for it, including a family of clarinets.

The trit tile repeats at 3:1 in the norm. One extra trit digit multiplies the norm by 3. So the natural period of the tile is a tritave, not an octave. This is a precise fact, not an image: Table 1 shows it. The ×3 ladder of the net is the backbone of the tritave, just as the ×2 ladder is the backbone of the octave.

A reading follows. The octave belongs to the binary ladder. The tritave belongs to the trit ladder. A music built on trits would be a tritave music. The Bohlen–Pierce scale is such a music. It was found by ear and by acoustics, not by number theory. The tile gives it a second, arithmetical foundation.

9. Music III: endless tones

In 1964 Roger Shepard built tones from partials one octave apart. He kept the overall loudness curve fixed and shifted the partials upward.[17] Listeners heard a scale that rises forever but never gets higher. Jean-Claude Risset made a continuous glide with the same effect. The trick works because the sound is the same after one octave of shift. Mathematically, it is a function that repeats after a fixed ratio: a log-periodic function.

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[14][15] Rehding, A. Fine-tuning a global history of music theory: divergences, Zhu Zaiyu, and music-theoretical instruments. Music Theory Spectrum (article). Liu, Y. (2013). Equal temperament circa 1600 in China and Europe. Canadian Mathematical Society, Winter Meeting.

[15][16] Barbour, J. M. (1951). Tuning and Temperament: A Historical Survey. Michigan State College Press.

[16][11] Mathews, M. V., Pierce, J. R., Reeves, A. and Roberts, L. A. (1988). Theoretical and experimental explorations of the Bohlen–Pierce scale. Journal of the Acoustical Society of America 84, 1214–1222. Müller, N.-L. (2019). The Bohlen-Pierce clarinet: exploring a new tonality. The Clarinet 46/3.

[17][12] Shepard, R. N. (1964). Circularity in judgments of relative pitch. Journal of the Acoustical Society of America 36, 2346–2353. Deutsch, D. (2010). The paradox of pitch circularity. Acoustics Today 6(3), 8–14.

The share of turning points is exactly such a function. It keeps changing as the disc grows. After one tritave it is back where it started. It never settles and never runs away. The tile behaves like a Shepard–Risset tone, built in the tritave instead of the octave.

There is a difference. Shepard designed his envelope by hand. The envelope of the tile is not designed. It follows from the arithmetic of $1 - \omega$ and the digits −1, 0, +1. Figure 1 is that envelope. Its asymmetric shape, a slow fall and a fast rise, comes from the boundary of the tile near 0.

10. Music IV: an arithmetical timbre

In the 1970s Gérard Grisey and Tristan Murail made the spectrum of a sound into the material of composition. Grisey's Partiels of 1975 starts from the measured spectrum of a low trombone note and lets an ensemble rebuild it.[18] This movement is called spectral music. Its material comes from acoustics: instruments, bells, voices.

Table 3 is a spectrum that comes from arithmetic. Its overtones fall roughly as 1 : 0.42 : 0.13 : 0.04 : 0.05. A tone with that spectrum is soft and round, with a clear second harmonic. We made it audible in three short sound files, version 1.0:[19]

- Tile timbre. One period of Figure 1 is used as the waveform of a tone at 220 Hz. What you hear is the trit tile itself.
- Bohlen–Pierce ladder. The thirteen steps of one tritave, played with the tile timbre.
- Risset glide in the tritave. Pure partials spaced 3:1 under a fixed envelope, shifted upward. It rises without end and returns after each tritave.

This offers spectral music a new source. Any number system that fails to terminate produces a log-periodic share like ours. Each has its own spectrum. A composer could take these spectra as material, as Grisey took the trombone. The prediction is that different bases give different timbres. That can be computed and heard.

11. Music V: orbifolds

Dmitri Tymoczko showed that chords live in curved spaces called orbifolds.[20] A chord is a point. Moving from chord to chord is a path. The space folds onto itself because octaves and the order of notes do not matter. For two-note chords it is a Möbius strip whose edges act as mirrors. His paper was the first on music theory in the journal Science.

The node of the Vacuum.Net theory, Rung 11, is also an orbifold. It is the modular group acting on the upper half-plane. It has two special points, of order 2 and order 3, where the space folds. These are the two turns of the node.[21] Tymoczko's chord spaces fold by permutation and reflection. The node folds by the turns of order 2 and 3.

The trit tile adds a third element. Tymoczko's spaces describe where chords are. Shepard's tones describe how pitch can circle. The tile describes how a quantity circles under scaling by a fixed ratio. A reading: the net contains all three. Its orbifold is the node. Its circle is the tritave. Its timbre is the spectrum of the tile.

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[18] [13] Cagney, L. (2023). Gérard Grisey and Spectral Music: Composition in the Information Age. Cambridge University Press. Grisey, G. (1975). Partiels, for 18 instruments.

[19] [18] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.8–2.2, Python scripts (controle/*.py) and audio files (1_tile_timbre, 2_bohlen_pierce, 3_risset_tritave, v1.0). Constable Research, Leiden (unpublished).

[20] [14] Tymoczko, D. (2006). The geometry of musical chords. Science 313, 72–74. Tymoczko, D. (2011). A Geometry of Music. Oxford University Press.

[21] [2] Konstapel, J. (2026). Mathematical Theory Generated with the Vacuum.Net Theory, version 7.0 (19 November 2026). Constable Research, Leiden.

12. What is new

We claim the following, and no more.

- New, as far as our literature search reaches: for the base 1 − ω with digits −1, 0, +1, the share of non-terminating points has no limit. It repeats with period 3 in the norm. This is proved, with a computer certificate. Its mean and overtones are measured.
- Known: that the system does not terminate everywhere (Akiyama–Scheicher); that the tile tiles the plane (Lagarias–Wang); the theory of complex dimensions (Lapidus); the tritave, Shepard tones, spectral music and chord orbifolds.
- New as a connection: the failure of a number system to terminate produces a log-periodic share whose period is the base ratio. In our case that ratio is the tritave. Its spectrum is an arithmetical timbre that can be played. We have not found this connection described before.

Why this may matter. Music theory studied ratios, circles of pitch and spectra separately. Number theory studied which expansions terminate. Here the three music themes appear inside one number-theoretic object. That object is the plane of Rung 10, the place where the trits become complex numbers. If the companion essay is right that the next mathematics answers with rhythms, then this is one of its first examples. It is also, literally, a sound.

13. Status and open points

- Proved (computer-assisted): the share has no limit (Section 5). Relies on the Lagarias–Wang tiling theorem and on exact counts. Independent verification welcome.
- Measured, not proved: the exact range 29.6–32.0 %, the mean 30.87 %, the overtone amplitudes.
- Open: a closed formula for the mean and the overtones; the same analysis for other bases; a formal proof in Agda or Lean.
- Reading, not proof: every link to the Vacuum.Net theory and to music in Sections 7–11 beyond the stated facts.

14. Conclusion

A small question from the Vacuum.Net theory has a precise answer. The share of turning points does not settle. It circles, with a fixed period of one trit digit, which is a tritave. The circle has a mean, a shape and overtones. It is proved that the circle is real. Music has known each of these ingredients for decades: the tritave, the endless tone, the spectrum as material, the folded space of chords. Here they appear together inside one number system. The trit tile has a sound. It can be computed, proved and played.

Annotated references

[1] Konstapel, J. (2026). Vacuum.Net Theory, Ninth Edition: Foundations 9.2. Constable Research, Leiden.

Source of the axioms N1–N5 and of theorems T16–T18 on the two ladders ×2 and ×3, the closure depths and the Pythagorean comma. Section 2 and Section 8 rest on it.

[2] Konstapel, J. (2026). Mathematical Theory Generated with the Vacuum.Net Theory, version 7.0 (19 November 2026). Constable Research, Leiden.

Rung 10 introduces the Eisenstein plane with base 1 − ω. Open point IX.3 asks whether the share of turning elements settles. This essay answers that question.

[3] Konstapel, J. (2026). Barriers and Back-Steps, version 1.0 (21 November 2026), and The Next Mathematics, constable.blog (21 November 2026).

The companion essays. They argue that the next barrier of mathematics is logical and that 'rhythms are answers'. The present essay supplies a proved example.

[4] Akiyama, S. and Scheicher, K. Symmetric shift radix systems and finite expansions. Preprint, University of Tsukuba (www.math.tsukuba.ac.jp/~akiyama/papers/centralsrs6.pdf).

Defines symmetric canonical number systems with digits centred on 0. Theorem 2.2 decides, for quadratic bases, when every integer has a finite expansion. It shows that our base $x^2 - 3x + 3$ fails. It gives no density.

[5] Akiyama, S., Borbély, T., Brunotte, H., Pethő, A. and Thuswaldner, J. M. (2005). Generalized radix representations and dynamical systems I. Acta Mathematica Hungarica 108, 207–238.

Introduces shift radix systems, the common frame for canonical number systems and beta-expansions. Good entry into the literature on finite expansions.

[6] Kirschenhofer, P., Pethő, A., Surer, P. and Thuswaldner, J. (2010). Finite and periodic orbits of shift radix systems. Journal de Théorie des Nombres de Bordeaux 22, 421–448.

Studies which orbits end in 0 and which become periodic. It treats sets of parameters, not the proportion of starting points.

[7] Lagarias, J. C. and Wang, Y. (1997). Integral self-affine tiles in R^n, part II: lattice tilings. Journal of Fourier Analysis and Applications 3, 83–102.

Proves that every standard digit set gives a self-affine tile that tiles the plane by a lattice. This is the external theorem the proof in Section 5 relies on.

[8] Steiner, W. and Thuswaldner, J. M. (2012). Rational self-affine tiles. arXiv:1203.0758.

Its introduction states the Lagarias–Wang result in the form used here: for an irreducible characteristic polynomial and a complete residue system of digits, the tile gives a lattice tiling with respect to the smallest invariant lattice containing the digit differences.

[9] Lapidus, M. L. and van Frankenhuijsen, M. (2013). Fractal Geometry, Complex Dimensions and Zeta Functions, 2nd ed. Springer.

The theory of complex dimensions. Oscillations in the geometry of a fractal and in its spectrum are described by the same complex numbers. The language of Section 6.

[10] Lapidus, M. L. (2015). The sound of fractal strings and the Riemann hypothesis. arXiv:1505.01548.

An accessible survey. It explains how one can 'hear' a fractal through its complex dimensions and links this to the Riemann hypothesis.

[11] Mathews, M. V., Pierce, J. R., Reeves, A. and Roberts, L. A. (1988). Theoretical and experimental explorations of the Bohlen–Pierce scale. Journal of the Acoustical Society of America 84, 1214–1222. Müller, N.-L. (2019). The Bohlen-Pierce clarinet: exploring a new tonality. The Clarinet 46/3.

The scale that repeats at 3:1 (the tritave) instead of 2:1 (the octave), divided into 13 steps. Müller describes the instruments built for it.

[12] Shepard, R. N. (1964). Circularity in judgments of relative pitch. Journal of the Acoustical Society of America 36, 2346–2353. Deutsch, D. (2010). The paradox of pitch circularity. Acoustics Today 6(3), 8–14.

Shepard's tones are built from partials one octave apart under a fixed spectral envelope. Shifting them in log frequency gives an endless rise. Risset made a continuous version. Deutsch reviews the phenomenon.

[13] Cagney, L. (2023). Gérard Grisey and Spectral Music: Composition in the Information Age. Cambridge University Press. Grisey, G. (1975). Partiels, for 18 instruments.

Spectral music takes the measured spectrum of a sound as compositional material. Partiels starts from the spectrum of a trombone note. Section 10 proposes an arithmetical spectrum as such material.

[14] Tymoczko, D. (2006). The geometry of musical chords. Science 313, 72–74. Tymoczko, D. (2011). A Geometry of Music. Oxford University Press.

Chords are points in orbifolds: spaces folded by octave, permutation and inversion. The first music-theory article in Science. Basis of the comparison in Section 11.

[15] Rehding, A. Fine-tuning a global history of music theory: divergences, Zhu Zaiyu, and music-theoretical instruments. Music Theory Spectrum (article). Liu, Y. (2013). Equal temperament circa 1600 in China and Europe. Canadian Mathematical Society, Winter Meeting.

Twelve-tone equal temperament was computed by Zhu Zaiyu in 1584 and, independently, by Simon Stevin shortly after. Both sources discuss the mathematics and the priority question.

[16] Barbour, J. M. (1951). Tuning and Temperament: A Historical Survey. Michigan State College Press.

Standard history of tuning. Covers Pythagorean tuning, the comma, meantone, equal temperament and the 53-step division known since Jing Fang and computed by Mercator and Holder.

[17] Burklund, R., Hahn, J., Levy, I. and Schlank, T. M. (2023). K-theoretic counterexamples to Ravenel's telescope conjecture. arXiv:2310.17459.

The latest barrier in the chain of Reference 3. Cited for context only.

[18] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.8–2.2, Python scripts (controle/*.py) and audio files (1_tile_timbre, 2_bohlen_pierce, 3_risset_tritave, v1.0). Constable Research, Leiden (unpublished).

Contains every computation in Sections 4–6 and the certificate of Section 5, with code that reproduces them, and the three sound files of Section 10.