The Trit Tile

A reference work on a number system that never settles: mathematics, literature, structure, music, applications, and its place in the Vacuum.Net theory

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Version 1.1 · 21 November 2026

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How to use this reference

This document is meant to be consulted, not only read once. It collects in one place everything known so far about one small object: a number system on the triangular lattice whose share of non-terminating points never settles. Later projects can cite its sections directly.

The document has six parts. Part I explains the Vacuum.Net theory and the notation, so that no earlier text is needed. Part II gives the mathematics: the system, the measurements, the theorem with its proof, the spectrum, and how all this sits in the literature. Part III places the result in category theory, logic and the higher windings of the theory. Part IV treats music at length. Part V lists applications, including a concrete procedure for detecting cycles in data. Part VI states exactly what is proved, what is measured, what is known and what is only a reading.

Every claim carries one of four labels where it matters. **Proved** means a proof is given or cited. **Measured** means a computation that can be repeated with the scripts.[1] **Known** means a published result of others. **Reading** means an interpretation in terms of the Vacuum.Net theory or of music, which may guide work but is not a theorem.

Abstract

Take a point of the triangular lattice a + bω, with ω a cube root of one. Subtract a digit −1, 0 or +1 so that the result is divisible by 1 − ω. Divide. Repeat. Some points reach 0. All others end in one of two points that turn forever. The Vacuum.Net theory asked whether the share of turning points settles. It does not. We prove that the share in discs has no limit. Its lower limit is at most 30.2 % and its upper limit at least 31.5 %. The share repeats exactly whenever the norm grows by a factor 3. We measure the full periodic shape, its mean of 30.87 % and its overtones.

The result belongs to a known family: the periodic fluctuations of digit counts found by Delange and studied for complex bases by Grabner, Kirschenhofer, Prodinger and Thuswaldner. It has the same structure as Benford's law: no natural density, but a logarithmic one. The theory of complex dimensions of Lapidus and coauthors explains why such an oscillation must occur. Category theory supplies a frame: finite expansions form an initial algebra and infinite ones a final coalgebra; the theorem measures the gap between them. In music the period 3:1 is the tritave of the Bohlen–Pierce scale, the oscillation has the form of a Shepard–Risset tone, and its overtones give a playable timbre. In physics the same structure is called discrete scale invariance. Throughout, the document shows how each element follows from the Vacuum.Net theory: the trits, the ladder ×3, closure, and the plane of Rung 10.

Part I — Foundations

1. The Vacuum.Net theory

1.1 One strand and five axioms

The Vacuum.Net theory starts from movement, not from sets or numbers.[2] There is exactly one strand. It moves and returns. Every distinction is a loop of this strand. Five axioms fix the rules.

- N1. There is exactly one strand.
- N2, closure. A loop persists only if it closes. What does not close passes on as tension.

[1] [5] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.5–2.4; Python scripts in controle/ (onafh.py, vergelijk.py, bewijs.py, klank.py, stap2_tegel.py, zeshoek.py); audio files v1.0. Constable Research, Leiden (unpublished).

[2] [1] Konstapel, J. (2026). Vacuum.Net Theory, Ninth Edition: Foundations 9.2 and Physics 9.0. Constable Research, Leiden.

- N3. The first passage through a point fixes a direction.
- N4. The response of the medium depends on its state.
- N5. Coarse blocks of the strand obey the same rules again.

Closure is the central idea. It separates what stays from what moves on. This document is, at bottom, a study of closure in one concrete system: which points close (reach 0) and which keep moving (turn forever).

1.2 Trits and their two readings

Each step of the strand is a trit. It is R (forward, +1), O (standing still, 0) or L (backward, −1). An infinite stream of trits can be read in two ways. As a balanced ternary address it names a place. As a signed calculation it gives a value. Two different streams can name the same place. Deciding whether they do may need a principle called LLPO: of two numbers, at least one is zero, even when one cannot tell which.[3] Constructive mathematics does not grant LLPO for free. So the theory has an edge built in: the point where calculation and address part ways.[4]

1.3 Two ladders, closure depths and the comma

Trits combine into ladders. Two ladders are basic: doubling (×2) and tripling (×3). No power of 2 equals a power of 3. The theory proves this and studies how close they come.[5] The best near meetings occur at the closure depths 1, 2, 5, 12, 41, 53, 306, 665 and so on. These are the denominators of the continued fraction of $\log_2 3$. At depth 12 the miss is the Pythagorean comma, $3^{12}/2^{19} \approx 1.0136$. These numbers return in Part IV, because they are also the classical tunings of music.

1.4 Rung 10 and Rung 11

A book derives mathematics from these rules in 22 rungs.[6] Two rungs matter here. Rung 10 places the trits in the plane. It uses the Eisenstein integers $a + b\omega$, with $\omega = (-1 + i\sqrt{3})/2$. They form a triangular lattice. The base is $\beta = 1 - \omega$. Multiplying by β stretches distances by $\sqrt{3}$ and turns the plane by −30 degrees. The identity $3 = -\omega^2(1 - \omega)^2$ shows that 3 is, up to a unit, the square of β. In number theory one says that 3 ramifies. The book left open whether the share of points that keep turning under this system settles. The open point is called IX.3.

Rung 11 is the node. Its group is the modular group, built from a turn of order 2 and a turn of order 3. Over the three trits it becomes a group with 24 elements, the symmetry group of the 24-cell. Section 12 returns to it.

1.5 Windings and barriers

The theory sorts loops by how they wind, from the plain circle (W0) to the irregular windings of spheres (W5).[7] A companion essay reads the history of mathematics as a chain of barriers.[8] A barrier is an operation whose inverse fails inside the current domain. Mathematics steps back to the point of failure, adds a new object and moves on, paying with a lost property. The essay predicts that some of the next

[3] [32] Bridges, D. and Richman, F. (1987). Varieties of Constructive Mathematics. Cambridge University Press.
[4] [2] Konstapel, J. (2026). Mathematical Theory Generated with the Vacuum.Net Theory, version 7.0 (reference edition, 19 November 2026). Constable Research, Leiden.
[5] [1] Konstapel, J. (2026). Vacuum.Net Theory, Ninth Edition: Foundations 9.2 and Physics 9.0. Constable Research, Leiden.
[6] [2] Konstapel, J. (2026). Mathematical Theory Generated with the Vacuum.Net Theory, version 7.0 (reference edition, 19 November 2026). Constable Research, Leiden.
[7] [3] Konstapel, J. (2026). Barriers and Back-Steps, version 1.0, and The Next Mathematics (constable.blog), 21 November 2026.
[8] [3] Konstapel, J. (2026). Barriers and Back-Steps, version 1.0, and The Next Mathematics (constable.blog), 21 November 2026.

answers will be rhythms rather than numbers. Part II gives a proved example. Here the operation is division by β. Its inverse, writing every point as a finite trit expansion, fails for about 31 % of points. The repair is to replace the missing limit by a rhythm.

2. Notation and glossary

Symbol or term	Meaning
ω	$(-1 + i\sqrt{3})/2$, a primitive cube root of one; $\omega^2 + \omega + 1 = 0$
Z[ω]	Eisenstein integers a + bω: the triangular lattice
N(a + bω)	Norm $a^2 - ab + b^2$ = squared distance from 0
β = 1 − ω	The base; N(β) = 3, |β| = √3
Digit d	The value −1, 0 or +1 with z ≡ d modulo β (d ≡ a + b mod 3)
T(z)	(z − d)/β, one step of the system
Finite point	A point whose orbit reaches 0; it has a finite expansion $\Sigma\, d_i\beta^i$
Turning point	A point whose orbit ends in +(1 + ω) or −(1 + ω)
U	The set of finite points
F	The tile: all sums $\Sigma_{i\geq 1}\, d_i\beta^{-i}$
W	Union of the tiles u + F over u in U
h(N)	Share of turning points among lattice points of norm ≤ N
G(R)	Fraction of a disc of radius R covered by W
Tritave	The ratio 3:1; one period of h in the norm
Log-periodic	Periodic as a function of the logarithm of the scale
Discrete scale invariance	Invariance under one fixed magnification factor only
Initial algebra	Least fixed point of a construction; built by induction
Final coalgebra	Greatest fixed point; unfolded by coinduction
LLPO	Lesser limited principle of omniscience (Section 1.2)

Table 1. Notation used throughout.

Part II — The mathematics

3. The number system

3.1 Definition and example

Every Eisenstein integer z = a + bω leaves a remainder modulo β. Because ω ≡ 1 modulo β, that remainder is a + b modulo 3. Choose it in {−1, 0, +1} and call it the digit d. The step is T(z) = (z − d)/β. In coordinates, if z − d = x + yω, then T(z) = ((2x − y) + (x + y)ω)/3.

An example. The number 2 has digit −1, because 2 ≡ −1 modulo 3. So T(2) = 3/β = 2 + ω. Next, 2 + ω has digit 0, and T(2 + ω) = 1 + ω. Then 1 + ω has digit −1, and T(1 + ω) = 1 + ω again. The orbit is caught. It turns forever.

3.2 Three endpoints

Each step shrinks the distance to 0 by a factor of about √3. Every orbit therefore reaches the small region around 0 and stays there. Only three points there are fixed: 0 itself and the two points ±(1 + ω). These are the units ∓ω². The other four units reach 0 in at most two steps. So every orbit ends in exactly one of three places. **Proved** (elementary; checked by computation[9]). The map z ↦ −z exchanges the two turning endpoints. So the turning points split exactly half and half.

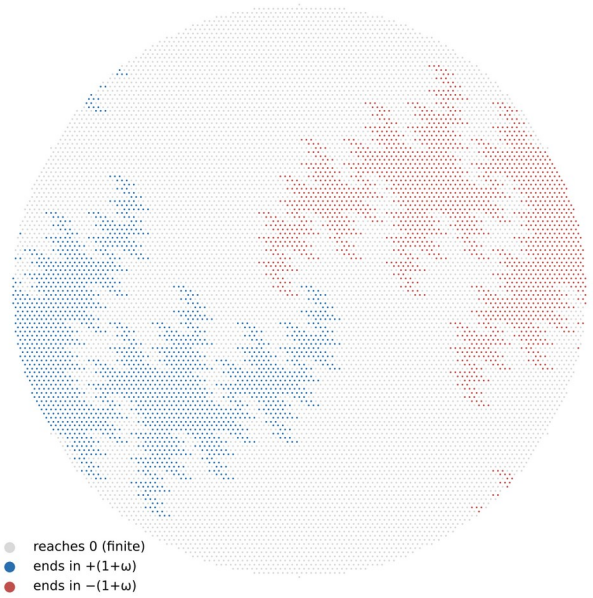

Figure 1. Lattice points of norm up to about 3700, coloured by the end of their orbit. Grey points reach 0. Blue and red points turn forever, in +(1 + ω) and −(1 + ω). The coloured regions form two opposite fans with ragged, self-similar edges.

3.3 Finite expansions

A point reaches 0 exactly when it has a finite expansion z = $d_0 + d_1\beta + \ldots + d_k\beta\text{^}k$ with digits in {−1, 0, +1}. The expansion is unique. The finite points form the set U. Two simple facts drive everything that follows. If u is in U, then βu + d is in U for every digit d, because T maps it back to u. And every nonzero u in U arises in this way from T(u). So U is built from 0 by repeating 'multiply by β, add a trit'.

4. Place in the literature of number systems

4.1 Canonical and symmetric number systems

A number system in which every integer has a finite expansion is called canonical. The study of such systems in rings of algebraic integers goes back to Knuth, Kátai, Kovács and Pethő. Akiyama and Scheicher studied the symmetric case, where the digits are centred on 0, as here.[10] Their Theorem 2.2 decides finiteness for quadratic bases. For a base with minimal polynomial $x^2 + Ax + B$, every integer is finite exactly when $|A| < 1 + B/2$ and $|B| \geq 2$, or $A = 1 + B/2$ and $|B| > 2$.

Our base satisfies $\beta^2 - 3\beta + 3 = 0$. So A = −3 and B = 3. The first condition fails because 3 is not less than 2.5. The second fails because −3 ≠ 2.5. **Known**: the system is not canonical. Turning points must exist. The book could have cited this for IX.3.

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[9] [5] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.5–2.4; Python scripts in controle/ (onafh.py, vergelijk.py, bewijs.py, klank.py, stap2_tegel.py, zeshoek.py); audio files v1.0. Constable Research, Leiden (unpublished).

[10] [6] Akiyama, S. and Scheicher, K. Symmetric shift radix systems and finite expansions. Preprint, University of Tsukuba (www.math.tsukuba.ac.jp/~akiyama/papers/centralsrs6.pdf).

4.2 Shift radix systems

Shift radix systems give a common frame for number systems and their orbits.[11] Work in this frame studies which parameters give finite orbits and which give cycles.[12] It describes sets of parameters. We found no study of the share of starting points that end in 0, and no report that this share fails to settle.

4.3 The role of the origin

For tilings generated by number systems, a key question is whether the origin is an inner point of the central tile. If it is, every point has a finite expansion and the tiles behave well.[13] In our system the origin lies on the boundary of the tile (Figure 2). This is the geometric reason for the turning points, and, as Section 7 shows, for the oscillation.

5. The tile

Let F be the set of all infinite sums $d_1\beta^{-1} + d_2\beta^{-2} + \ldots$ with digits −1, 0, +1. It is compact and has a fractal boundary. Fractal tiles of this kind have been studied since the work of Gilbert on complex bases.[14] The tile obeys the set equation $\beta F = (F - 1) \cup F \cup (F + 1)$.

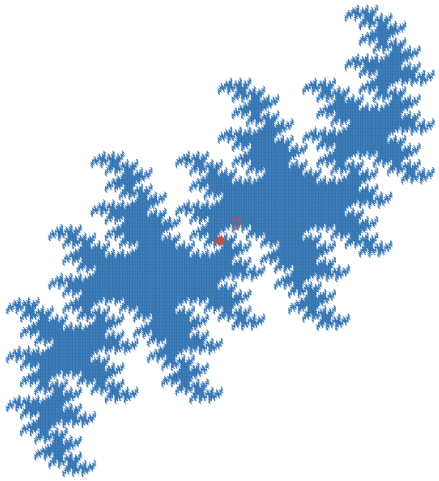

Figure 2. The tile F, drawn from 3^{11} points. The red dot is the origin. It sits in a notch of the boundary, not inside.

Lagarias and Wang proved that tiles from a complete set of digit residues tile the plane by a lattice.[15] When the characteristic polynomial is irreducible, the lattice is the smallest invariant lattice containing the digit differences.[16] Here the differences include 1, and the smallest β-invariant lattice containing 1 is Z[ω]. So the translates u + F, for u in Z[ω], cover the plane and overlap only on sets of zero area. Each has the area of one lattice cell, √3/2. The boundary has zero area.[17] **Known.**

[11][7] Akiyama, S., Borbély, T., Brunotte, H., Pethő, A. and Thuswaldner, J. M. (2005). Generalized radix representations and dynamical systems I. Acta Mathematica Hungarica 108, 207–238.
[12][8] Kirschenhofer, P., Pethő, A., Surer, P. and Thuswaldner, J. (2010). Finite and periodic orbits of shift radix systems. Journal de Théorie des Nombres de Bordeaux 22, 421–448.
[13][9] Akiyama, S. (2002). On the boundary of self affine tilings generated by Pisot numbers. Journal of the Mathematical Society of Japan 54(2).
[14][10] Gilbert, W. J. (1982). Fractal geometry derived from complex bases. Mathematical Intelligencer 4, 78–86.
[15][11] Lagarias, J. C. and Wang, Y. (1997). Integral self-affine tiles in R^n, part II: lattice tilings. Journal of Fourier Analysis and Applications 3, 83–102.
[16][12] Steiner, W. and Thuswaldner, J. M. (2012). Rational self-affine tiles. arXiv:1203.0758.
[17][13] Lau, K.-S. and Xu, Y. (2000). On the boundary of attractors with non-void interior. Proceedings of the American Mathematical Society 128, 1761–1768.

Every point of F lies within ρ = 1/(√3 − 1) ≈ 1.366 of 0, because $|\beta^{-i}| = 3^{(-i/2)}$ and the digits have size at most 1.

6. Measurements

6.1 Two independent programs

We counted with two programs written independently.[18] The first starts from each lattice point and applies T until the orbit ends. The second works forwards. It lists all finite expansions with up to 16 digits: 3^{16} = 43 046 721 points. All are distinct. It then counts how many lie in each disc. Both use exact integer arithmetic. The lists with 15 and 16 digits give identical counts up to norm 1.1 million, so the counts are complete there.

6.2 Results

Norm N	Share turning h(N)	Note
8 100	0.3077	low phase
14 029	0.3102	high phase
72 900	0.3076	N × 9
126 266	0.3102	N × 9
656 100	0.3076	N × 9
1 136 398	0.3102	N × 9

Table 2. Share of turning points in discs of norm N. The two methods agree to four decimals. Multiplying N by 3 returns the same value; the table shows steps of 9 for compactness.

The share does not drift towards a limit. On a fine grid it ranges from about 29.6 % to 32.0 %. Each factor 3 in the norm brings it back. Hexagons instead of discs give other values (30.5 % to 32.1 %) but the same kind of swing. **Measured.**

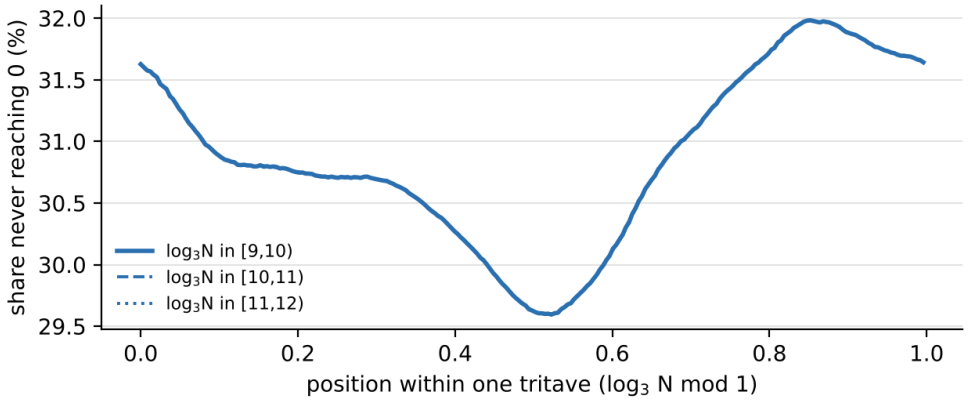

Figure 3. The share of turning points over one period, sampled at 243 points in three consecutive periods. The curves coincide. The shape falls slowly and rises fast.

7. The theorem and its proof

Theorem. For the base 1 − ω with digits −1, 0, +1, the share h of Eisenstein integers of norm at most N that never reach 0 has no limit as N grows. Its lower limit is at most 0.302 and its upper limit at least 0.315.

[18] [5] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.5–2.4; Python scripts in controle/ (onafh.py, vergelijk.py, bewijs.py, klank.py, stap2_tegel.py, zeshoek.py); audio files v1.0. Constable Research, Leiden (unpublished).

7.1 Self-similarity of W

Let W be the union of the tiles u + F over all finite points u. From Section 3.3, the map (u, d) ↦ βu + d sends U × {−1, 0, 1} onto U. Combined with the set equation of F this gives βW = W. So W looks the same after stretching by √3 and turning by −30 degrees. Let G(R) be the fraction of the disc of radius R that W covers. Discs are unchanged by turning. Hence G(√3·R) = G(R) exactly.

7.2 Counting equals area

Let A(r) be the number of finite points in the disc of radius r. The tiles u + F do not overlap (Section 5). Each tile lies within ρ of its point u. So the area of W inside the disc of radius R is at least (√3/2)·A(R − ρ) and at most (√3/2)·A(R + ρ). The number of lattice points in a disc is its area divided by √3/2, up to an error of order R. Therefore the share of finite points equals G(R) up to an error of order 1/R.

7.3 The certificate

It remains to show that G is not constant. We bound it at two radii by exact counts. First we check that the counts are complete. One step of T reduces the distance from |z| to at most (|z| + 1)/√3. After j steps a point that started within R lies within R·3^(−j/2) + ρ of 0. For the radii used, j = 13 steps bring it within √3. There only 0 and the six units remain, and these reach their end in at most two further steps. So every finite point in range has at most 15 digits, and the list of 16-digit expansions contains them all.

Radius R	Lower bound of G(R)	Upper bound of G(R)
389.73 (high phase of turning)	0.67540	0.68513
321.57 (low phase of turning)	0.69797	0.70998

Table 3. Certified bounds on the covered fraction G. The intervals are separated by 0.0128. Two further pairs of radii give margins 0.0130 and 0.0116.

The upper bound at the first radius lies below the lower bound at the second. So G is not constant. Since G repeats under scaling by √3, the share of finite points comes within O(1/R) of values at most 0.6851 and at least 0.6980 infinitely often. This proves the theorem.

Status. Computer-assisted proof. It relies on the Lagarias–Wang tiling theorem and on exact integer counts. The certificate is reproduced by the script bewijs.py.[19] An independent check by a second party is recommended before formal publication.

7.4 What the proof shows

The proof shows more than non-convergence. The share of finite points in a disc equals the area density of the tile F in small discs around its boundary point 0. **Proved** (Sections 7.1–7.2). The oscillation of IX.3 therefore says exactly this: the tile has no Lebesgue density at the point 0. The density circles with every factor √3 of zoom.

8. The spectrum

8.1 Fourier decomposition

Because the oscillation repeats exactly, it can be split into a mean and overtones. We sampled one period at 243 points in three consecutive periods and took the Fourier transform.

Component	Amplitude	Phase
Mean	30.87 %	—
Overtone 1	0.90 %	23.9°

[19][5] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.5–2.4; Python scripts in controle/ (onafh.py, vergelijk.py, bewijs.py, klank.py, stap2_tegel.py, zeshoek.py); audio files v1.0. Constable Research, Leiden (unpublished).

Component	Amplitude	Phase
Overtone 2	0.38 %	131.0°
Overtone 3	0.12 %	−4.7°
Overtone 4	0.03 %	148.6°
Overtone 5	0.05 %	−46.8°
Overtones 6–8	< 0.03 % each	—

Table 4. Spectrum of the oscillation. The three periods agree to 0.00001 in amplitude and within 4 degrees in phase. Measured.

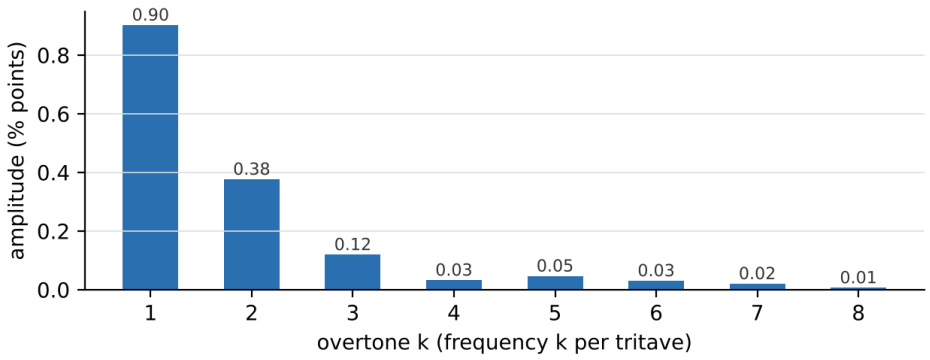

Figure 4. Amplitudes of the first eight overtones.

8.2 The mean as a logarithmic density

A plain limit does not exist, but an average does. Weight each finite point u by $1/|u|^2$ and count in annuli. This logarithmic density of the finite points is 0.6911, so the turning share has logarithmic density 0.3089. The average of Table 4 over one period gives 0.3087. **Measured.** This is the same situation as in Benford's law (Section 9.2).

9. Place in analysis and number theory

9.1 Delange-type fluctuations

In 1975 Delange showed that the average sum of digits of the integers below N equals a smooth main term plus a periodic function of log N.[20] The periodic function is continuous and nowhere differentiable. Many digit counts behave this way. Flajolet, Grabner, Kirschenhofer, Prodinger and Tichy showed how the Mellin–Perron formula produces the Fourier series of such fluctuations.[21] For complex bases, Grabner, Kirschenhofer and Prodinger studied digit sums in large circles.[22] Gittenberger and Thuswaldner found periodic fluctuations of period 1 in counts over Gaussian integers with $|z|^2 < N$.[23] Grabner and Liardet studied the harmonic properties of such functions.[24] Heuberger and Kropf gave general formulas for the Fourier coefficients of fluctuations produced by digit automata.[25]

[20] [14] Delange, H. (1975). Sur la fonction sommatoire de la fonction 'somme des chiffres'. L'Enseignement Mathématique 21, 31–47.
[21] [15] Flajolet, P., Grabner, P., Kirschenhofer, P., Prodinger, H. and Tichy, R. F. (1994). Mellin transforms and asymptotics: digital sums. Theoretical Computer Science 123, 291–314.
[22] [16] Grabner, P. J., Kirschenhofer, P. and Prodinger, H. (1998). The sum-of-digits function for complex bases. Journal of the London Mathematical Society 57, 20–40.
[23] [17] Gittenberger, B. and Thuswaldner, J. M. (1999). The moments of the sum-of-digits function in number fields. Canadian Mathematical Bulletin 42, 68–77.
[24] [18] Grabner, P. J. and Liardet, P. (1999). Harmonic properties of the sum-of-digits function for complex bases. Acta Arithmetica 91, 329–349.
[25] [19] Heuberger, C. and Kropf, S. (2015). Output sum of transducers: limiting distribution and periodic fluctuation. Electronic Journal of Combinatorics 22(2), #P2.19.

Known: oscillations of this type are the rule in digit counting, not the exception. Our theorem is a new member of the family. What is new is the set being counted, the finite expansions of a non-canonical system, and the period, the tritave. A specialist would classify the result here.

9.2 Benford's law

The set of positive integers with leading digit 1 has no natural density. Its share among 1 … N swings between 1/9 and 5/9. Its logarithmic density exists and equals $\log_{10} 2 \approx 0.301$.[26] The same holds, by recent work, for the leading digits of the Ramanujan τ-function at primes. Our theorem has exactly this form: no natural density, a logarithmic density of 0.3089, and a swing in between. The resemblance of 0.301 and 0.309 is a coincidence of numbers, not of structure.

9.3 Complex dimensions

Lapidus and van Frankenhuijsen developed a theory of complex dimensions.[27] The oscillations of a fractal, in its geometry and in its spectrum, are governed by complex numbers attached to it. Lapidus and Pearse extended this to self-similar tilings.[28] They distinguish a lattice case, where all scaling ratios are powers of one number, from a non-lattice case. In the lattice case the complex dimensions lie on vertical lines, evenly spaced, and each line produces a multiplicatively periodic function. Our tile has one ratio, $1/\sqrt{3}$. It is a lattice case. **Known theory, applied as a reading**: the oscillation of IX.3 is the lattice-case periodic function of the trit tile. Its complex dimensions lie at $1 + 2\pi i k/\ln 3$ in the norm ($2 + 2\pi i k/\ln \sqrt{3}$ in the radius). Table 4 gives their strengths. Lapidus speaks of hearing the shape of a fractal.[29]

9.4 Towards exact overtones

Table 4 is measured. The Mellin–Perron method of Section 9.1 may turn it into exact formulas. The plan: write the count of finite points as a Dirichlet series over the digit structure, locate its poles on the line of the complex dimensions, and read off the residues as Fourier coefficients. This is open work.

Part III — Structure

10. Category theory

10.1 Self-similarity as a universal property

Peter Freyd observed that the unit interval is the universal space made of two copies of itself glued end to end. In category theory it is the terminal coalgebra of the 'glue two copies' construction, with multiplication by 2 as its structure. Tom Leinster turned this into a general theory of self-similar spaces as universal solutions of systems of equations.[30] Moss and coauthors showed that many such fractals arise as the completion of an initial algebra.[31]

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[26][20] Hill, T. P. (1995). A statistical derivation of the significant-digit law. Statistical Science 10, 354–363. Jameson, M., Thorner, J. and Ye, L. (2016). Benford's law for coefficients of newforms. arXiv:1407.1577.
[27][21] Lapidus, M. L. and van Frankenhuijsen, M. (2013). Fractal Geometry, Complex Dimensions and Zeta Functions, 2nd ed. Springer.
[28][22] Lapidus, M. L. and Pearse, E. P. J. Tube formulas and complex dimensions of self-similar tilings. arXiv:math/0605527.
[29][23] Lapidus, M. L. (2015). The sound of fractal strings and the Riemann hypothesis. arXiv:1505.01548.
[30][27] Leinster, T. (2011). A general theory of self-similarity. Advances in Mathematics 226, 2935–3017.
[31][28] Bhattacharya, P., Moss, L. S., Ratnayake, J. and Rose, R. (2014). Fractal sets as final coalgebras obtained by completing an initial algebra. LNCS 8464, 146–167.

10.2 The trit tile in this frame

Reading, simple to check. The finite points U are built from 0 by 'multiply by β, add a trit'. They form the initial algebra of this construction: everything that can be reached by finitely many steps, proved by induction. The tile F, with all infinite trit streams, is the final coalgebra: everything that can be unfolded digit by digit forever, handled by coinduction.

In this frame the theorem measures the gap between the two. The share of lattice points reached by the initial algebra does not settle. The turning points are the lattice points that exist only coinductively: their unfolding never stops. The gap between least and greatest fixed point is where the rhythm lives.

10.3 Relation to the theory

Reading. Closure (N2) is the initial algebra: what closes, stays. The tension that passes on is the coalgebra. The edge of the trits, where two streams may name the same place, is the comparison of coalgebraic objects. Category theory is therefore not an outside tool here. It names the move the theory makes at every rung: from what closes to what continues.

11. Logic and computation

11.1 Signed digits

In exact real arithmetic, real numbers are coded as infinite streams of signed digits −1, 0, +1.[3233] Berger extracts programs from coinductive proofs about such streams. Schwichtenberg and Wiesnet treat signed digits and Gray code with inductive and coinductive definitions. These are the trits of the net, in their computational form.

11.2 The productivity problem and the LLPO edge

The same literature names a basic difficulty. When two streams are added, the first digit of the result may not be decidable from any finite part of the inputs. The cure is delay digits. **Reading**: this is the LLPO edge of the theory. Deciding the first digit of a sum is deciding whether a quantity is at most zero or at least zero without knowing which. In 2024 it was shown that homotopy type theory with four added axioms proves LLPO.[34] So the edge can be crossed by an axiom, not by a computation. The non-settling share is a numerical shadow of this edge: no finite stage fixes the answer, and what remains is a rhythm.

12. Relation to the node and the higher windings

The ramified prime $\beta = 1 - \omega$ connects this document to the rest of the windings research. Over the trits, the node of Rung 11 becomes a group of order 24, the symmetry group of the 24-cell.[35] At the prime 3 the irregular windings of stable homotopy theory (W5) are organised by a different group of order 24, built inside a quaternion algebra in which 3 has a square root. Inside that algebra the order-3 element is the ω of Rung 10, and $3 = -\omega^2(1 - \omega)^2$ holds literally.[36]

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[32][29] Berger, U. (2011). From coinductive proofs to exact real arithmetic: theory and applications. Logical Methods in Computer Science 7(1:8).
[33][30] Schwichtenberg, H. and Wiesnet, F. (2021). Logic for exact real arithmetic. Logical Methods in Computer Science (arXiv:1904.12763).
[34][31] Cherubini, F., Coquand, T., Geerligs, F. and Moeneclaey, H. (2024). A Foundation for Synthetic Stone Duality. arXiv:2412.03203.
[35][42] Coxeter, H. S. M. (1973). Regular Polytopes, 3rd ed. Dover.
[36][5] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.5–2.4; Python scripts in controle/ (onafh.py, vergelijk.py, bewijs.py, klank.py, stap2_tegel.py, zeshoek.py); audio files v1.0. Constable Research, Leiden (unpublished).

The most recent barrier in that field is the failure of Ravenel's telescope conjecture.[37] Ravenel's 2026 work on it uses ramified cyclotomic extensions of the p-adic numbers as its classical model.[38] At p = 3 the uniformiser is ω − 1. So the base of the trit tile is also the classical model of the tool now used on that barrier. **Known facts, reading of the connection.**

A further link concerns the book's Open 7, the choice between ω and its conjugate. Inside the node over the 3-adic trits the two cannot be exchanged without a mirror. Inside the quaternion algebra an element of order 4 exchanges them, and doing so twice gives −1. **Computed.**[39]

Part IV — Music

13. Ratios, the comma and temperament

Music is the oldest applied mathematics of ratios. Two strings sound well together when their lengths have a simple ratio. The octave is 2:1. The fifth is 3:2. These are the two ladders of the theory: ×2 and ×3.

A circle of twelve fifths overshoots seven octaves by the Pythagorean comma, about 23.5 cents. The theory proves that this miss never vanishes.[40] In its language: the loop does not close, and what does not close passes on as tension. Musicians met this barrier for two thousand years. Their answers follow the same pattern as the barriers of mathematics.

Barrier in music	Resolution	Property lost	Link to the net
Twelve fifths do not return to the octave	Temperament: spread the comma	Pure fifths	Closure fails (T16–T18)
Even spreading needs 2^(1/12)	Equal temperament (Zhu Zaiyu 1584; Stevin)	Rational ratios	Closure depth 12
Twelve steps give poor thirds	41 and 53 steps (Jing Fang; Mercator; Holder)	A simple keyboard	Closure depths 41 and 53
The octave as frame	Tritave 3:1 (Bohlen–Pierce)	Octave equivalence	The ×3 ladder alone
Pitch as fixed height	Circular pitch (Shepard; Risset)	Top and bottom	Log-periodicity
The note as unit	Spectrum as material (Grisey; Murail)	The note	Spectrum of the tile
A fixed melody	Self-similar series (Nørgård)	Linear form	Self-similarity
Chords as lists	Chords in orbifolds (Tymoczko)	Flat chord space	The node is an orbifold

Table 5. Barriers in the history of tuning and composition. The last column is a reading.

Twelve-tone equal temperament was computed by Zhu Zaiyu in 1584 and, independently, by Simon Stevin.[41] The 53-step division goes back to Jing Fang; Mercator computed its comma and Holder praised

[37][33] Burklund, R., Hahn, J., Levy, I. and Schlank, T. M. (2023). K-theoretic counterexamples to Ravenel's telescope conjecture. arXiv:2310.17459.
[38][34] Ravenel, D. C. (2026). Cyclotomic extensions in stable homotopy theory. Preprint, University of Rochester (18 May 2026).
[39][5] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.5–2.4; Python scripts in controle/ (onafh.py, vergelijk.py, bewijs.py, klank.py, stap2_tegel.py, zeshoek.py); audio files v1.0. Constable Research, Leiden (unpublished).
[40][1] Konstapel, J. (2026). Vacuum.Net Theory, Ninth Edition: Foundations 9.2 and Physics 9.0. Constable Research, Leiden.
[41][39] Rehding, A. Fine-tuning a global history of music theory: divergences, Zhu Zaiyu, and music-theoretical instruments. Music Theory Spectrum. Liu, Y. (2013). Equal temperament circa 1600 in China and Europe. Canadian Mathematical Society.

its thirds.[42] The numbers 12, 41 and 53 are closure depths of the theory and classical tunings of music. This is the continued fraction of $\log_2 3$, known for centuries. It shows that the net and music share their first barrier.

14. The tritave

Most music repeats at the octave. The Bohlen–Pierce scale repeats at 3:1, the tritave, and divides it into 13 equal steps of ratio 3^(1/13).[43] Its consonances use odd ratios such as 3:5:7. It was described in the 1970s and 1980s, and instruments, including a family of clarinets, now exist for it.

The trit tile repeats at 3:1 in the norm. One extra trit digit multiplies the norm by 3. So the natural period of the tile is a tritave. **Measured** (Table 2). **Reading**: the octave belongs to the ladder ×2; the tritave belongs to the ladder ×3. The Bohlen–Pierce scale, found by ear and acoustics, receives here a second, arithmetical foundation.

15. Endless tones

In 1964 Roger Shepard built tones from partials an octave apart under a fixed loudness curve.[44] Shifting the partials upward produced a scale that rises forever but never gets higher. Jean-Claude Risset made a continuous glide. The effect works because the sound is the same after one octave of shift: it is a log-periodic function.

The share of turning points is such a function. It keeps changing as the disc grows and returns after each tritave. **Reading**: the tile behaves like a Shepard–Risset tone in the tritave. The difference is that Shepard designed his envelope by hand. The tile's envelope, Figure 3, follows from arithmetic alone. Its slow fall and fast rise come from the notch at 0 in Figure 2.

16. Spectral music and an arithmetical timbre

In the 1970s Gérard Grisey and Tristan Murail made the measured spectrum of a sound into compositional material.[45] Grisey's Partiels (1975) begins from the spectrum of a low trombone note. Spectral music takes its material from acoustics.

Table 4 is a spectrum taken from arithmetic. Its overtones fall roughly as 1 : 0.42 : 0.13 : 0.04 : 0.05. We made it audible in three sound files, version 1.0:[46]

- Tile timbre: one period of Figure 3 used as the waveform of a 220 Hz tone.
- Bohlen–Pierce ladder: the thirteen steps of a tritave played with the tile timbre.
- Risset glide in the tritave: pure partials spaced 3:1 under a fixed envelope, rising without end.

Reading and proposal: every number system that fails to terminate produces its own log-periodic share and its own spectrum. Composers could use these arithmetical timbres as Grisey used the trombone. Different bases should give different timbres. This can be computed and heard.

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[42][40] Barbour, J. M. (1951). Tuning and Temperament: A Historical Survey. Michigan State College Press.

[43][35] Mathews, M. V., Pierce, J. R., Reeves, A. and Roberts, L. A. (1988). Theoretical and experimental explorations of the Bohlen–Pierce scale. Journal of the Acoustical Society of America 84, 1214–1222.

[44][36] Shepard, R. N. (1964). Circularity in judgments of relative pitch. Journal of the Acoustical Society of America 36, 2346–2353. Deutsch, D. (2010). The paradox of pitch circularity. Acoustics Today 6(3), 8–14.

[45][37] Cagney, L. (2023). Gérard Grisey and Spectral Music: Composition in the Information Age. Cambridge University Press.

[46][5] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.5–2.4; Python scripts in controle/ (onafh.py, vergelijk.py, bewijs.py, klank.py, stap2_tegel.py, zeshoek.py); audio files v1.0. Constable Research, Leiden (unpublished).

17. Self-similar composition

Per Nørgård discovered his infinity series around 1959.[47] It generates a melody that contains itself: the second, third and fourth notes return as the fifth, ninth and thirteenth, and so on at every level. He used it for melody, harmony and rhythm. The series is self-similar under a factor 2. The trit tile is self-similar under a factor 3 in the norm. **Reading:** a 'tritave Nørgård series' built from the finite trit expansions, ordered by norm, is a concrete compositional experiment suggested by this work.

18. Chord geometry

Dmitri Tymoczko showed that chords live in orbifolds: spaces folded by octave equivalence, permutation and inversion.[48] Two-note chords form a Möbius strip with mirror edges. His 2006 paper was the first on music theory in Science.

The node of the theory, Rung 11, is also an orbifold: the modular group acting on the upper half-plane, with folding points of order 2 and 3. **Reading**: Tymoczko's spaces describe where chords are. Shepard's tones describe how pitch can circle. The tile describes how a quantity circles under a fixed ratio. The theory contains all three: the node as orbifold, the tritave as circle, the spectrum of the tile as timbre.

Part V — Applications

19. Discrete scale invariance in nature and markets

In physics the structure found here is called discrete scale invariance. A system is invariant only under one magnification factor λ and its powers. Sornette reviewed how this produces complex exponents and log-periodic corrections to power laws.[49] He lists rupture, earthquakes, growth processes and other cases. Log-periodic structures matter because they reveal a preferred scaling ratio of the underlying process.

Johansen, Sornette and Ledoit claimed log-periodic signatures before major financial crashes.[50] These claims are influential and debated. Zhou and Sornette showed numerically that lattice fractals display clear log-periodicity, while non-lattice ones do not.[51] This gives a practical test: a visible log-periodic rhythm points to a single preferred ratio.

Reading: the trit tile is a clean, exactly solvable example of discrete scale invariance with λ = 3 in the norm. It can serve as a calibration case for methods that search for log-periodicity in data. Its preferred ratio is not fitted; it is given by arithmetic.

20. Computation

Trit streams are a standard representation in exact real arithmetic (Section 11). The tile adds a quantitative fact: for the plane version of balanced ternary with base 1 − ω, about 31 % of lattice points have no finite expansion, and this share oscillates. Any implementation that uses this system for exact

[47][41] Anderson, M. (1997). The many patterns of Per Nørgård. Tempo 202, 3–7.
[48][38] Tymoczko, D. (2006). The geometry of musical chords. Science 313, 72–74. Tymoczko, D. (2011). A Geometry of Music. Oxford University Press.
[49][24] Sornette, D. (1998). Discrete scale invariance and complex dimensions. Physics Reports 297, 239–270 (extended version arXiv:cond-mat/9707012).
[50][25] Johansen, A., Sornette, D. and Ledoit, O. (1999). Predicting financial crashes using discrete scale invariance. Journal of Risk 1(4) (arXiv:cond-mat/9903321).
[51][26] Zhou, W.-X. and Sornette, D. (2007). Numerical investigations of discrete scale invariance in fractals and multifractal measures. arXiv:cond-mat/0408600.

complex arithmetic must handle the two turning endpoints, for instance by adding them as extra digits or by switching base. **Known structure, measured share.**

21. The Guide of Humanity: a cycle detector

The Guide of Humanity, built separately, is meant to find where cycles occur in a person's or an organisation's history. This document offers a procedure based on the mathematics above. It is a proposal. Its use on human data is a reading that has to prove itself in practice.

1. Take a quantity that grows or shrinks over time or over scale: counts, sizes, durations, intervals.
2. Re-sample it on a logarithmic scale of the growing variable.
3. Test two preferred ratios first: 2 (octave, the ladder ×2) and 3 (tritave, the ladder ×3). Then the closure depths 12, 41 and 53 as candidate cycle lengths.
4. For each ratio, fold the data onto one period and compute the mean and the overtones, as in Table 4.
5. Report a cycle only if the overtones repeat across several periods, as they do for the tile, and only with a statistical test against shuffled data.
6. Report three things: the ratio (which ladder), the mean (where things stand on average) and the spectrum (the character of the cycle).

Two cautions. First, log-periodic fits can find patterns in noise; the literature on financial crashes shows how contested this is.[52] The repeat test of step 5 is therefore essential. Second, the tile is exact and clean. Human data are not. The tile serves as the known reference case against which a detector can be checked.

22. Research programme

1. Exact overtones: derive Table 4 by the Mellin–Perron method (Section 9.4).
2. Other bases: compute shares, means and spectra for other non-terminating bases; test whether the spectrum is determined by the base polynomial and the position of 0 on the tile boundary.
3. Formal proof: state and prove the theorem in cubical Agda or Lean, with the counts as a certified computation.
4. Category theory: make the initial-algebra / final-coalgebra reading precise, following Leinster and Moss.
5. Homotopy link: test whether the ramified prime $1 - \omega$ of the tile plays a role in the telescopic questions at $p = 3$ (Section 12).
6. Music: compose with arithmetical timbres and a tritave self-similar series; test perception of the tile timbre against Bohlen–Pierce material.
7. Guide: implement the detector of Section 21 and calibrate it on the tile.

Part VI — Status

23. What is proved, measured, known and read

Statement	Status	Where
Every orbit ends in 0 or $\pm(1 + \omega)$	Proved (elementary)	3.2
The system is not canonical	Known (Akiyama–Scheicher)	4.1

[52] [25] Johansen, A., Sornette, D. and Ledoit, O. (1999). Predicting financial crashes using discrete scale invariance. Journal of Risk 1(4) (arXiv:cond-mat/9903321).

Statement	Status	Where
The tile tiles the plane by Z[ω]	Known (Lagarias–Wang)	5
βW = W; share equals tile density at 0 up to O(1/R)	Proved	7.1–7.2, 7.4
The share has no limit; liminf ≤ 0.302, limsup ≥ 0.315	Proved (computer-assisted)	7.3
Range 29.6–32.0 %; period 3 in the norm	Measured	6
Mean 30.87 %; log density 30.89 %	Measured	8
Overtone amplitudes and phases	Measured	8.1
Oscillation is a lattice-case complex-dimension effect	Known theory, applied	9.3
Result belongs to the Delange family; Benford analogy	Known context	9.1–9.2
Initial algebra / final coalgebra reading	Reading	10
Signed digits = trits; productivity = LLPO edge	Known facts, reading	11
Link to 24, W5 at p = 3, telescope model	Known facts, reading	12
Tritave, Shepard–Risset, spectral, Nørgård, Tymoczko links	Reading on known facts	13–18
Discrete scale invariance; cycle detector	Known theory; proposal	19–21

Table 6. Status of all statements in this reference.

24. Conclusion

A small open question of the Vacuum.Net theory has a precise answer. The share of points that never close does not settle. It circles with a fixed period of one trit digit. The circle has a mean, a shape and overtones, and it is proved to be real.

The answer is not isolated. In number theory it joins the family of Delange and of Benford. In fractal geometry it is the lattice-case oscillation of a tile whose origin lies on its edge. In category theory it measures the gap between an initial algebra and a final coalgebra. In logic it is a shadow of the LLPO edge. In physics it is discrete scale invariance with ratio 3. In music it is a tritave, an endless tone and a timbre.

What the Vacuum.Net theory adds is the reason. The ratio is 3 because the strand steps in trits. The gap exists because closure separates what stays from what moves on. The origin lies on the edge because the plane of Rung 10 is built on the ramified prime 1 − ω. This reference collects these threads so that later projects can start from them.

Annotated references

[1] Konstapel, J. (2026). Vacuum.Net Theory, Ninth Edition: Foundations 9.2 and Physics 9.0. Constable Research, Leiden.

Primary source of the theory. Axioms N1–N5; theorems T16–T18 on the ladders ×2 and ×3, the closure depths and the Pythagorean comma. Read first for any later project.

[2] Konstapel, J. (2026). Mathematical Theory Generated with the Vacuum.Net Theory, version 7.0 (reference edition, 19 November 2026). Constable Research, Leiden.

The book of 22 rungs. Rung 10 (Eisenstein plane, base 1 − ω), Rung 11 (the node), the LLPO edge and the open points Open 7 and IX.3. This reference answers IX.3.

[3] Konstapel, J. (2026). Barriers and Back-Steps, version 1.0, and The Next Mathematics (constable.blog), 21 November 2026.

The history of mathematics as a chain of barriers, read through the windings, with three predictions. The present work supplies the first worked example of 'rhythms as answers'.

[4] Konstapel, J. (2026). The Sound of the Trit Tile, version 1.0, 21 November 2026. Constable Research, Leiden.

The shorter essay on which this reference builds. It contains the first statement of the theorem and the musical reading.

[5] Konstapel, J. (2026). Research notes 'Hogere windingen', versions 1.5–2.4; Python scripts in controle/ (onafh.py, vergelijk.py, bewijs.py, klank.py, stap2_tegel.py, zeshoek.py); audio files v1.0. Constable Research, Leiden (unpublished).

All computations of Parts II and IV with code that reproduces them, including the certificate of Section 7 and the three sound files.

[6] Akiyama, S. and Scheicher, K. Symmetric shift radix systems and finite expansions. Preprint, University of Tsukuba (www.math.tsukuba.ac.jp/~akiyama/papers/centralsrs6.pdf).

Defines symmetric canonical number systems. Theorem 2.2 decides finiteness for quadratic bases; applied in Section 4. No density results.

[7] Akiyama, S., Borbély, T., Brunotte, H., Pethő, A. and Thuswaldner, J. M. (2005). Generalized radix representations and dynamical systems I. Acta Mathematica Hungarica 108, 207–238.

Introduces shift radix systems as a common frame for number systems. Entry point to the literature on finite and periodic expansions.

[8] Kirschenhofer, P., Pethő, A., Surer, P. and Thuswaldner, J. (2010). Finite and periodic orbits of shift radix systems. Journal de Théorie des Nombres de Bordeaux 22, 421–448.

Which orbits end in 0 and which cycle, studied over parameter regions. Treats parameters, not the share of starting points.

[9] Akiyama, S. (2002). On the boundary of self affine tilings generated by Pisot numbers. Journal of the Mathematical Society of Japan 54(2).

Explains why it matters whether the origin is an inner point of the central tile. In our system it is not; Section 5 builds on this contrast.

[10] Gilbert, W. J. (1982). Fractal geometry derived from complex bases. Mathematical Intelligencer 4, 78–86.

Early, readable account of the fractal tiles that arise from complex number bases. Good background for Figure 4.

[11] Lagarias, J. C. and Wang, Y. (1997). Integral self-affine tiles in R^n, part II: lattice tilings. Journal of Fourier Analysis and Applications 3, 83–102.

Standard digit sets give self-affine tiles that tile by a lattice. The one external theorem used in the proof of Section 7.

[12] Steiner, W. and Thuswaldner, J. M. (2012). Rational self-affine tiles. arXiv:1203.0758.

States the Lagarias–Wang result in the form used here: irreducible characteristic polynomial and complete residue digits give a lattice tiling with respect to the smallest invariant lattice containing the digit differences.

[13] Lau, K.-S. and Xu, Y. (2000). On the boundary of attractors with non-void interior. Proceedings of the American Mathematical Society 128, 1761–1768.

The boundary of a self-affine tile has Lebesgue measure zero, and smaller Hausdorff dimension. Supports the area argument in Section 7.

[14] Delange, H. (1975). Sur la fonction sommatoire de la fonction 'somme des chiffres'. L'Enseignement Mathématique 21, 31–47.

The founding example of a periodic fluctuation in a digit count: a continuous, nowhere differentiable periodic function of log N.

[15] Flajolet, P., Grabner, P., Kirschenhofer, P., Prodinger, H. and Tichy, R. F. (1994). Mellin transforms and asymptotics: digital sums. Theoretical Computer Science 123, 291–314.

The Mellin–Perron method that produces the Fourier series of such fluctuations. The proposed route to exact overtones (Section 9.4).

[16] Grabner, P. J., Kirschenhofer, P. and Prodinger, H. (1998). The sum-of-digits function for complex bases. Journal of the London Mathematical Society 57, 20–40.

Digit sums for complex bases counted in large circles. The closest published setting to ours.

[17] Gittenberger, B. and Thuswaldner, J. M. (1999). The moments of the sum-of-digits function in number fields. Canadian Mathematical Bulletin 42, 68–77.

Periodic fluctuations of period 1 in counts over Gaussian integers with $|z|^2 < N$. Shows the method that would apply to our count.

[18] Grabner, P. J. and Liardet, P. (1999). Harmonic properties of the sum-of-digits function for complex bases. Acta Arithmetica 91, 329–349.

Spectral (harmonic) properties of digit functions in complex bases; uniform distribution and singular spectrum.

[19] Heuberger, C. and Kropf, S. (2015). Output sum of transducers: limiting distribution and periodic fluctuation. Electronic Journal of Combinatorics 22(2), #P2.19.

General formula for the Fourier coefficients of periodic fluctuations produced by digit automata. A possible general frame for other bases.

[20] Hill, T. P. (1995). A statistical derivation of the significant-digit law. Statistical Science 10, 354–363.
Jameson, M., Thorner, J. and Ye, L. (2016). Benford's law for coefficients of newforms. arXiv:1407.1577.

Benford's law: the set of integers with leading digit 1 has no natural density but has logarithmic density $\log_{10} 2$. The newform paper finds the same pattern for the Ramanujan τ-function. The best-known relatives of our theorem.

[21] Lapidus, M. L. and van Frankenhuijsen, M. (2013). Fractal Geometry, Complex Dimensions and Zeta Functions, 2nd ed. Springer.

Complex dimensions describe oscillations in the geometry and the spectrum of fractals. The language of Section 9.3.

[22] Lapidus, M. L. and Pearse, E. P. J. Tube formulas and complex dimensions of self-similar tilings. arXiv:math/0605527.

In the lattice case each line of complex dimensions gives a multiplicatively periodic function. Our tile is a lattice case; this explains why the oscillation must exist.

[23] Lapidus, M. L. (2015). The sound of fractal strings and the Riemann hypothesis. arXiv:1505.01548.

Accessible survey of how a fractal can be 'heard'. Bridge between Part II and Part IV.

[24] Sornette, D. (1998). Discrete scale invariance and complex dimensions. Physics Reports 297, 239–270 (extended version arXiv:cond-mat/9707012).

Review of discrete scale invariance in physics. Log-periodic corrections reveal preferred scaling ratios. Main source for Part V.

[25] Johansen, A., Sornette, D. and Ledoit, O. (1999). Predicting financial crashes using discrete scale invariance. Journal of Risk 1(4) (arXiv:cond-mat/9903321).

Claims log-periodic signatures before market crashes. Influential and debated; cited as an application, not as an endorsement.

[26] Zhou, W.-X. and Sornette, D. (2007). Numerical investigations of discrete scale invariance in fractals and multifractal measures. arXiv:cond-mat/0408600.

Lattice fractals show visible log-periodicity; non-lattice ones do not. A practical test for Section 21.

[27] Leinster, T. (2011). A general theory of self-similarity. Advances in Mathematics 226, 2935–3017.

Category-theoretic theory of self-similar spaces as universal solutions. Includes Freyd's theorem that [0,1] is a terminal coalgebra.

[28] Bhattacharya, P., Moss, L. S., Ratnayake, J. and Rose, R. (2014). Fractal sets as final coalgebras obtained by completing an initial algebra. LNCS 8464, 146–167.

Fractals obtained as the completion of an initial algebra. Basis of the reading in Section 10.

[29] Berger, U. (2011). From coinductive proofs to exact real arithmetic: theory and applications. Logical Methods in Computer Science 7(1:8).

Real numbers as coinductive streams of signed digits −1, 0, +1, with program extraction. The trits of the net in their computational form.

[30] Schwichtenberg, H. and Wiesnet, F. (2021). Logic for exact real arithmetic. Logical Methods in Computer Science (arXiv:1904.12763).

Signed-digit and Gray-code streams; inductive and coinductive definitions. Discusses the productivity problem that corresponds to the LLPO edge.

[31] Cherubini, F., Coquand, T., Geerligs, F. and Moeneclaey, H. (2024). A Foundation for Synthetic Stone Duality. arXiv:2412.03203.

Homotopy type theory plus four axioms, which prove LLPO. Shows how the edge of the trits can be crossed by an axiom.

[32] Bridges, D. and Richman, F. (1987). Varieties of Constructive Mathematics. Cambridge University Press.

Defines LLPO and the other omniscience principles.

[33] Burklund, R., Hahn, J., Levy, I. and Schlank, T. M. (2023). K-theoretic counterexamples to Ravenel's telescope conjecture. arXiv:2310.17459.

The current barrier in stable homotopy theory. Context for Section 12.

[34] Ravenel, D. C. (2026). Cyclotomic extensions in stable homotopy theory. Preprint, University of Rochester (18 May 2026).

Uses ramified cyclotomic extensions of the p-adic numbers as classical model. At p = 3 its uniformiser is ω − 1, the base of Rung 10.

[35] Mathews, M. V., Pierce, J. R., Reeves, A. and Roberts, L. A. (1988). Theoretical and experimental explorations of the Bohlen–Pierce scale. Journal of the Acoustical Society of America 84, 1214–1222.

The scale that repeats at 3:1 (the tritave), divided into 13 steps. Section 14.

[36] Shepard, R. N. (1964). Circularity in judgments of relative pitch. Journal of the Acoustical Society of America 36, 2346–2353. Deutsch, D. (2010). The paradox of pitch circularity. Acoustics Today 6(3), 8–14.

Tones from octave-spaced partials under a fixed envelope; endless rise. Risset's continuous version. Section 15.

[37] Cagney, L. (2023). Gérard Grisey and Spectral Music: Composition in the Information Age. Cambridge University Press.

History and technique of spectral music, including Partiels (1975). Section 16.

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