Upward in the Windings

What the net makes by itself, and which coupling makes which level

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Introduction

A winding is a movement that comes back to where it began. It is not a number that is counted. It is a movement that is remembered. Go once around a circle and you have made a winding of the first level. A loop of loops sweeps over a surface: a winding of the second level. This book asks one question about such windings on the Vacuum.Net ladder.

> **Central question.** Which windings does the net make by itself, and which way of coupling nets makes which level?

Version 1.0 asked a wider question: does the net carry all windings? That question has no end, because the windings of spheres are not all known. This version replaces it by the question above. It has a finite answer: a table with one row per level. The table is in the chapter “The answer”. Every cell has a status.

The ladder

The Vacuum.Net theory starts from movement. One step is a direction: forward, standing still, or back. We write it as a trit: R, O or L. An infinite row of trits is a stream. Read in balanced base three (L = −1, O = 0, R = 1), a stream is an address: a cell inside a cell inside a cell. Read in signed base two (digits −1, 0, 1), the same stream is a number one can add and multiply with: it computes. A place with two base-three names is an edge, for example 1/6 = 0.RLLL… = 0.ORRR….

The book Mathematical Theory Generated with the Vacuum.Net Theory proves this ladder by machine.[1] It ends at the edge: going back from computing to an address needs a decision that no computation gives. Research Note 1 summarises that result.[2]

The bare net

The bare net, Wind, has a point for every stream and a path from every stream to its tail (the stream without its first trit). It is the homotopy coequalizer of the identity and the tail map: every stream is glued to its tail by a path, not by an equation. A path can be walked; an equation can only be true or false. That difference is what lets the net remember turns.

Terms

- **Level.** The homotopy group $\pi_n(X)$ holds the n-dimensional loops in X up to deformation. A type with no loops above level n is an n-type.
- **Retract.** A is a retract of B when A maps into B and back, and the round trip is the identity on A. Then every winding of A survives inside B.
- **Suspension.** Two poles, and a path from pole to pole for every point.
- **Join.** A path from every point of one type to every point of the other.
- **Glued cell.** A disk whose rim is fastened to a loop, so that the loop becomes trivial.
- **drop and tail.** The tail of a stream is the stream without its first trit; drop n s removes the first n trits.
- **Rest point c0.** The constant stream O O O …; it is its own tail, so its step is a closed loop.
- **Merely.** A statement holds merely when it is known that a witness exists, without the witness being recoverable (propositional truncation).
- **Homotopy coequaliser.** Given two maps f, g from A to B: the points of B, plus a path from f(a) to g(a) for every a.

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[1] J. Konstapel, Mathematical Theory Generated with the Vacuum.Net Theory, version 7.0, reference edition, 19 November 2026; Agda package vacuumnet-mathematics-v7.0.

[2] J. Konstapel, Edges, Choice and Countable Cycles, Research Note 1, version 1.0, 20 November 2026.

- **Retract at the level of loops.** The loops of A map into the loops of B and back, and the round trip is the identity.
- **LLPO, WLPO, continuity.** Constructive principles. LLPO: for a binary sequence with at most one 1, either all even or all odd places are 0. WLPO: a binary sequence is all 0 or not. Continuity principles (Brouwer) say that functions on streams depend on finitely many trits; under them, such decisions fail.
- **Brunerie number.** The integer n with $\pi_4(S^3) \cong \mathbb{Z}/n$, defined and computed inside type theory; its value is ±2.
- **Guide of Humanity.** The author's advisory system on swarp.nl that helps a client (a person or organisation) find their cycles and the decisions at their edges.

How the book is made

- Each rung has three parts: Seen (the picture), Reasoned (what is proved), Back to the whole (what it means).
- Every claim has a status: proved in Agda (name in a footnote), known theorem (with source), argument on paper, reading, or open.
- All Agda files check with Agda 2.8.0 and the cubical library v0.9 under --safe, without postulates.[3] The framework is homotopy type theory.[4]
- Every file was reviewed by an independent reviewer: an AI agent that did not write it. All its corrections are applied.

[3] A. Vezzosi, A. Mörtberg and A. Abel, "Cubical Agda: a dependently typed programming language with univalence and higher inductive types", Journal of Functional Programming 31 (2021) e8. The cubical library v0.9 contains $\Omega S^1 \equiv \mathbb{Z}$, $S^2 \simeq \mathrm{Susp} S^1$, $S^3 \simeq$ join S^1 S^1, $\pi_3 S^2 \cong \mathbb{Z}$, $\pi_4 S^3 \simeq \mathbb{Z}/2\mathbb{Z}$ and π_1(Bouquet A) ≡ FreeGroup A.

[4] The Univalent Foundations Program, Homotopy Type Theory: Univalent Foundations of Mathematics, Institute for Advanced Study, 2013. Chapter 6 (higher inductive types), Chapter 8: circle, suspension, Hopf fibration (§8.5), Freudenthal (§8.6).

Part I · Level 0 and level 1: the bare net

1 · Parts: which streams belong together

Seen

Before we count turns, we ask which points can reach each other at all.

Reasoned

Theorem 1.1. The set of parts of Wind is the set of streams up to a common tail. Two points pt s and pt t are connected exactly when there are m and n with drop m s = drop n t (merely: the numbers exist, but cannot be read back from a path).[5]

Back to the whole

Reading: the net computes its own parts. Two histories belong to one process exactly when they end in the same way, even if one took longer to get there.

2 · Turns: one cycle

Reasoned

Theorem 2.1. Let tel : Wind → S^1 be the counting map that sends every step to one turn around the circle. If a stream returns to itself after k steps, the k steps close into a loop whose image under tel winds exactly k times. At a stream that is its own tail, every integer gives a loop, and $\mathbb{Z}$ is a retract of the loops there.[6]

Argument on paper (checked by the reviewer; turning it into homotopy type theory needs a result like Wärn 2024): every stream has exactly one step out, so each part of Wind contains at most one cycle. A part with an eventually periodic stream has $\pi_1 = \mathbb{Z}$; any other part is a tree and has no turns.

Known theorem: the loops around the circle are exactly the integers.[7]

Back to the whole

Reading: turns arise exactly where a pattern returns.

3 · Order: two kinds of step

Seen

Give every stream two different steps, A and B, to the same tail. Now there are two ways to walk two steps.

Reasoned

Theorem 3.1. For every stream s: first A then B is a different path from first B then A. At the rest point c0 (the constant stream O O O …, which is its own tail) the commutator $(A \bullet B) \bullet (B \bullet A)^{-1}$ is not trivial.[8]

Theorem 3.2. The figure eight (two circles at one point) is a retract of this net, at the level of loops.[9]

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[5] Agda: componenten, verbonden, verbonden←, staart-trans, file VacuumNetComponents.

[6] Agda: periode-winding, lus-niet-triviaal, retract-ℤ, vast-retract, file VacuumNetWinding.

[7] D. R. Licata and M. Shulman, "Calculating the Fundamental Group of the Circle in Homotopy Type Theory", LICS 2013, 223–232.

[8] Agda: volgorde, commutator, file VacuumNetEight.

[9] Agda: acht-retract, lus-retract, file VacuumNetEight.

Known theorem: the loops of the figure eight form the free group on two letters (in the library). The part of the net at the rest point presumably has a free group of countably infinite rank (on paper, not proved); the eight is only a piece of it.

Back to the whole

Reading: one process with two ways to move already knows order. Order is a level-1 phenomenon.

4 · The ceiling of the bare net

Reasoned

> **Theorem 4.1.** Streams form a set.[10]
>
> **Theorem 4.2 (known).** A coequaliser of two maps between sets is a 1-type.[11] Hence the bare net, with any set of step types, has no windings above level 1.

Consequence (classical argument, not formalised): in the bare net every π_1 is free, and free groups have no torsion. So the bare net has no turn that vanishes after n rounds.

Back to the whole

Reading: a single process, however long, knows parts, turns and order. Nothing more.

5 · Twist: a glued cell

Seen

Glue one disk at the rest point, fastened to the step taken twice. Then two rounds are nothing, while one round is still something.

Reasoned

> **Theorem 5.1.** In this net the step at the rest point is not trivial, but the step taken twice is. Its order is exactly 2. The space with one point, one loop and this disk (homotopy type of the projective plane, classical) is a retract.[12]

This torsion is not made by the net. It is made by the cell we glue on.

6 · A test case: Tymoczko's chord spaces

Seen

Tymoczko represents a chord as a point in a geometric space called an orbifold: a space that is locally a quotient of ordinary space by a finite symmetry group. At a point fixed by a symmetry, the group of symmetries fixing it is its isotropy.[13] Ordered chords of n notes lie on an n-torus; forgetting the order of the voices gives an orbifold. For two notes it is a Möbius strip, whose rim holds the unisons, where the two voices coincide.[14][15]

Reasoned

On the net: a dyad is a pair of streams. Each voice can step. The voices may swap. Swapping twice is nothing (a glued cell), and swapping fits with stepping (a glued square). This is a two-dimensional

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[10] Agda: isSetStroom, file VacuumNetStreamSet.

[11] D. Wärn, "Path spaces of pushouts", arXiv:2402.12339 (2024; v2, 2025). Theorem 5.4 and the remark after it: any pushout or coequaliser of 0-truncated types is 1-truncated.

[12] Agda: orde-twee-niet-een, orde-twee, M2-retract, file VacuumNetTwist.

[13] D. Tymoczko, "The Geometry of Musical Chords", Science 313 (2006), 72–74.

[14] Tymoczko 2006 (see references).

[15] C. Callender, I. Quinn and D. Tymoczko, "Generalized Voice-Leading Spaces", Science 320 (2008), 346–348.

presentation of the homotopy quotient, not the full quotient: the square that makes the two voices' steps commute and the higher coherences are left out. For the statement about the unison this suffices. The comparison with Tymoczko's continuous torus is an analogy: here the voices move on the net.

> **Theorem 6.1.** For every stream s, the swap at the unison (s, s) is not trivial, while swapping twice is. At the unison the swap has order exactly 2.[16]

In the underlying space of the coarse quotient, the Möbius strip, the unison is a boundary without a turn of its own. The orbifold structure keeps $\mathbb{Z}/2$ there as isotropy; the homotopy quotient makes it visible as a loop of order 2. Interpretation, not proved: everything in this chapter stays at level 1, and its torsion comes from the glued cell, not from the bare net.

Back to the whole

Reading: voice leading lives at level 1. Its richness is order and twist, not higher windings. Where two voices meet, the meeting keeps a memory of which voice was which.

[16] Agda: unisono-draai, unisono-orde-twee, file VacuumNetVoices.

Part II · The jumps: suspension and join

7 · Suspension lifts a choice to a turn

Seen

Hang a type between two poles. A path from the north pole through a to the south pole, and back through b, is a loop. If a and b lie in different parts, the loop goes around something.

Reasoned

Theorem 7.1. In the suspension of any type X: if a property f : X → Bool separates a and b, then the loop through a and back through b is not trivial.[17]

Theorem 7.2. Take two separate nets and hang them between the same poles. Then every loop through the first and back through the second is not trivial.[18]

Caveat: inside one bare net, telling two parts apart means deciding a property of tails, such as "eventually only L". Such properties are not provably decidable (under a continuity principle they are even constant). With the method of this book the jump from a choice to a turn needs a separation. Whether it can be done without one is open; it is a question of Kraus and Altenkirch.[19] Argument on paper: for a set X, the loops of the suspension of X form a free group (Wärn 2024 settles the needed truncation),[20] and a set embeds in its free group constructively.[21] Mapping the net to its set of parts, this gives: for streams in different parts the loop is not trivial, with no decision needed. Status: on paper, not in Agda. The decision is needed only for the method used in Agda here.

Back to the whole

Reading: two processes that are separate, held between the same two poles, form a turn. In Agda this is proved when the separation is given; on paper it holds without that.

8 · Suspension lifts a turn to a sheet

Reasoned

Theorem 8.1. At a stream that is its own tail, its loop becomes a sheet in the suspended net. Its image is the standard sheet of the sphere S^2; the n-fold loop gives the n-fold sheet.[22]

Known theorem, also in the library: $\pi_2(S^2) \cong \mathbb{Z}$, generated by the standard sheet.[23] So this jump needs no decision, only a stream that is its own tail. That the sheet of Theorem 8.1 is a generator is not formalised.

9 · Join: the Hopf winding

Reasoned

[17] Agda: scheiding, file VacuumNetLift.

[18] Agda: twee-processen, file VacuumNetLift.

[19] N. Kraus and T. Altenkirch, "Free Higher Groups in Homotopy Type Theory", LICS 2018, 599–608.

[20] Wärn 2024 and Kraus–Altenkirch 2018 (see references).

[21] R. Mines, F. Richman and W. Ruitenburg, A Course in Constructive Algebra, Springer, 1988. Chapter X: free groups, constructively.

[22] Agda: vlak, vlak-beeld, vlak-n-beeld, file VacuumNetSphere.

[23] HoTT Book, §8.5–8.6 (see references).

Theorem 9.1. For two streams that are their own tails, S^3 is a retract of the join of the net with itself.[24]

Known theorem: the Hopf map $S^3 \to S^2$ generates $\pi_3(S^2) \cong \mathbb{Z}$; any two of its fibres are circles that link once.[25]

Back to the whole

Reading: cycles that grip into each other, each point to each point, carry a third-level winding.

10 · All spheres, and torsion without a cell

Reasoned

Theorem 10.1. Suspension keeps retracts. Hence, given two streams that are their own tails, the n-fold suspension of S^3 is a retract of the n-fold suspension of the join of the net with itself. That this suspension is the sphere of dimension n + 3 is known, not formalised.[26]

Known theorem, also in the library: $\pi_4(S^3) \cong \mathbb{Z}/2$, via the Brunerie number.[27][28] Known: $\pi_5(S^3) \cong \mathbb{Z}/2$; its Brunerie-type number is not yet computed by machine.[29]

Observation 10.2 (from 4.2, 5.1, 9.1 and the library). At level 1 torsion appears only when we glue a cell. At higher levels torsion appears from coupling alone: S^3 is a retract of the join of the net with itself (join only, given two streams that are their own tails), so π_4 of that join contains $\mathbb{Z}/2$. No cell is glued.

Back to the whole

Reading: a single process cannot produce a turn that cancels itself. Coupled processes can, at a higher level, without anyone imposing it.

[24] Agda: joinretract, S^3-retract, file VacuumNetHopf.

[25] HoTT Book, §8.5–8.6 (see references).

[26] Agda: susp-retract, Σ^n-retract, hogere-sfeer, file VacuumNetHigher.

[27] G. Brunerie, On the homotopy groups of spheres in homotopy type theory, PhD thesis, Université Nice Sophia Antipolis, 2016.

[28] A. Ljungström and A. Mörtberg, "Formalizing $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ and Computing a Brunerie Number in Cubical Agda", LICS 2023 (arXiv:2302.00151).

[29] T. Jack and A. Ljungström, "Yet another homotopy group, yet another Brunerie number", abstract, TYPES 2025.

Part III · Everything, but not by itself

11 · Presentations and cell complexes

Reasoned

Theorem 11.1. For every group presentation ⟨A | R⟩, the presentation complex is a retract of the net with A step types and the same relation disks glued at the rest point.[30]

Theorem 11.2. Any finite or infinite sequence of cell attachments, starting from a bouquet of circles, gives a retract of the same attachments made on the net.[31]

Classical (not formalised): every connected CW complex is homotopy equivalent to one with a single 0-cell, so by 11.2 it is, up to homotopy, a retract of a net version.[32] The reviewer's remark is essential: in substance this is pushout functoriality plus one fact about the net (the bouquet is a retract of the bare net). The cells are glued by us.

Back to the whole

Reading: the net can carry anything one builds on it. That is not the same as making it.

[30] Agda: presentatie-retract, Zmod-op-net, Klein-op-net, Klaver-op-net, file VacuumNetPresent.

[31] Agda: plak, bouw, colim-retract, file VacuumNetCells, VacuumNetColim.

[32] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002. Prop. 0.17, Cor. 0.20, Prop. 1A.2 (maximal trees), Prop. 1.26 / Cor. 1.28 (attaching 2-cells).

The answer

The table answers the central question. Each row is a level or a jump; each cell has a status.

Level	What appears	Made by	Status
0 · components	which streams belong together	the bare net	proved in Agda: two points are connected exactly when their streams merely share a tail (W11a)
1 · turns	a cycle that can be counted ($\mathbb{Z}$)	the bare net	proved in Agda: $\mathbb{Z}$ is a retract of the loops (W0); every loop is an integer loop: on paper
1 · order	two cycles whose order matters (free groups)	the bare net with two step types	proved in Agda: $A \bullet B \neq B \bullet A$ for every stream (W6)
1 · ceiling	nothing higher than level 1; no torsion	the bare net	known theorem (Wärn 2024, not formalised) plus streams form a set (Agda): the bare net is a 1-type; no torsion: classical argument, not formalised
1 · twist	torsion: a turn that vanishes after n rounds	a glued cell	proved in Agda for n = 2 (W7, W13); general n: retract only (W9), $\pi_1 = \mathbb{Z}/n$ classical, nontriviality not in Agda
0 → 1 jump	a choice between parts becomes a turn	suspension	proved in Agda once a separation is given (W14); without one: on paper (Wärn 2024 + Mines–Richman–Ruitenburg)
1 → 2 jump	a turn becomes a sheet	suspension	image proved in Agda (W1b); nontriviality from $\pi_2(S^2) = \mathbb{Z}$, known
3 · Hopf	interlocked cycles	join	proved in Agda: S^3 is a retract (W2); $\pi_3(S^2) = \mathbb{Z}$ known
≥ 3 · spheres	all windings of spheres, also irregular	join, then suspension	proved in Agda: retracts (W3/W4); the groups themselves known only in part
higher · twist	torsion without any glued cell: $\pi_4(S^3) = \mathbb{Z}/2$	join alone	known theorem (Brunerie; library proof Ljungström–Mörtberg); carried by the net via W2
all	every connected CW complex, up to homotopy	glued cells	proved in Agda as pushout functoriality (W10, W10b); classical transfer on paper
stable	periodic order (Bott)	—	open (W5)

In one sentence: the bare net makes parts, turns and order; suspension lifts a separated choice to a turn and a turn to a sheet; join makes interlocked cycles; and only coupling, not the bare net, makes torsion appear by itself.

Bott periodicity: open

The windings of spheres are irregular. The windings of the classical groups repeat with period 2 or 8.[33] The author's Spiral Navigator uses this periodicity.[34] A literature search (November 2026) found no formalisation in homotopy type theory. This row of the table stays open.

[33] R. Bott, "The stable homotopy of the classical groups", Annals of Mathematics 70 (1959), 313–337.

[34] J. Konstapel, "The Spiral Navigator", constable.blog, 10 March 2026.

Relation to the Guide of Humanity (reading)

The Guide finds a client's cycles, their edges and the decision at each edge. The table gives it a grammar. Only the mathematics is proved; that human cycles behave like this is a reading.

- **One process** knows parts, turns and order. Its history decides which part it is in.
- **A turn that cancels itself** is never made inside one process. Inside one process it must be imposed, like a glued cell.
- **Two separate processes held between the same poles** form a new turn. On paper this holds even when one cannot tell them apart by computation.
- **Interlocked processes** that each return to themselves carry higher windings, including self-cancelling ones, without anyone imposing them.

Open points

- Every loop at a periodic point of Wind is an integer loop (level 1, on paper).
- A machine proof of the jump from parts to turns without a given separation (on paper via Wärn and Mines–Richman–Ruitenburg).
- A machine proof of Wärn's theorem for the net, and of "π_1 free per part".
- Computing any irregular winding from the net itself.
- Bott periodicity.

Annotated references

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A. Vezzosi, A. Mörtberg and A. Abel, "Cubical Agda: a dependently typed programming language with univalence and higher inductive types", Journal of Functional Programming 31 (2021) e8. The cubical library v0.9 contains $\Omega S^1 \equiv \mathbb{Z}$, $S^2 \simeq \mathrm{Susp} S^1$, $S^3 \simeq$ join S^1 S^1, $\pi_3 S^2 \cong \mathbb{Z}$, $\pi_4 S^3 \simeq \mathbb{Z}/2\mathbb{Z}$ and π_1(Bouquet A) ≡ FreeGroup A. *Proof assistant and library used.*

D. R. Licata and M. Shulman, "Calculating the Fundamental Group of the Circle in Homotopy Type Theory", LICS 2013, 223–232. *Loops around the circle are the integers.*

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A. Ljungström and A. Mörtberg, "Formalizing $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ and Computing a Brunerie Number in Cubical Agda", LICS 2023 (arXiv:2302.00151). *Machine computation of the Brunerie number.*

T. Jack and A. Ljungström, "Yet another homotopy group, yet another Brunerie number", abstract, TYPES 2025. *The current frontier: $\pi_5(S^3)$.*

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R. Bott, "The stable homotopy of the classical groups", Annals of Mathematics 70 (1959), 313–337. *Periodicity of the classical groups.*

J. Konstapel, "The Spiral Navigator", constable.blog, 10 March 2026. *The author's navigation design using Bott.*