BLOWUP FOR THE EULER EQUATIONS WITH SMOOTH FORCING

Abstract. We construct a finite-time singularity for the incompressible Euler equations on $\mathbb{R}^3$ with a force smooth in space and time up to and including the blowup time. The initial velocity and force are axisymmetric and supported in one fixed solid torus about a circle of arbitrarily prescribed radius and height. The initial velocity is smooth, with nonzero swirl and zero meridional component. Circulation (angular momentum) and meridional velocity remain bounded, whereas the L^∞ norms of the circulation gradient and full vorticity tend to infinity. The time integral of the vorticity L^∞ norm also diverges. On every closed time interval before blowup, the solution is smooth and unique in a three-dimensional finite-energy, locally space-time Lipschitz class with bounded spatial gradients; competitors need not be axisymmetric. The construction continues the program of Diego Córdoba and Luis Martínez-Zoroa on singularity formation through the interaction of successively finer scales. We introduce localized circulation and velocity oscillations transported by the exact nonlinear flow of the preceding fields. Each addition amplifies a small circulation seed and returns its principal azimuthal-vorticity amplitude at its material center to zero, preparing the next addition. Higher-order corrections make the physical force increments summable together with all mixed space-time derivatives.

Contents

1. The Euler theorem and the amplification mechanism

This paper is part of the program of Diego Córdoba and Luis Martínez-Zoroa on singularity formation through interacting layers at successively smaller scales [6, 7, 5]. The preceding layers generate the flow that amplifies a new localized perturbation, while cancellations and corrections control the force needed to sustain this evolution. Here we carry out this construction for axisymmetric Euler with swirl, starting from smooth data and keeping the force smooth in every mixed space-time derivative through blowup.

1.1. The equations and the main result.

The forced incompressible Euler equations are

$$\partial_t u + (u\cdot\nabla)u + \nabla p = f, \qquad \operatorname{div} u = 0 \quad \text{on } \mathbb{R}^3\times[0,T_*). \tag{1.1}$$

In cylindrical coordinates write $u = u^r e_r + u^\varphi e_\varphi + u^z e_z$. Axisymmetry means that the three components are independent of φ. Set $\Gamma = ru^\varphi$, $\boldsymbol{\omega} = \operatorname{curl} u$, and $\omega = \boldsymbol{\omega} \cdot e_\varphi$. For $r_0 > 0$, $z_0 \in \mathbb{R}$, and $0 < R < r_0/2$, put

$$\mathcal{T}_R = \{(r, \varphi, z) : (z - z_0)^2 + (r - r_0)^2 < R^2\}.$$

Smoothness on a closed time interval includes compatible one-sided derivatives at both endpoints. The uniqueness class below allows non-axisymmetric competitors and a distributional pressure.

Theorem 1.1. *For every $r_0 > 0$ and $z_0 \in \mathbb{R}$ there exist $T_* > 0$, $0 < R < r_0/2$, a divergence-free axisymmetric field $u_0 \in C_c^\infty(\mathcal{T}_R; \mathbb{R}^3)$ with nonzero swirl and zero meridional velocity, and an axisymmetric force*

$$f \in C^\infty(\mathbb{R}^3 \times [0, T_*]; \mathbb{R}^3), \qquad \operatorname{supp} f(\cdot, t) \subset \overline{\mathcal{T}_R},$$

such that (1.1) has a solution $(u, p) \in C^\infty(\mathbb{R}^3 \times [0, T_*))$ with $u(\cdot, 0) = u_0$. For every $\tau < T_*$,

$$u \in L^\infty([0, \tau]; L^2(\mathbb{R}^3)), \qquad \nabla u \in L^\infty(\mathbb{R}^3 \times [0, \tau]).$$

The solution is unique among divergence-free, locally space-time Lipschitz solutions with the same data and force, solving (1.1) almost everywhere for some distributional pressure, and satisfying both of these bounds. The competing solution need not be axisymmetric. Moreover,

$$\operatorname{supp}(\Gamma, \omega)(\cdot, t) \subset \overline{\mathcal{T}_R}, \qquad \sup_{t < T_*} \left(\|\Gamma(t)\|_\infty + \|u^r(t)\|_\infty + \|u^z(t)\|_\infty \right) < \infty, \tag{1.2}$$

$$\lim_{t \uparrow T_*} \|\nabla \Gamma(t)\|_\infty = \infty, \qquad \lim_{t \uparrow T_*} \|\boldsymbol{\omega}(t)\|_\infty = \infty, \tag{1.3}$$

$$\int_0^{T_*} \|\boldsymbol{\omega}(t)\|_\infty \, dt = \infty. \tag{1.4}$$

1.2. **The amplification mechanism.** The circulation transports the swirl. Its square generates azimuthal vorticity through the fixed axial derivative, and the resulting meridional velocity can amplify the circulation gradient. The construction repeats this amplification with localized oscillations, returning the principal azimuthal-vorticity amplitude to zero before the next oscillation grows. The amplitudes themselves can be small while their spatial derivatives increase. Corrections to the two equations make the complete physical force summable in every mixed derivative.

The companion Boussinesq construction of Alpöge and Buckmaster [1] develops this mechanism with smooth space-time forcing. Its local two-field wave calculation [1, Lemma 2.1] takes place on an affine background with a constant elliptic symbol. For Euler the symbol and the buoyancy coefficient vary across the support. The amplitudes below therefore depend on the material position, and the transporting velocity includes all preceding corrections.

1.3. **Related work.** The Beale–Kato–Majda criterion [2] places vorticity growth at the center of the continuation problem for smooth Euler flow. The time-integrated vorticity conclusion in (1.4) is proved directly here, separately from the norm limits in (1.3). For unforced Euler on $\mathbb{R}^3$, Elgindi [10] constructed finite-time singularities in the $C^{1,\alpha}$ class for small positive α; Elgindi, Ghoul and Masmoudi [11] obtained finite-energy examples through stability and localization. Elgindi and Pasqualotto [12] developed related whole-space blowup mechanisms for Boussinesq and axisymmetric Euler with swirl, again with $C^{1,\alpha}$ rather than everywhere-smooth initial data. In a setting with a solid boundary, Luo and Hou [13] provided numerical evidence for singularity formation in smooth axisymmetric Euler flow. Chen and Hou [3, 4] proved smooth-data, finite-energy blowup in a domain with boundary by a computer-assisted argument.

The layered approach also produces non-self-similar singularities. Córdoba, Martínez-Zoroa and Zheng [9] constructed finite-energy, unforced Euler solutions that are $C^{1,\alpha}$, smooth away from the origin, and axisymmetric without swirl. For forced Euler, Córdoba and Martínez-Zoroa [6] constructed finite-time singularities with a force uniformly bounded in the spatial space $C^{1,1/2-\epsilon} \cap L^2$. Their stated preterminal velocity regularity is $C^{3,1/2} \cap L^2$. A related construction of Córdoba, Martínez-Zoroa and Zheng [8] gives blowup for hypodissipative Navier–Stokes with a sufficiently small fractional dissipation exponent and force in $L^1_t C^{1,\epsilon}_x \cap L^\infty_t L^2_x$.

For the two-dimensional incompressible porous media equation, Córdoba and Martínez-Zoroa [7] constructed singularities of smooth solutions with a source uniformly smooth in the spatial variables. Their theorem controls every spatial order; regularity of all time derivatives is a separate issue. For the Boussinesq system on the plane, Córdoba, Laín-Sanclemente and Martínez-Zoroa [5] introduced a multi-layer degenerate-pendulum construction with momentum force in $C^0_t C^{1,\alpha}_x \cap C^0_t L^2_x$, where $0 < \alpha < \sqrt{4/3} - 1$. The momentum force in that construction is noncompact, although its curl and the scalar force are compactly supported. These Euler, IPM and Boussinesq works supply the successive-layer amplification underlying the present construction. The smooth-forcing companion [1] and the Euler argument below require summability of every mixed derivative with one common choice of scales. In Theorem 1.1, the resulting physical force also retains one fixed compact support. The proof treats the Euler-specific radial coefficients, nonlinear transport and both residual equations.

2. Axisymmetric coordinates and compact fields

The volume coordinate removes the radial weight from transport and puts the remaining radial coefficients explicitly in the vorticity equation. Write

$$y = (z, r^2/2), \quad r = (2y_2)^{1/2}, \quad v = (u^z, ru^r), \quad J(a_1, a_2) = (-a_2, a_1), \quad \xi = \omega/r, \quad \Theta = \Gamma^2. \tag{2.1}$$

Thus $dy = r\,dz\,dr$ and $\partial_{y_2} = r^{-1}\partial_r$. All two-dimensional derivatives below refer to y unless another variable is indicated. The radial formulas are used on $y_2 > 0$; fields supported away from its boundary are extended by zero. Set

$$\mathsf{A}(y) = \begin{pmatrix} (2y_2)^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \qquad L = \operatorname{div}_y(\mathsf{A}\nabla_y) = \partial^2_{y_2} + (2y_2)^{-1}\partial^2_{y_1}, \qquad W(y) = (2y_2)^{-2}. \tag{2.2}$$

Choose concentric open balls $\mathcal{D}_0 \Subset \mathcal{D}$ about $y_* = (z_0, r_0^2/2)$ whose closures lie in the volume-coordinate image of the meridional disk defining $\mathcal{T}_R$. This is possible for every prescribed r_0, z_0. Constants depending on these fixed balls and on r_0, R are allowed throughout.

Lemma 2.1 (Coordinate and force identities)**.** *Let I be a compact time interval and let Γ, Ψ be smooth on $\mathbb{R}^2 \times I$, with support in $\overline{\mathcal{D}_0}$ at every time. Define $v = J\nabla_y\Psi$ and $\xi = L\Psi$. Their physical velocity is*

$$u^z = -\Psi_{y_2}, \qquad u^r = \Psi_{y_1}/r, \qquad u^\varphi = \Gamma/r. \tag{2.3}$$

It extends by zero to a smooth divergence-free axisymmetric field on $\mathbb{R}^3 \times I$. With $D_v = \partial_t + v \cdot \nabla_y$, put

$$F^\Gamma = D_v\Gamma, \qquad E^\xi = D_v(L\Psi) - W\partial_{y_1}(\Gamma^2). \tag{2.4}$$

Then $\int E^\xi\,dy = 0$. There is a fixed linear operator $\mathcal{R}_y$ for which

$$(f^z, f^r/r) = \mathcal{R}_y E^\xi, \qquad f^\varphi = F^\Gamma/r, \qquad \operatorname{curl}_y \mathcal{R}_y E^\xi = E^\xi, \tag{2.5}$$

and all three force components have support in the lift of $\overline{\mathcal{D}}$. For each k, this recovery has a finite C^k bound, including mixed time derivatives, depending only on k and the fixed geometry. The resulting fields solve Euler with a smooth pressure on I.

Proof. The divergence identity is $\partial_z u^z + r^{-1}\partial_r(ru^r) = \operatorname{div}_y v = 0$. Direct differentiation gives $r^{-1}(\partial_z u^r - \partial_r u^z) = L\Psi$. The cylindrical circulation and reduced-vorticity equations are

$$D_v\Gamma = rf^\varphi, \qquad D_v\Theta = 2\Gamma rf^\varphi, \qquad D_v\xi = W\partial_{y_1}\Theta + (\partial_z f^r - \partial_r f^z)/r. \tag{2.6}$$

The terms $\partial_t L\Psi$, $\operatorname{div}_y(vL\Psi)$ and $W(y_2)\partial_{y_1}\Gamma^2$ each have integral zero.

Use the divergence operator $\mathscr{B}$ of [1, Lemma 5.2], translated and rescaled so its fixed smooth bump has integral one and lies in $\mathcal{D}$. The asserted divergence identity and derivative bounds hold for every smooth input; the parity clause is unnecessary here. With $\mathcal{R}_y = J\mathscr{B}$, zero integral gives $\operatorname{curl}_y \mathcal{R}_y E^\xi = E^\xi$. Its support is contained in the convex hull of the input and the fixed bump, hence in $\overline{\mathcal{D}}$. Finally,

$$\partial_{y_1}(f^r/r) - \partial_{y_2} f^z = (\partial_z f^r - \partial_r f^z)/r.$$

The alternate cylindrical prescription is $(f^z, f^r) = \mathcal{R}_{z,r}(rE^\xi)$, since $\int rE^\xi \, dz \, dr = \int E^\xi \, dy$.

Let $h = f - \partial_t u - (u \cdot \nabla)u$. Its azimuthal component vanishes by the circulation equation. The reduced-vorticity equation gives $\partial_z h^r - \partial_r h^z = 0$. The meridional half-plane is simply connected, so $h^r \, dr + h^z \, dz$ has a smooth potential. It is constant near the axis, where u and f vanish, and therefore lifts to a smooth axisymmetric pressure. Multiplication by $r^{\pm 1}$ and the coordinate change have bounded derivatives on the fixed support. $\square$

A compact rotating base identifies the angular control. Choose smooth cutoffs χ_0, χ_1 supported in $\mathcal{D}_0$, with $\chi_1 = 1$ on $\operatorname{supp} \chi_0$ and both equal to one near y_*. Let H be smooth and compactly supported in $\mathcal{D}_0$, equal to $|y - y_*|^2/2$ on a smaller neighborhood. For $\Gamma_0, A_0, N_0 > 0$, a unit vector e_0, and a smooth angle ϑ with $\vartheta(0) = \dot\vartheta(0) = 0$, put

$$\begin{gathered} e(t) = \exp(\vartheta(t)J)e_0, \qquad \Psi^b = \dot\vartheta H, \\ \Gamma^b(y,t) = \Gamma_0 \chi_1(y) - \frac{A_0}{N_0} F\big(N_0 e(t) \cdot (y - y_*)\big)\chi_0(y). \end{gathered} \tag{2.7}$$

The periodic profile F is defined below. At $t = 0$ this gives nonzero swirl and zero meridional velocity. Near y_*, $v^b = \dot\vartheta J(y - y_*)$ and $\xi^b = \dot\vartheta(1 + (2y_2)^{-1})$; the reduced vorticity is therefore spatially variable even there. On a return interval a real trial parameter μ will index angular velocities $\alpha_\mu = \dot\vartheta_\mu$. Their admissible range, dependence on the complete fields, and sufficient bounds for opposite return signs are established in Section 7. Uniform realization of those bounds is proved in Proposition 4.2. For the complete increments (γ_j, ψ_j) constructed in Section 5, write

$$v_j = J\nabla_y \psi_j, \qquad (\Gamma_{<m}, \Psi_{<m}) = (\Gamma^b, \Psi^b) + \sum_{1 \le j < m} (\gamma_j, \psi_j). \tag{2.8}$$

Positivity of older circulation is required only on regions containing new oscillations. The coefficient there is always $2W\Gamma_{<m}$.

2.1. Normalization of the ring and the construction scale.

2.1. Normalization of the ring and the construction scale. The physical radius can be fixed before constructing the increments. The following reduction preserves the entire uniqueness class and the time-integrated vorticity conclusion.

Lemma 2.2 (Reduction to the unit ring)**.** *If all conclusions of Theorem 1.1 hold for $r_0 = 1, z_0 = 0$, they hold for every prescribed $r_0 > 0, z_0 \in \mathbb{R}$.*

Proof. For a solution with unit ring radius write $\widehat{x} = (x - z_0 e_z)/r_0$, and set

$$u(x,t) = r_0 \widehat{u}(\widehat{x}, t), \qquad p(x,t) = r_0^2 \widehat{p}(\widehat{x}, t), \qquad f(x,t) = r_0 \widehat{f}(\widehat{x}, t).$$

Time is unchanged. Every term of the momentum equation is r_0 times its original value, and divergence is unchanged. The torus of meridional radius $\widehat{R} < 1/2$ maps to the prescribed torus of radius $R = r_0 \widehat{R} < r_0/2$. Axisymmetry, compact containment of the initial datum, nonzero swirl and zero initial meridional velocity are preserved. Direct differentiation gives

$$\Gamma = r_0^2 \widehat{\Gamma}, \qquad \nabla\Gamma = r_0 \nabla\widehat{\Gamma}, \qquad \boldsymbol{\omega} = \widehat{\boldsymbol{\omega}}, \qquad \nabla u = \nabla \widehat{u}, \qquad \|u\|_2 = r_0^{5/2} \|\widehat{u}\|_2, \tag{2.9}$$

with composition by $\widehat{x}$ understood. These formulas preserve the support and boundedness conclusions, both full norm limits, and exactly the integral of the full vorticity norm. For each space multi-index γ and time order b,

$$\partial_t^b \partial_x^\gamma f = r_0^{1-|\gamma|} (\partial_t^b \partial_{\widehat{x}}^\gamma \widehat{f})(\widehat{x}, t).$$

Thus force smoothness through the terminal time is preserved, without asserting a terminal state or pressure. Finally the inverse scaling is a bijection on the divergence-free, locally space-time Lipschitz competitors in Theorem 1.1, preserving their preterminal L^2 and gradient bounds. The a.e. momentum equation and its distributional pressure transform by the same affine change. Hence uniqueness in that class transfers, without requiring axisymmetry of the competitor. ☐

For the construction it therefore suffices to use $y_* = (0, 1/2)$. The finite identities for arbitrary fixed geometry remain valid, but the estimates for the actual sequence will be taken in these normalized units. The final physical dilation is performed only after constructing the solution. In the original volume coordinates it sends the round quadratic base to

$$\frac{\alpha}{2}\big[r_0(z - z_0)^2 + r_0^{-1}(y_2 - r_0^2/2)^2\big].$$

Indeed $y = (z_0, 0) + S\widehat{y}$, with $S = \operatorname{diag}(r_0, r_0^2)$, and the streamfunction scales as $\Psi = r_0^3 \widehat{\Psi}$. The resulting linear velocity matrix is $S(\alpha J)S^{-1}$. No round-base assertion in unscaled volume coordinates is needed.

Choose one fixed geometric length $\varepsilon_{\rm geom} > 0$. Primes in the following lemma designate magnified coordinates and fields; they do not denote time derivatives. Put

$$y = y_* + \varepsilon_{\rm geom} x', \qquad \Gamma(y,t) = \varepsilon_{\rm geom}\Gamma'(x',t), \qquad \Psi(y,t) = \varepsilon_{\rm geom}^2 \Psi'(x',t). \tag{2.10}$$

This is a change of variables in the reduced equations, not a symmetry that leaves their coefficient functions unchanged. Define

$$c_{\rm geom}(x') = 1 + 2\varepsilon_{\rm geom} x_2', \quad \mathsf{A}' = \operatorname{diag}(c_{\rm geom}^{-1}, 1), \quad L' = \partial_{x_2'}^2 + c_{\rm geom}^{-1}\partial_{x_1'}^2, \quad W' = \varepsilon_{\rm geom} c_{\rm geom}^{-2}.$$

The scalar $c_{\rm geom}$ is distinct from the principal coupling c_m in (3.4).

Lemma 2.3 (Magnified equations and support)**.** *Let Γ, Ψ be the smooth compact field pairs of Lemma 2.1, with $y_* = (0, 1/2)$. Under (2.10), with $v' = J\nabla'\Psi'$ and $D' = \partial_t + v' \cdot \nabla'$, the exact reduced residuals satisfy*

$$D'\Gamma' = F^{\Gamma'}, \qquad D'L'\Psi' - W'\partial_{x_1'}(\Gamma'^2) = E^{\xi'}, \qquad F^\Gamma = \varepsilon_{\rm geom} F^{\Gamma'}, \quad E^\xi = E^{\xi'}. \tag{2.11}$$

Moreover $\xi = L'\Psi'$ and $\nabla_y \Gamma = \nabla'\Gamma'$.

Let $\mathcal{R}'$ be a fixed compact curl inverse in the primed coordinates. Suppose Γ', $L'\Psi'$, $F^{\Gamma'}$, and $\mathcal{R}'E^{\xi'}$ have support in $\overline{B}_{L_F}$ for their respective time intervals, and the initial datum, expressed in primed coordinates, has compact support there. If $0 < \varepsilon_{\rm geom} \le (16L_F)^{-1}$, their physical supports are contained in the torus of meridional radius $R = 4\varepsilon_{\rm geom} L_F \le 1/4$ about the unit ring. Their images lie inside the smaller torus of meridional radius $R/2$.

The normalized choice $\Gamma_0 = 1/2$ corresponds to $\Gamma_0' = (2\varepsilon_{\rm geom})^{-1}$. At the initial ring,

$$\mathsf{A}'(0) = \mathrm{Id}, \qquad 2W'(0)\Gamma_0' = 1.$$

If $|\Gamma' - \Gamma_0'| \le B_\Gamma$ on a tube and $2\varepsilon_{\rm geom} B_\Gamma \le 1/2$, then on that tube

$$2W'\Gamma' \in [\,\tfrac{1}{2}c_{\rm geom}^{-2}, \tfrac{3}{2}c_{\rm geom}^{-2}\,]. \tag{2.12}$$

On B_{L_F}, assuming $2\varepsilon_{\rm geom} L_F \le 1/2$, the full metric satisfies

$$\big\|\mathsf{A}' - \mathrm{Id}\big\| \le 4\varepsilon_{\rm geom} L_F, \qquad |\partial_{x_2'}^k c_{\rm geom}^{-1}| \le 2k!(4\varepsilon_{\rm geom})^k. \tag{2.13}$$

Proof. One has $\nabla_y = \varepsilon_{\rm geom}^{-1}\nabla'$, $v = \varepsilon_{\rm geom}v'$, and $L\Psi = L'\Psi'$. Transport is unchanged, while

$$W\partial_{y_1}\Gamma^2 = c_{\rm geom}^{-2}\varepsilon_{\rm geom}^{-1}\partial_{x_1'}(\varepsilon_{\rm geom}^2\Gamma'^2) = W'\partial_{x_1'}\Gamma'^2.$$

This proves (2.11). The zero integral of each reduced-vorticity residual is preserved, since $dy = \varepsilon_{\rm geom}^2 dx'$. If $\mathcal{R}'$ is a fixed compact curl inverse in the primed ball, its corresponding physical force pair is

$$(f^z, f^r/r)(y,t) = \varepsilon_{\rm geom}\mathcal{R}'E^{\xi'}(x',t), \qquad f^\varphi = \varepsilon_{\rm geom}F^{\Gamma'}/r.$$

Its y-curl is E^ξ, so the force and pressure argument of Lemma 2.1 applies. The force inverse is fixed once for all increments.

On the indicated ball, $z = \varepsilon_{\rm geom}x_1'$ and $r = \sqrt{1+2\varepsilon_{\rm geom}x_2'}$. Hence

$$|r-1| = \frac{2\varepsilon_{\rm geom}|x_2'|}{\sqrt{1+2\varepsilon_{\rm geom}x_2'}+1} \le 2\varepsilon_{\rm geom}|x_2'|, \qquad z^2+(r-1)^2 \le 4\varepsilon_{\rm geom}^2|x'|^2.$$

The closed ball therefore maps inside radius $R/2$. This proves the support statement for the state as well as the force; force support alone would not imply the later support of the state. The coupling formula follows by multiplying

$$2W'\Gamma' = c_{\rm geom}^{-2}[1+2\varepsilon_{\rm geom}(\Gamma'-\Gamma_0')].$$

Finally $c_{\rm geom} \ge 1/2$ and direct differentiation of its reciprocal gives (2.13). □

The same fixed scale relates the finite increments in the two charts. We use here the profile notation of Section 5. With primed material labels a', let

$$a = \varepsilon_{\rm geom}a', \qquad N_m = N_m'/\varepsilon_{\rm geom}, \qquad \ell_m = \varepsilon_{\rm geom}\ell_m'.$$

The phase $N_m p_m \cdot a = N_m' p_m \cdot a'$ is unchanged. The material maps obey $X_m(a,t) = y_* + \varepsilon_{\rm geom}X_m'(a',t)$, and thus $M_m = M_m'$, $H_m = H_m'$, $\zeta_m = \zeta_m'$, and $\kappa_m = \kappa_m'$, with composition in labels understood. Take corresponding principal data $T_m = \varepsilon_{\rm geom}T_m'$, $\widehat{\Omega}_m = \varepsilon_{\rm geom}\widehat{\Omega}_m'$ at insertion, and corresponding envelopes and activations $g_m(a) = g_m'(a')$, $w_m = w_m'$. In the suppressed-index notation of (5.2),

$$C = C', \qquad \mathfrak{l} = \varepsilon_{\rm geom}^{-1}\mathfrak{l}', \qquad h = h',$$
$$q = \varepsilon_{\rm geom}q', \qquad b = b', \qquad \varpi = \varepsilon_{\rm geom}^{-1}\varpi'. \tag{2.14}$$

The normalized pair and stream profiles transform as

$$(V_{m,n}, \widetilde{W}_{m,n}) = \varepsilon_{\rm geom}(V_{m,n}', \widetilde{W}_{m,n}'), \qquad U_{m,n} = \varepsilon_{\rm geom}^2 U_{m,n}'. \tag{2.15}$$

The central Ω_m, the parameters σ_{m-1}, the fibre matrix $\mathcal{B}_m$ and its propagator are unchanged: $d_m = d_m'$ and $c_m = 2(W'\Gamma'_{<m})\circ X_m'$. Substitution in the phase inverse and in the finite equations proves (2.15) successively in the correction level. Indeed (2.14) and the displayed label and carrier scalings give, in (5.11),

$$W_n = W_n', \qquad Z_n = Z_n', \qquad S_n^\Gamma = \varepsilon_{\rm geom}S_n^{\Gamma'}, \qquad S_n^\xi = S_n^{\xi'}.$$

Thus both components of the normalized pair forcing scale by $\varepsilon_{\rm geom}$, as does its insertion data. The phase mean and mean-zero primitive commute with this fixed change. The explicit label-vector primitives satisfy

$$Q_n^\Gamma = \varepsilon_{\rm geom}^2 Q_n^{\Gamma'}, \qquad Q_n^\xi = \varepsilon_{\rm geom}Q_n^{\xi'}.$$

Their label divergences have the required residual scalings. Both terminal composite levels are retained.

Every primed estimate is converted back with this one fixed scale. A label derivative of order k costs $\varepsilon_{\rm geom}^{-k}$; time orders do not change. On the fixed off-axis support, physical-coordinate derivatives also have fixed geometry-dependent bounds. These factors preserve every proved mixed-norm summability estimate without changing its carrier exponents or allowing a different scale for each order. The construction must still establish those estimates for the complete primed fields.

3. A wave and its amplification

We first describe the principal cancellation; localization and lower order terms will be treated in Section 5. Let $m \geq 1$ index an increment with frequency $N_m > 0$ and material radius $\ell_m > 0$. Suppose the smooth older fields $\Gamma_{<m}, \Psi_{<m}$ have already been prescribed on a finite interval, and set $v_{<m} = J\nabla\Psi_{<m}$. For a starting time τ_m, let

$$\partial_t X_m(a,t) = v_{<m}(X_m(a,t),t), \qquad X_m(a,\tau_m) = Z(\tau_m) + a, \qquad A_m(\cdot,t) = X_m(\cdot,t)^{-1}. \tag{3.1}$$

Here Z is the point carried by the full constructed velocity, as specified in Section 4. Fix $p_m \in \mathbb{R}^2$ with $|p_m| = 1$, and write

$$s_m = N_m p_m \cdot A_m(y,t), \quad H_m = (D_a X_m)^{-1}, \quad \zeta_m = H_m^T p_m, \quad \kappa_m = \zeta_m^T \mathsf{A}(X_m)\zeta_m. \tag{3.2}$$

The last three quantities are functions of (a,t). Let F be a smooth odd 2π-periodic function, with $F(s) = s$ near zero, and let P be its mean-zero periodic primitive. For a smooth envelope g_m supported in $|a| < \ell_m$, equal to one near zero, consider

$$\gamma_{m,0} = (w_m T_m F g_m)^\natural, \qquad \psi_{m,0} = (w_m B_m P g_m)^\natural, \qquad \Omega_m = N_m^2 \kappa_m B_m. \tag{3.3}$$

The superscript $\natural$ substitutes $(s_m, A_m(y,t), t)$ into a profile. The smooth activation $w_m(t)$ is zero before τ_m and eventually one. The amplitudes T_m, B_m depend on (a,t).

Let $d_m = (J\zeta_m)\cdot(\nabla_y\Gamma_{<m})\circ X_m$, $c_m = 2(W\Gamma_{<m})\circ X_m$, and choose a positive constant σ_{m-1} on this increment. Define $\widehat{\Omega}_m = \sigma_{m-1}\Omega_m/N_m$. The principal amplitude law is

$$\partial_t \begin{pmatrix} T_m \\ \widehat{\Omega}_m \end{pmatrix} = \mathcal{B}_m(a,t)\begin{pmatrix} T_m \\ \widehat{\Omega}_m \end{pmatrix}, \qquad \mathcal{B}_m = \begin{pmatrix} 0 & -d_m/(\sigma_{m-1}\kappa_m) \\ \sigma_{m-1}c_m\zeta_{m,1} & 0 \end{pmatrix}. \tag{3.4}$$

Indeed, the leading circulation coefficient is $\partial_t T_m + N_m B_m d_m$, and the leading vorticity coefficient is $\partial_t\Omega_m - N_m c_m \zeta_{m,1} T_m$. Both vanish by (3.4). Derivatives of the envelope, material amplitudes and metric remain in the residual. On a time interval where the two off-diagonal entries are fixed and positive, the growing eigenvalue is the square root of their product. This is the elementary amplification in [1, Lemma 2.1], with the physical Euler coefficient retained.

Write $R_m = (T_m, \widehat{\Omega}_m)$ and let $\mathcal{P}_m(a;t,t')$ be the propagator of (3.4). For nonzero growing initial data define the central weight $\mathfrak{w}_m(t) = |R_m(0,t)|/|R_m(0,\tau_m)|$.

3.1. An estimate for a family of amplitude equations.

Fix one increment and suppress its index, with $\sigma = \sigma_{m-1}$ and $R = (T, \widehat{\Omega})$. All derivative orders below are total orders in the indicated variables. The following assumptions concern supplied older fields on a compact interval. Write $I = [\tau, t_1]$, $B = \{|a| \leq \ell\}$, $\lambda = \ell^{-1}$ and $\Pi \geq \max(1,\lambda)$. A time derivative is taken at fixed label. Let $\mathcal{B}(a,t)$ be the matrix in (3.4), let $\mathcal{P}(a;t,s)$ be its propagator, and put $R(a,t) = \mathcal{P}(a;t,\tau)R^{\mathrm{in}}(a)$. Assume $R^{\mathrm{in}}(0) \neq 0$. Its central solution never vanishes, by uniqueness for the linear equation. Put

$$Y_* = \sup_I |R(0,t)|, \qquad \rho(t) = |R(0,t)|/Y_* \in (0,1].$$

The ratio $\rho(t)/\rho(s)$ equals the ratio of the previously defined weights $\mathfrak{w}_m(t)/\mathfrak{w}_m(s)$.

Here are precise supplied assumptions through an integer order $r \geq 1$. The positive constants $C_R, c_b, c_{\alpha,b}, e_\alpha$ are specified before applying the assertion; increase them to at least one.

$$\begin{gathered}
\|\mathcal{P}(0;t,s)\| \leq C_R\rho(t)/\rho(s), \qquad \|\partial_t^b \mathcal{B}(0,t)\| \leq c_b \Pi^{b+1}, \quad b<r, \\
\ell^{|\alpha|}\|\partial_a^\alpha \mathcal{B}(\cdot,t)\|_\infty \leq c_{\alpha,0}\mathfrak{m}(t), \qquad c_* := \int_I \mathfrak{m}, \quad 1 \leq |\alpha| \leq r, \\
\ell^{|\alpha|}\|\partial_a^\alpha \partial_t^b \mathcal{B}\|_\infty \leq c_{\alpha,b}\Pi^{b+1}\mathfrak{n}, \quad 1 \leq |\alpha|,\ |\alpha|+b \leq r, \quad 0 \leq \mathfrak{n} \leq 1, \\
\ell^{|\alpha|}\|\partial_a^\alpha R^{\text{in}}\|_\infty \leq e_\alpha c_{\text{dat}}|R^{\text{in}}(0)|, \quad 1 \leq |\alpha| \leq r, \\
c_* + c_{\text{dat}} \leq \frac{\log 2}{6C_R^2}.
\end{gathered} \tag{3.5}$$

At first label order use the norm of the full differential, and normalize $c_{1,0} = e_1 = 1$ by putting these factors in $\mathfrak{m}, c_{\text{dat}}$. The same first-order normalization is understood in the remaining coefficient bounds. No integral-only bound for a time derivative is assumed.

Lemma 3.1 (Weighted material amplitude estimates)**.** *Under* (3.5)*, for $s \leq t$ in I,*

$$\begin{gathered}
\|\mathcal{P}(a;t,s)\| \leq 2C_R\rho(t)/\rho(s), \\
\|\mathcal{P}(a;t,s) - \mathcal{P}(0;t,s)\| \leq 2C_R^2 c_*\rho(t)/\rho(s), \\
|R(a,t) - R(0,t)| \leq q_*|R(0,t)|, \qquad q_* = 3C_R^2(c_* + c_{\text{dat}}) < \tfrac{1}{2}, \\
\left|\log\frac{|R(a,t)|}{|R(0,t)|}\right| \leq 6C_R^2(c_* + c_{\text{dat}}).
\end{gathered} \tag{3.6}$$

There are explicitly recursive order constants such that, for $|\alpha| + b \leq r$,

$$\begin{aligned}
\|\partial_a^\alpha \partial_t^b \mathcal{P}(a;t,s)\| &\leq C_{\alpha,b}\lambda^{|\alpha|}\Pi^b(c_* + \mathbf{1}_{b>0}\mathfrak{n})^{\min(1,|\alpha|)}\rho(t)/\rho(s), \\
|\partial_a^\alpha \partial_t^b R(a,t)| &\leq E_{\alpha,b}\lambda^{|\alpha|}\Pi^b(c_* + \mathbf{1}_{b>0}\mathfrak{n} + c_{\text{dat}})^{\min(1,|\alpha|)}|R(a,t)|.
\end{aligned} \tag{3.7}$$

The constants depend only on the displayed order constants, C_R and the requested orders, not on $N, \sigma, \ell, \Pi, |I|$ separately.

Proof. The mean-value formula gives $\|\mathcal{B}(a,t) - \mathcal{B}(0,t)\| \leq \mathfrak{m}(t)$. Variation of constants around the central equation gives

$$\mathcal{P}(a;t,s) = \mathcal{P}(0;t,s) + \int_s^t \mathcal{P}(0;t,u)[\mathcal{B}(a,u) - \mathcal{B}(0,u)]\mathcal{P}(a;u,s)\,du.$$

After division by $\rho(t)/\rho(s)$, its supremum $\mathfrak{z}$ on $[s,t]$ satisfies $\mathfrak{z} \leq C_R + C_R c_*\mathfrak{z}$, so $\mathfrak{z} \leq C_R/(1 - C_R c_*) \leq 2C_R$. Inserting this once more gives the second estimate. The initial-data difference is at most $c_{\text{dat}}|R^{\text{in}}(0)|$, and its size is at most $(1 + c_{\text{dat}})|R^{\text{in}}(0)|$. Splitting the difference of the two propagated data therefore gives its relative bound $2C_R^2 c_*(1 + c_{\text{dat}}) + C_R c_{\text{dat}} \leq q_*$. Since $q_* < 1/2$, $|\log(1+x)| \leq 2|x|$ on $|x| \leq 1/2$ proves the last estimate, including nonvanishing of every label solution.

For clarity, all derivative constants can be generated as follows. Put $C_{0,0} = 2C_R$ and for $\alpha \neq 0$ set

$$C_{\alpha,0} = 2C_R \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} c_{\beta,0} C_{\alpha-\beta,0}.$$

Indeed, $G_\alpha = \partial_a^\alpha \mathcal{P}$ has zero initial data and source $\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}(\partial_a^\beta \mathcal{B})G_{\alpha-\beta}$. The weighted Duhamel formula integrates $\mathfrak{m}$ exactly once; all additional factors $c_* \leq 1$ may be discarded. This proves the first line of (3.7) when $b = 0$.

For time derivatives define noncommutative matrix polynomials $P_0 = \text{Id}$, $P_{b+1} = \partial_t P_b + P_b\mathcal{B}$. Their words $\prod_{i=1}^j \partial_t^{h_i}\mathcal{B}$ satisfy $\sum_i(h_i+1) = b$, and their coefficient sum is at most $b!$. Consequently $\partial_t^b \mathcal{P} =$

$P_b\mathcal{P}$. The mean-value bound and the pointwise label hypothesis give $\|\partial_t^h \mathcal{B}(a,t)\| \le (c_h + c_{1,h})\Pi^{h+1}$. Let $A_{u,v}$ be the maximum of these constants and of $c_{\gamma,j}$ with $1 \le |\gamma| \le u$, $j \le v$. Then an admissible bound for these polynomials is

$$L_{b,0} = b!A_{0,b-1}^b, \qquad L_{b,u} = b!\max(1,b)^u A_{u,b-1}^b \quad (u > 0),$$

with $L_{0,0} = 1, L_{0,u} = 0$ for $u > 0$. Every differentiated word with a label derivative contains a factor $\mathfrak{n}$; its label multinomial sum is at most $b^{|\alpha|}$. Leibniz in $P_b\mathcal{P}$ thus gives

$$C_{\alpha,b} = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} L_{b,|\beta|} C_{\alpha-\beta,0}.$$

The differentiated factor either is in the propagator, contributing c_*, or is in these polynomials, contributing $\mathfrak{n}$. This proves the claimed separate small factors. Finally differentiate $\mathcal{P}(a;t,\tau)R^{\text{in}}(a)$ and apply the datum bounds. Division by $|R(a,t)| \ge |R(0,t)|/2$ gives, for example,

$$E_{\alpha,b} = 4\sum_{\beta \le \alpha} \binom{\alpha}{\beta} C_{\alpha-\beta,b} \max(1, e_\beta), \qquad e_0 = 1.$$

For $\alpha = 0$ the sharper pointwise polynomial estimate applies directly to $R(a,t)$, so no initial-weight lower bound is needed. □

The central weighted hypothesis has a concrete sufficient condition. Partition I, in this order, into growth, transition, holding and the entire remaining interval. The central entries $\widehat{a}, \widehat{b}$ are positive C^1 functions on growth. Put $\gamma = \sqrt{\widehat{a}\widehat{b}}$, $r_e = \sqrt{\widehat{a}/\widehat{b}}$ and $\epsilon_e = \dot{r}_e/(2r_e)$. Assume the nonzero initial vector satisfies $R_2(0,\tau) = \sqrt{\widehat{b}(\tau)/\widehat{a}(\tau)}R_1(0,\tau)$, and that $r_0 = \sup\max(r_e, r_e^{-1}) < \infty$ and $2\gamma\rho_* > |\epsilon_e|(1+\rho_*^2)$ for some $0 < \rho_* < 1$. On holding assume $\widehat{b} = 0$ and $R_2(0,t) = 0$. If

$$E = \int_{\text{growth}} |\epsilon_e|, \quad S = \int_{\text{transition}} \|\mathcal{B}(0,t)\|, \quad H_0 = \int_{\text{holding}} |\widehat{a}|, \quad D_0 = \int_{\text{remaining}} (|\widehat{a}| + |\widehat{b}|),$$

then one can take

$$C_R = r_0^2\sqrt{1+\rho_*^2}(1+H_0)\exp((1+\rho_*)E + 2S + 2D_0).$$

These are exactly the hypotheses and conclusion of [1, Lemma 2.13], applied only to the central two-dimensional ODE. Its proof uses the invariant cone for $u_\pm = r_e^{-1/2}R_1 \pm r_e^{1/2}R_2$, followed by forward and backward estimates on the remaining three intervals. The physical Euler coefficient is still the coefficient in (3.4).

Positive initial entries $\widehat{a}(a), \widehat{b}(a)$ give the growing datum $R^{\text{in}}(a) = \theta(1, \sqrt{\widehat{b}/\widehat{a}})$ for a fixed $\theta \ne 0$. The datum hypothesis follows by differentiating this expression, provided the entries have positive lower bounds and the resulting relative derivatives meet its stated smallness condition. It is not a consequence of positivity alone. On an activation interval, if $|R(0,t)| \le Y_{\text{act}}$ and $|R_2(0,t)| \le B_{\text{act}}|R_1(0,t)|$, put $A_{\text{act}} = \sqrt{1+B_{\text{act}}^2}$. If $A_{\text{act}}q_* < 1$, then

$$|R_1(a,t)| \ge (1 - A_{\text{act}}q_*)|R_1(0,t)|, \qquad \frac{|R_2(a,t)|}{|R_1(a,t)|} \le \frac{B_{\text{act}} + A_{\text{act}}q_*}{1 - A_{\text{act}}q_*}.$$

These assertions follow componentwise from (3.6). The same derivative estimates apply on this interval with Y_{act} in place of Y_*.

For later use, all profile suprema are on $\mathbb{S}^1 \times \overline{B}_\ell \times I$, with smooth data on a containing collar. Define the weighted profile norm

$$[Q]_r = \max_{i+|\alpha|+b \le r} \sup_{s,a,t} \rho(t)^{-1}\lambda^{-|\alpha|}\Pi^{-b}|\partial_s^i\partial_a^\alpha\partial_t^b Q|. \tag{3.8}$$

For coefficients the same norm without s and without ρ^{-1} is denoted $[c]_r^c$. The weight is not differentiated in this definition. Leibniz gives $[PQ]_r \le 2^r[P]_r[Q]_r$, since $\rho^2 \le \rho$. Coefficient multiplication has the same rule, while $\partial_s, \nabla_a, \partial_t$ cost respectively $1, \lambda, \Pi$ and one order. The phase operators of Section 5 have the same bounds in this norm.

Lemma 3.2 (Inhomogeneous amplitude estimate). *Put $\mathcal{T} = \max(|I|, \Pi^{-1})$. If $y_t = \mathcal{B}y + f$ with zero initial data, then*

$$[y]_r \le D_r \mathcal{T}[f]_r.$$

One admissible finite recursion for D_r is the following. Write $c^\sharp_{u,j} = \max_{|\alpha|=u} c_{\alpha,j}$ for $u > 0$, $c^\sharp_{0,j} = c_j + c_{1,j}$, and set

$$d_{0,u} = 2C_R\left[1 + \sum_{v=1}^{u}\binom{u}{v} c^\sharp_{v,0} d_{0,u-v}\right], \qquad d_{0,0} = 2C_R,$$

where the first formula is used for $u \ge 1$, and

$$d_{b+1,u} = 1 + \sum_{j=0}^{b}\sum_{v=0}^{u}\binom{b}{j}\binom{u}{v} c^\sharp_{v,j} d_{b-j,u-v}, \qquad D_r = \max_{b+u\le r} d_{b,u}.$$

Proof. Differentiate first only in phase and label. The differentiated equation contains $\partial_s^i\partial_a^\alpha f$ and the label-Leibniz terms $(\partial_a^\beta \mathcal{B})\partial_s^i\partial_a^{\alpha-\beta} y$. Their weighted integrals cost respectively $|I| \le \mathcal{T}$ and $\int \mathfrak{m} = c_* \le 1$, giving the recursion for $d_{0,u}$. For positive time order differentiate the equation itself. The source term costs $\Pi^{-1} \le \mathcal{T}$, and each matrix derivative costs $c^\sharp_{v,j}\Pi^{j+1}\lambda^v$. The displayed time recursion follows by the full mixed Leibniz formula. There is no loss of an additional source derivative and no factor exponential in $\Pi|I|$. ☐

Lemma 3.3 (Material dependence of the amplitudes). *For the selected sequence constructed in Proposition 12.3, take $R_m(a, \tau_m)$ on the growing eigenline at each label, with controlled relative label variation and positive off-diagonal coefficients at insertion. The amplitudes satisfy (3.6) and (3.7) on their material balls at the admitted orders returned by Proposition 11.1, uniformly over finite restrictions of their preterminal lifetimes. The estimates cost one factor ℓ_m^{-1} per material derivative and compare $|R_m(a,t)|$ with $|R_m(0,t)|$. Propagator bounds use the weight $\mathfrak{w}_m(t)/\mathfrak{w}_m(t')$. For each fixed layer, boundedness at every fixed order is the separate conclusion of Proposition 11.1, with constants depending on the layer and the order.*

Proof. Proposition 12.3 constructs one infinite selected history satisfying the scalar conditions of Proposition 11.1. The latter returns (3.5), including the central weighted history and relative growing data, on every finite history satisfying its scalar conditions. Lemma 3.1 then gives the admitted relative estimates on each finite restriction, with the same lifetime upper integrals. Their bounds persist on the union. For each fixed layer, all-order finiteness follows from the separate fixed-layer argument in Section 10, whose clock-moment and eventual-admission hypotheses are verified in Proposition 12.3. ☐

4. Material coordinates and lifetime bounds

The full older flow is needed because material-dependent amplitudes produce nonlinear older velocities. Its linear deformation at one point is useful for estimates, but the oscillations are defined by the full inverse map. Set $Y_m(t) = X_m(0,t)$ and $M_m = D_aX_m(0,t)$. Then

$$\dot{Y}_m = v_{<m}(Y_m, t), \qquad \dot{M}_m = D_mM_m, \qquad D_m = \nabla_y v_{<m}(Y_m, t), \qquad \det M_m = 1. \tag{4.1}$$

There is a uniquely defined nonlinear remainder Ξ_m such that

$$X_m(a,t) = Y_m(t) + M_m(t)(a + \Xi_m(a,t)), \qquad \Xi_m(0,t) = 0, \quad D_a\Xi_m(0,t) = 0.$$

The determinant identity follows from $\operatorname{div}_y v_{<m} = 0$ and $M_m(\tau_m) = \mathrm{Id}$.

On each finite continuation let

$$\dot Z(t) = v(Z(t), t), \qquad Z(0) = y_*, \qquad v = v^b + \sum_j v_j, \tag{4.2}$$

where the sum runs over the finitely many activated complete increments defined in (2.8). Its axisymmetric lift is the moving circle used to evaluate the gradient. The starting rule (3.1) gives $Y_m(\tau_m) = Z(\tau_m)$. For all subsequent times on a finite continuation,

$$\frac{d}{dt} A_m(Z(t), t) = D_y A_m(Z(t), t) \sum_{j \ge m} v_j(Z(t), t). \tag{4.3}$$

This follows by subtracting the transport equation for A_m from the derivative along Z. The bound needed for nesting concerns this displacement and the phase $N_m p_m \cdot A_m(Z(t), t)$.

The central velocity gradient also distinguishes the elliptic symbol from the Euclidean phase length. For any streamfunction profile U, its pure phase contribution is

$$N_m^2 U_{ss} J \zeta_m \otimes \zeta_m = \frac{|\zeta_m|^2}{\kappa_m} (N_m^2 \kappa_m U_{ss}) J e_m \otimes e_m, \qquad e_m = \zeta_m / |\zeta_m|. \tag{4.4}$$

This identity is evaluated at the same (s, a, t), and gives the indicated contribution at $y = X_m(a, t)$ after $s = N_m p_m \cdot a$. Material derivatives of U and derivatives of the inverse map give the other Hessian terms. All these terms enter the complete center equations.

4.1. **Bounds for the nonlinear material map.** We give sufficient estimates on a containing region rather than on an unknown image. Suppress the index m. On I suppose the central path Y and matrix M solve (4.1). On $Y(t) + M(t) B_{2\ell}$ define

$$\widetilde v(a', t) = M^{-1}\{v_<(Y + Ma', t) - v_<(Y, t) - DMa'\}.$$

Assume this region lies in the fixed off-axis domain of the supplied older fields. In particular $\widetilde v(0, t) = D_{a'} \widetilde v(0, t) = 0$. Let $r_F \ge 2$. Bounds below use Fréchet spatial derivative norms; coordinate derivative norms satisfy them after fixed dimensional factors. Take $C_0 = 1$ by enlarging the size letters if needed. Throughout this subsection $K \ge 1$, $0 < \zeta_- \le \zeta_+$, and all size and rate letters are nonnegative; any reciprocal size has a strictly positive lower window. Assume for $b + n \le r_F$

$$\begin{gathered} \|\partial_t^b M^{\pm 1}\| \le C_b K \Pi^b, \quad \|\partial_t^b D\| \le C_b \Pi^{b+1}, \quad |\partial_t^b \dot Y| \le C_b V_c \Pi^b, \\ |\partial_t^b (M^{-T} p)| \le C_b \zeta_+ \Pi^b, \qquad \zeta_- \le |M^{-T} p| \le \zeta_+, \\ \|D_{a'}^n \widetilde v(\cdot, t)\| \le C_{0,n}^{\mathrm{nl}} \eta \mathfrak{d}(t) \ell^{1-n}, \quad 0 \le \mathfrak{d} \le \Pi, \quad \int_I \mathfrak{d} \le 1, \\ \|\partial_t^b D_{a'}^n \widetilde v\| \le C_{b,n}^{\mathrm{nl}} \eta \Pi^{b+1} \ell^{1-n} \quad (b \ge 1), \qquad \eta \max(C_{0,0}^{\mathrm{nl}}, C_{0,1}^{\mathrm{nl}}, 1) \le \tfrac{1}{8}. \end{gathered} \tag{4.5}$$

The bounds for $M, D, \dot Y$ are needed only through the derivative orders appearing in the formulas below; assuming them through r_F is harmless. Let $\eta(t) = \eta \int_\tau^t \mathfrak{d}$.

Lemma 4.1 (Small nonlinear deformation)**.** *There are finite constants $\mathfrak{x}_{b,n}$, defined by the partition recursion in the proof, such that the exact older flow exists for every label in B_ℓ throughout I, and*

$$\begin{gathered} |D_a^n \Xi(a, t)| \le \mathfrak{x}_{0,n} \eta(t) \ell^{1-n}, \\ |\partial_t D_a^n \Xi(a, t)| \le \mathfrak{x}_{1,n} \eta \mathfrak{d}(t) \ell^{1-n}, \\ |\partial_t^b D_a^n \Xi| \le \mathfrak{x}_{b,n} \eta \Pi^b \ell^{1-n} \quad (b \ge 1), \qquad b + n \le r_F. \end{gathered} \tag{4.6}$$

Moreover $|a + \Xi| \le 9\ell/8$, $\|D_a \Xi\| < 1/4$, and

$$D_a X = M(\mathrm{Id} + D_a \Xi), \qquad H = (\mathrm{Id} + J (D_a \Xi)^T J^T) M^{-1}. \tag{4.7}$$

Thus $\partial_t^b D_a^n (D_a X)^{\pm 1}$ *is bounded by* $\mathfrak{c}_{b,n} K \Pi^b \ell^{-n}$ *for* $b+n \le r_F - 1$*; after subtraction of* $M^{\pm 1}$ *it has an additional factor* η*. At* $b=0$ *this factor is* $\eta(t)$*.*

Proof. Differentiating $X = Y + M(a+\Xi)$ gives $\Xi_t = \widetilde{v}(a+\Xi,t)$ and zero initial data. Until exit from $B_{2\ell}$, $|\Xi| \le C^{\mathrm{nl}}_{0,0}\eta(t)\ell \le \ell/8$, excluding exit. The Jacobian $\mathcal{G} = D_a(a+\Xi)$ solves $\mathcal{G}_t = D_{a'}\widetilde{v}(a+\Xi,t)\mathcal{G}$, so $\|\mathcal{G} - \mathrm{Id}\| \le e^{C^{\mathrm{nl}}_{0,1}\eta(t)} - 1 < 1/4$. This proves the low-order claims without replacing an integral by a lifetime supremum. Smooth ODE dependence then permits differentiation.

Here is a fully finite definition of the constants. Scale a by ℓ and time by Π^{-1}, so the differentiated vector field has bounds $C^{\mathrm{nl}}_{b,n}\eta$; at time order zero retain the integrable factor $\mathfrak{d}/\Pi$. Put $\mathfrak{x}_{0,0} = C^{\mathrm{nl}}_{0,0}$, $\mathfrak{x}_{0,1} = e^{1/8}C^{\mathrm{nl}}_{0,1}$. For $n \ge 2$, let $\mathrm{Part}_*(n)$ be the partitions of $\{1,\dots,n\}$ other than the one-block partition, and set

$$\mathfrak{x}_{0,n} = e^{1/8} \sum_{\pi \in \mathrm{Part}_*(n)} C^{\mathrm{nl}}_{0,|\pi|} \prod_{B\in\pi} \mathfrak{a}_{|B|}, \qquad \mathfrak{a}_1 = 5/4, \quad \mathfrak{a}_j = \mathfrak{x}_{0,j} \ (j \ge 2).$$

All j on the right are smaller than n. The excluded term is $D_{a'}\widetilde{v}\, D_a^n \Xi$, treated by its integrating factor; all others integrate one factor $\eta\mathfrak{d}$. Induction and $\eta \le 1$ give the first estimate with its time-resolved factor.

For mixed derivatives use distinguishable differentiation positions. If a set S contains n label differentiation positions and $b-1$ time differentiation positions, the mixed chain rule for $\widetilde{v}(G(a,t),t)$, with $G = a + \Xi$, is

$$\partial^S(\widetilde{v}(G,t)) = \sum_{T\subset S_t} \sum_{\pi\in\mathrm{Part}(S\setminus T)} (\partial_t^{|T|} D^{|\pi|}\widetilde{v})(G,t)[\partial^{B_1}G, \dots, \partial^{B_{|\pi|}}G]. \tag{4.8}$$

Set $\mathfrak{a}_{0,1} = 5/4$ and $\mathfrak{a}_{h,j} = \mathfrak{x}_{h,j}$ for every other nonempty block type. Define

$$\mathfrak{x}_{b,n} = \sum_{T\subset S_t} \sum_{\pi\in\mathrm{Part}(S\setminus T)} C^{\mathrm{nl}}_{|T|,|\pi|} \prod_{B\in\pi} \mathfrak{a}_{b(B),n(B)} \quad (b \ge 1).$$

The empty product is one. Every block has total order at most $b+n-1$, so this is a recursion, not an implicit bound. For $b=1$ all differentiation positions are spatial, and the common factor is $\eta\mathfrak{d}(t)$; for $b>1$ the common factor is bounded by $\eta\Pi^b$. Products of further η factors are at most one. The powers of ℓ and Π telescope in each term, proving (4.6). Formula (4.8) itself follows by adding one derivative: it either differentiates the explicit time variable of $\widetilde{v}$, creates a new singleton block, or enlarges one block.

Both D_aX and M have determinant one because $v_<$ is divergence free. Hence $\det(\mathrm{Id} + D_a\Xi) = 1$, and its inverse is its two-dimensional adjugate, namely $\mathrm{Id} + J(D_a\Xi)^T J^T$. This proves (4.7). Leibniz gives, for example, $\mathfrak{c}_{b,n} = C_b \mathbf{1}_{n=0} + \sum_{j=0}^b \binom{b}{j} C_j \mathfrak{x}_{b-j,n+1}$, with enlargement for the direct matrix. Every term of either matrix difference contains a derivative of Ξ, proving the final assertion. □

For a coefficient write $c \in \mathcal{C}(S,\Lambda;r)$ when

$$|\partial_t^b D_a^n c| \le S c_{b,n} \Pi^b \Lambda^{\min(1,n)} \ell^{-(n-1)_+}, \qquad b+n \le r. \tag{4.9}$$

The finite list $(c_{b,n})$ is part of its bound. The class retains the first-label variation instead of bounding every derivative ℓ^{-1}. The closure rules below use elementary input rates satisfying $\Lambda\ell \le 1$. Output rates may be fixed finite sums of these rates, with their multiplicities absorbed in the order-dependent constants. Products add first-label rates and multiply sizes, with the binomial convolution of constants. This follows by Leibniz: if both factors are differentiated in label, $\Lambda_1\Lambda_2\ell \le \Lambda_1 + \Lambda_2$ when $\Lambda_i\ell \le 1$. If $c \ge S_- > 0$, its inverse has size S_-^{-1} and the same rate. For a set $\mathcal{S}$ of b time and n label differentiation positions its constants are

$$c^{\mathrm{inv}}_{b,n} = \sum_{\pi\in\mathrm{Part}(\mathcal{S})} |\pi|!(S/S_-)^{|\pi|} \prod_{B\in\pi} c_{b(B),n(B)}. \tag{4.10}$$

This is the chain rule for x^{-1}. Exponentials of a bounded scalar use the same partition formula without $|\pi|!$, with the appropriate powers of their fixed exponent. These proofs are the product and composition arguments in [1, Lemma 4.2], with label differentiation positions added.

If $\Lambda\ell \le 1$ and $h(a',t) \in \mathcal{C}(S,\Lambda;r_F)$ on $B_{2\ell}$, then $h(a+\Xi,t)$ has the same size and rate through r_F-1, with constants given by (4.8) and the just defined $\mathfrak{a}_{b,n}$. For a term with $q \ge 1$ blocks, its label scale is $\Lambda\ell^{1-q}\ell^{q-n} = \Lambda\ell^{1-n}$ if $n>0$. If $n=0$, its extra factor $\Lambda\ell$ is at most one. This proves the assertion and makes its constants explicit. At time order zero one may keep time-dependent first-label rates pointwise, including $\eta(t)$ from Lemma 4.1; no time supremum is needed.

For the following coefficients use the suppressed-index definitions

$$q = \Gamma_< \circ X, \quad b = \xi_< \circ X, \quad \varpi = W \circ X, \quad h = He_1, \quad C = H\mathsf{A}(X)H^T,$$
$$\mathfrak{l} = \operatorname{div}_a C, \qquad c = 2\varpi q, \qquad d = Jp\cdot\nabla_a q.$$

They are also collected in (5.2). To apply this calculus assume the following older scalar bounds in the rigid coordinates $Y+Ma'$. For $\phi = \Gamma_<, \xi_<$ let $h_\phi = D_{a'}[\phi(Y+Ma',t)]$. Suppose

$$h_\Gamma \in \mathcal{C}(G_0,\mu_\Gamma;r_F-1), \qquad h_\xi \in \mathcal{C}(X_0,\mu_\xi;r_F-1), \qquad \mu_\phi\ell \le 1.$$

Assume $\Gamma_- \le \Gamma_<(Y+Ma',t) \le \Gamma_+$, with $\Gamma_- > 0$, $G_0\ell \le \Gamma_-$, and central time derivatives $|\partial_t^b\Gamma_<(Y,t)| \le C_b\Gamma_+\Pi^b$. For $u(a',t) = \log(2(Y+Ma')_2)$ assume all positive-order bounds of $\mathcal{C}(1,\Lambda_{\mathrm{met}};r_F)$, where $\Lambda_{\mathrm{met}}\ell \le 1$, and the metric window $a_-\operatorname{Id} \le \mathsf{A}(Y+Ma') \le a_+\operatorname{Id}$, where $0 < a_- \le a_+$. These are conditions on the supplied older fields on a known region.

Put $\mathfrak{x}_* = 1 + \max_{b+n\le r_F}\mathfrak{x}_{b,n}$ and require $\eta\mathfrak{x}_* \le 1/4$ and $K\eta\mathfrak{x}_* \le \zeta_-/2$. Let

$$\Lambda_\Xi = \mathfrak{x}_*\eta\lambda, \quad \Lambda_\zeta = K\Lambda_\Xi/\zeta_-, \quad \Lambda_\Gamma = G_0/\Gamma_-, \quad \Lambda_\kappa = 2\Lambda_\zeta + \Lambda_{\mathrm{met}}.$$

The following table gives bounds through order r_F-2; its constants are exactly those produced by the displayed product, partition and reciprocal rules. In the row for q the central value and the integral of its rigid gradient supply the pure-time bounds.

(4.11)

coefficient	size	first-label rate
$H,\ h$	K	Λ_Ξ
ζ	ζ_+	Λ_ζ
$\mathsf{A}(X),\ \mathsf{A}(X)^{-1}$	$a_+,\ a_-^{-1}$	Λ_{met}
$\varpi,\ \varpi^{-1}$	$W_+,\ W_-^{-1}$	$2\Lambda_{\mathrm{met}}$
C	K^2a_+	$2\Lambda_\Xi + \Lambda_{\mathrm{met}}$
$\mathfrak{l} = \operatorname{div}_a C$	$K^2a_+(2\Lambda_\Xi + \Lambda_{\mathrm{met}})$	λ
$\kappa,\ \kappa^{-1}$	$\kappa_+,\ \kappa_-^{-1}$	Λ_κ
$\nabla_a q,\ d$	G_0	$\mu_\Gamma + \Lambda_\Xi$
$\nabla_a b$	X_0	$\mu_\xi + \Lambda_\Xi$
q	Γ_+	Λ_Γ
$c = 2\varpi q$	$2W_+\Gamma_+$	$2\Lambda_{\mathrm{met}} + \Lambda_\Gamma$

Here $W_- = \inf W$ and $W_+ = \sup W$ on the containing region, and $\kappa_- = a_-\zeta_-^2/4$, $\kappa_+ = 9a_+\zeta_+^2/4$ are valid windows. Indeed $\zeta = M^{-T}p + M^{-T}J(D_a\Xi)J^Tp$, whose error has size at most $K\mathfrak{x}_*\eta \le \zeta_-/2$. The same decomposition gives its time and first-label bounds; it is important that the central covector has its own ζ_+ bound. The gradient of a pulled-back scalar is $(\operatorname{Id}+D_a\Xi)^T h_\phi(a+\Xi,t)$, which proves the gradient rows. The remaining rows follow by the indicated products; the reciprocal row uses its positive lower bound. The divergence row requires one further label derivative of H, and therefore two derivatives of Ξ.

For this metric the divergence has the sharper identity

$$\mathfrak{l}_i = \sum_{j,k}\mathsf{A}_{jj}(X)H_{kj}\partial_{a_k}H_{ij}. \tag{4.12}$$

Indeed, in $\partial_{a_k} C_{ki}$ the derivatives of the first H vanish by Piola, and the metric contribution is $\sum_k H_{kj}\partial_{a_k}[\mathsf{A}_{jj}(X)] = \partial_{y_j}\mathsf{A}_{jj} = 0$. Its size is therefore bounded by a finite constant times $a_+K^2\Lambda_\Xi$, with subsequent label derivatives at rate λ. At time order zero this refinement retains the time-resolved factor $\eta(t)$.

There is also a bound for the physical time differentiation coefficient. Set $\mathfrak{v} = H(v_< \circ X)$, so the vector ν of (5.31) is $-\mathfrak{v}\circ A$. Writing $S = \mathrm{Id} + D_a\Xi$ and $E = M^{-1}DM$ gives the exact identity

$$\mathfrak{v} = S^{-1}M^{-1}\dot{Y} + S^{-1}E(a+\Xi) + S^{-1}\Xi_t.$$

The calculus therefore bounds $\mathfrak{v}$ through $r_F - 1$ with size $V_X = KV_c + K^2\Pi\ell + \eta\Pi\ell$, first-label rate λ, and the displayed finite input constants. In particular the center drift is retained. Spatial differentiation of an inverse-map coefficient is $H^T\nabla_a$, and its material derivative is ∂_t at fixed label. Iterating these rules gives all spatial and material derivatives of D_yA; ordinary time derivatives use $\partial_t - \mathfrak{v}\cdot\nabla_a$. Thus these bounds concern the full inverse map, not only M^{-1}.

These are orders of the specified rigid gradients, not of the primitive circulation or streamfunction. A conservative way to supply all of them is to bound the mixed Eulerian derivatives of $\Gamma_<$ through r_F+1 and of $\Psi_<$ through r_F+3, on the containing region, together with the displayed central matrix and trajectory derivatives. The extra derivatives come from $\nabla_y\Gamma_<$, $v_< = J\nabla_y\Psi_<$, $D = \nabla_y v_<(Y,t)$ and $\nabla_y\xi_< = \nabla_y L\Psi_<$. The chain rule for composition with $Y + Ma'$ gives the stated rigid bounds with finite constants; their required small sizes remain separate hypotheses.

The coefficient table also provides explicit amplitude variation inputs. At time order zero replace η by $\eta(t)$ in $\Lambda_\Xi, \Lambda_\zeta$ and use pointwise supplied rates $\mu_\Gamma(t), \Lambda_{\mathrm{met}}(t), \Lambda_\Gamma(t)$ if available. Let $\zeta_{c,1} = (M^{-T}p)_1$ and $c_+ = 2W_+\Gamma_+$. For a finite constant C_{var} generated by the same rules, one common majorant for all $1 \le |\alpha| \le r_F - 2$ is

$$\mathfrak{m}(t) = C_{\mathrm{var}}\ell\left[\frac{G_0}{\sigma\kappa_-}(\mu_\Gamma(t) + \Lambda_\Xi(t) + \Lambda_\kappa(t)) + \sigma c_+\left\{|\zeta_{c,1}(t)|(2\Lambda_{\mathrm{met}}(t) + \Lambda_\Gamma(t)) + K\Lambda_\Xi(t)\right\}\right]. \tag{4.13}$$

The proof for the lower entry writes $c\zeta_1 = c\zeta_{c,1} + c(\zeta_1 - \zeta_{c,1})$. The second term has absolute variation $O(c_+K\Lambda_\Xi)$; there is no division by the possibly vanishing $\zeta_{c,1}$. The upper entry uses the gradient row and the reciprocal rule. For mixed derivatives with a positive time order, replace all pointwise rates by their fixed bounds and $|\zeta_{c,1}|$ by ζ_+. Divide the resulting expression by Π to obtain an admissible $\mathfrak{n}$, increasing C_{var} through the requested order. The inequalities $\int_I \mathfrak{m} + c_{\mathrm{dat}} \le \log 2/(6C_R^2)$ and $\mathfrak{n} \le 1$ are explicit supplied scalar tests. They are not consequences of the metric window alone.

Proposition 4.2 (Quantitative control of the amplifying sequence)**.** *For the selected sequence constructed in Proposition 12.3, every generic index $m \ge 2$ satisfies the common-family quantitative conditions of Proposition 7.7, including its entry data, complete older-field bounds and entire return parameter interval. At index one it satisfies the separate first-return conditions of Lemma 7.5 and Proposition 7.6. At every index, a return time t_m^b is selected from its family after growth such that $\Omega_m(0, t_m^b) = \zeta_{m,1}(0, t_m^b) = 0$. The component $\zeta_{m,1}(0,t)$ is held at zero while the selected continuation uses $\alpha = \mathcal{F}_m$, as in Lemma 7.4; the central pair $(T_m(0,t), \widehat{\Omega}_m(0,t))$ then stays constant by (3.4). This does not assert vanishing of the corrections. The selected continuation preserves the amplified complete circulation gradient on this holding interval by the complete-gradient estimates returned in Proposition 11.1; the sufficient bound (7.11) is given below. The next insertion is $\tau_{m+1} = t_m^b$. The family continues while its coefficient bounds, positive older circulation and interior support conditions hold.*

Proof. At $m \ge 2$, Proposition 11.1 returns the coefficient and entry bounds in (7.18)–(7.19), all-trial support and denominator margins, complete-gradient holding and quantitative older-field estimates for the same finite family under its explicit scalar conditions. It also returns the principal clock and the next growing datum. At $m = 1$, its separate (11.14) gives the opposite signs of Lemma 7.5 and the return and holding conclusions of Proposition 7.6. Proposition 12.3 realizes these scalar conditions on one sequence and applies the least return selection to the same successive histories. This gives the asserted continuation and complete-gradient holding at every index. ☐

Proposition 4.3 (Corrected fields over their full lifetimes)**.** *Under the finite scalar comparisons of Proposition 11.1, one joint induction bounds the older corrected fields, their nonlinear maps, and their amplitude coefficients at the admitted finite orders and on the domains specified there. Their transported supports remain inside $\mathcal{D}_0$, the point $Z(t)$ remains inside the required material regions, and older circulation stays positive on those regions. On the infinite selected history of Proposition 12.3, each fixed increment has bounded mixed force derivatives of every fixed order up to T_*; these bounds may depend on the layer and the order.*

Proof. Sections 9–11 prove the joint induction and return (3.5), (4.5), (4.11), (6.3) and the finite control bounds under the displayed scalar conditions. Proposition 11.1 separately proves fixed-layer all-order finiteness on an infinite selected history satisfying these conditions and eventual admission of every order. Proposition 12.3 constructs this history, verifies all clock moments, and bounds every fixed force increment at every mixed order. The finite admitted estimates and fixed-layer finiteness retain their stated quantifiers. ☐

5. The finite correction equations

We construct one finite circulation increment on a supplied smooth older state. The correction equations cancel nonzero phase modes successively; the phase averages of the residuals remain in the force. All statements in this section concern a fixed compact time interval.

5.1. **The supplied fields and the lifted equations.** Suppress the index m except in the depth J_m. Let $I = [\tau, t_1]$, and let $\Gamma_<, \Psi_<$ be smooth, supported in $\overline{\mathcal{D}_0}$ on I, with $v_< = J\nabla_y \Psi_<$ and $\xi_< = L\Psi_<$. Smoothness on a closed interval means smooth extendibility to a neighborhood. Use their exact older flow X in (3.1), starting from a prescribed point $Y(\tau)$, and its inverse A. Fix balls $B_\ell \Subset B_{\ell'}$ in label space such that

$$\bigcup_{t\in I} X(\overline{B}_{\ell'}, t) \Subset \mathcal{D}_0. \tag{5.1}$$

This containment is a hypothesis for the present finite construction. The older force is the force of this same pair recovered by Lemma 2.1. Fix $N, \sigma > 0$, a unit vector p, an integer $J_m \geq 0$, a smooth envelope $g \in C_c^\infty(B_\ell)$, and a smooth activation w flat at τ and zero before τ. The carrier F and its mean-zero primitive P are those of Section 3. Initial amplitude data may be any smooth pair on a neighborhood of $\overline{B}_{\ell'}$.

Write

$$\begin{gathered} H = (D_aX)^{-1}, \quad \zeta = H^T p, \quad h = He_1, \quad C = H\mathsf{A}(X)H^T, \quad \kappa = p^T Cp, \\ q = \Gamma_< \circ X, \quad b = \xi_< \circ X, \quad \varpi = W \circ X, \quad c = 2\varpi q, \quad \partial_\perp = Jp\cdot\nabla_a, \quad d = \partial_\perp q, \\ \mathfrak{l}_i = \sum_j \partial_{a_j} C_{ji} = (LA_i)\circ X, \qquad \mathfrak{l} = (\mathfrak{l}_1, \mathfrak{l}_2). \end{gathered} \tag{5.2}$$

Here A_i is the ith component of the inverse map. All these coefficients depend on (a,t) and are independent of the phase s. In particular $d = (J\zeta)\cdot(\nabla_y\Gamma_<)\circ X$, and $p \cdot h = \zeta_1$. Compactness and ellipticity give $\min_{\overline{B}_{\ell'}\times I} \kappa > 0$.

For periodic profiles put

$$\nabla_N = Np\partial_s + \nabla_a, \qquad \mathcal{J}_s(U,V) = U_s\partial_\perp V - V_s\partial_\perp U, \qquad \{U,V\}_a = J\nabla_a U\cdot\nabla_a V.$$

The notation $Q^\natural(y,t)$ means $Q(Np\cdot A(y,t), A(y,t), t)$, extended by zero when the profile has compact material support. Define

$$\mathcal{D}_1 U = 2(Cp)\cdot\nabla_a U_s + (p\cdot\mathfrak{l})U_s, \qquad \mathcal{D}_2 U = C : \nabla_a^2 U + \mathfrak{l}\cdot\nabla_a U.$$

The following identities include both first-order coefficient terms:

(5.3)
$$
\begin{aligned}
(\partial_t + v_< \cdot \nabla_y)Q^\natural &= (Q_t)^\natural, & \nabla_y Q^\natural &= (H^T \nabla_N Q)^\natural,\\
J\nabla_y U^\natural \cdot \nabla_y Q^\natural &= (N\mathcal{J}_s(U,Q) + \{U,Q\}_a)^\natural, &&\\
\partial_{y_1} Q^\natural &= (N\zeta_1 Q_s + h\cdot \nabla_a Q)^\natural, &&\\
LQ^\natural &= (\mathcal{L}_N Q)^\natural, & \mathcal{L}_N Q &= \nabla_N \cdot (C\nabla_N Q) = N^2\kappa Q_{ss} + N\mathcal{D}_1 Q + \mathcal{D}_2 Q.
\end{aligned}
$$

To prove them, the flow equation gives transport of A and $\det D_a X = 1$. The determinant identity

$$\det(H^T u, H^T v) = \det(u,v)$$

gives the bracket formula. The Piola identity

$$\operatorname{div}_a\bigl(H(G\circ X)\bigr) = (\operatorname{div}_y G)\circ X$$

gives the elliptic formula and, on applying it to $G = \mathsf{A}\nabla_y A_i$, the formula for $\mathfrak{l}_i$. Applying it to $G = e_1$ and to $G = We_1$ also proves

$$\operatorname{div}_a h = 0, \qquad h\cdot\nabla_a\varpi = 0, \qquad \operatorname{div}_a(\varpi h) = 0. \tag{5.4}$$

The matrix $D_a X(a,t)$ in these identities is the full nonlinear flow derivative. Its value $M = D_a X(0,t)$ is only the central matrix in (4.1).

Lemma 5.1 (Finite residual differences)**.** *For smooth compact increments γ,ψ, put $v' = J\nabla_y\psi$ and $\xi' = L\psi$. Subtracting the residuals of the older pair from those of $(\Gamma_< + \gamma, \Psi_< + \psi)$ gives*

$$
\begin{aligned}
F_m^\Gamma &= (\partial_t + v_<\cdot\nabla_y)\gamma + v'\cdot\nabla_y\Gamma_< + v'\cdot\nabla_y\gamma,\\
E_m^\xi &= (\partial_t + v_<\cdot\nabla_y)\xi' + v'\cdot\nabla_y\xi_< + v'\cdot\nabla_y\xi' - W\partial_{y_1}(2\Gamma_<\gamma + \gamma^2).
\end{aligned}\tag{5.5}
$$

For $\gamma = V^\natural$, $\psi = U^\natural$, these are $\mathcal{E}_\Gamma(V,U)^\natural$ and $\mathcal{E}_\xi(V,U)^\natural$, where

$$
\begin{aligned}
\mathcal{E}_\Gamma(V,U) &= V_t + NdU_s + \{U,q\}_a + N\mathcal{J}_s(U,V) + \{U,V\}_a,\\
\mathcal{E}_\xi(V,U) &= \partial_t(\mathcal{L}_N U) + NU_s\partial_\perp b + \{U,b\}_a + N\mathcal{J}_s(U,\mathcal{L}_N U) + \{U,\mathcal{L}_N U\}_a\\
&\quad - Nc\zeta_1 V_s - ch\cdot\nabla_a V - 2\varpi V h\cdot\nabla_a q - N\varpi\zeta_1\partial_s(V^2) - \varpi h\cdot\nabla_a(V^2).
\end{aligned}\tag{5.6}
$$

Proof. Linearity of L and $(\Gamma_< + \gamma)^2 - \Gamma_<^2 = 2\Gamma_<\gamma + \gamma^2$ give (5.5). Substitution of (5.3) and the product rule give every term in (5.6). In particular, the terms containing b are present even when the older streamfunction is locally quadratic. □

Every outer time derivative below acts on coefficients and profiles. For example,

$$
\begin{aligned}
\partial_t(N\mathcal{D}_1 U) &= N\bigl[2(C_t p)\cdot\nabla_a U_s + (p\cdot\mathfrak{l}_t)U_s + 2(Cp)\cdot\nabla_a U_{st} + (p\cdot\mathfrak{l})U_{st}\bigr],\\
\partial_t(\mathcal{D}_2 U) &= C_t : \nabla_a^2 U + \mathfrak{l}_t\cdot\nabla_a U + C:\nabla_a^2 U_t + \mathfrak{l}\cdot\nabla_a U_t.
\end{aligned}\tag{5.7}
$$

Higher time derivatives use the full Leibniz sums of these expressions.

5.2. **Phase inversion and the finite recursion.** For a mean-zero periodic Q, define its mean-zero primitive by

$$(\partial_s^{-1}Q)(s,a,t) = \frac{1}{2\pi}\int_0^{2\pi}(\pi - v)Q(s-v,a,t)\,dv. \tag{5.8}$$

Integration by parts gives $\partial_s\partial_s^{-1}Q = Q$, and its phase average is zero. It commutes with label and time derivatives and has operator norm at most $\pi/2$ in each maximum derivative norm. Put $\mathsf{P}_{\neq 0}Q = Q - \langle Q\rangle_s$.

Let $(T,\widehat{\Omega})$ solve (3.4) with the supplied initial data, that is, with

$$\mathcal{B}(a,t)=\begin{pmatrix}0 & -d/(\sigma\kappa)\\ \sigma c\zeta_1 & 0\end{pmatrix},\qquad \Omega=(N/\sigma)\widehat{\Omega},\qquad B=\Omega/(N^2\kappa).$$

Define $V_0=wTFg$ and $\widetilde{W}_0=w\widehat{\Omega}Fg$. For each level the streamfunction is recovered from the normalized first phase derivative:

$$W_n=(N/\sigma)\widetilde{W}_n,\qquad U_n=(N^2\kappa)^{-1}\partial_s^{-1}W_n,\qquad \widetilde{W}_n=N\sigma\kappa\,\partial_sU_n. \tag{5.9}$$

Thus $U_0=wBPg$ and $N^2\kappa\partial_s^2U_n=\partial_sW_n$. One phase primitive in (5.9) inverts the displayed first derivative; it is not an inverse of $N^2\kappa\partial_s^2$ on the input W_n.

Here is an explicit assignment of grades. A principal term containing level n has grade n. The slow older-circulation term has grade $i+1$; the fast and slow older-vorticity terms have grades $i+1$ and $i+2$. The two lower elliptic terms from U_i have grades $i+1$ and $i+2$. Both slow linear circulation terms have grade $i+1$. A fast quadratic term involving levels i,j has grade $i+j+1$, and a slow one has grade $i+j+2$.

Use zero for negative profile indices. After depth J_m is constructed, set $V_i=U_i=W_i=\widetilde{W}_i=0$ for $i>J_m$, but define for every $j\geq 0$

$$Z_j=\partial_sW_j+N\mathcal{D}_1U_{j-1}+\mathcal{D}_2U_{j-2},\qquad Z_j^\flat=N\mathcal{D}_1U_{j-1}+\mathcal{D}_2U_{j-2}. \tag{5.10}$$

In particular, $Z_{J_m+1}=N\mathcal{D}_1U_{J_m}+\mathcal{D}_2U_{J_m-1}$, $Z_{J_m+2}=\mathcal{D}_2U_{J_m}$, and $Z_j=0$ only for $j>J_m+2$. These definitions give $\mathcal{L}_N\sum_{i=0}^{J_m}U_i=\sum_{j=0}^{J_m+2}Z_j$.

For $n\geq 1$ the complete sources are

$$\begin{aligned}S_n^\Gamma&=\{U_{n-1},q\}_a+N\sum_{i+j=n-1}\mathcal{J}_s(U_i,V_j)+\sum_{i+j=n-2}\{U_i,V_j\}_a,\\ S_n^\xi&=\partial_tZ_n^\flat+N\partial_sU_{n-1}\partial_\perp b+\{U_{n-2},b\}_a-ch\cdot\nabla_aV_{n-1}-2\varpi V_{n-1}h\cdot\nabla_aq\\ &\quad+N\sum_{i+j=n-1}\mathcal{J}_s(U_i,Z_j)+\sum_{i+j=n-2}\{U_i,Z_j\}_a\\ &\quad-N\varpi\zeta_1\partial_s\sum_{i+j=n-1}V_iV_j-\varpi h\cdot\nabla_a\sum_{i+j=n-2}V_iV_j.\end{aligned} \tag{5.11}$$

Every sum uses nonnegative indices with the stated sum; the first index of a bracket and both circulation indices are at most J_m, while a composite index is at most J_m+2. Empty sums are zero. During the construction of level $n\leq J_m$, the formulas use only levels below n, since every contributing Z_j has $j\leq n-1$. All derivatives in these sources apply to the full profiles, including g, the label-dependent amplitudes, and κ^{-1}.

For $1\leq n\leq J_m$ solve

$$\begin{aligned}&\partial_t\begin{pmatrix}V_n\\ \widetilde{W}_n\end{pmatrix}=\mathcal{B}(a,t)\begin{pmatrix}V_n\\ \widetilde{W}_n\end{pmatrix}+\begin{pmatrix}-\mathsf{P}_{\neq 0}S_n^\Gamma\\ -\dfrac{\sigma}{N}\partial_s^{-1}\mathsf{P}_{\neq 0}S_n^\xi\end{pmatrix},\\ &(V_n,\widetilde{W}_n)|_{t=\tau}=(0,0).\end{aligned} \tag{5.12}$$

For the propagator $\mathcal{P}(a;t,t')$ of $\mathcal{B}$, its solution is the pointwise integral

$$\begin{pmatrix}V_n\\ \widetilde{W}_n\end{pmatrix}(s,a,t)=\int_\tau^t\mathcal{P}(a;t,t')\begin{pmatrix}-\mathsf{P}_{\neq 0}S_n^\Gamma\\ -\dfrac{\sigma}{N}\partial_s^{-1}\mathsf{P}_{\neq 0}S_n^\xi\end{pmatrix}(s,a,t')\,dt'. \tag{5.13}$$

This is the phase/Duhamel construction of [1, equation (5.8)] at each label. The sources are the full expressions (5.11), whose variable coefficients and additional terms have been computed here.

Proposition 5.2 (Exact finite corrections)**.** *For the supplied smooth data above, the recursion has a unique smooth solution to every finite depth. All $V_n, W_n, \widetilde{W}_n, U_n$ have zero phase mean, are supported in* $\operatorname{supp} g$ *in the label variable, and are flat at activation. Put $V = \sum_{n=0}^{J_m} V_n$ and $U = \sum_{n=0}^{J_m} U_n$. Their exact residuals are*

$$\mathcal{E}_\Gamma(V,U) = \dot{w}TFg + \sum_{n=1}^{J_m}\langle S_n^\Gamma\rangle_s + \sum_{n=J_m+1}^{2J_m+2} S_n^\Gamma,$$
$$\mathcal{E}_\xi(V,U) = \dot{w}\Omega F'g + \sum_{n=1}^{J_m}\langle S_n^\xi\rangle_s + \sum_{n=J_m+1}^{2J_m+4} S_n^\xi. \tag{5.14}$$

The fields $\gamma = V^\natural$ and $\psi = U^\natural$ are smooth compact physical increments with zero initial traces of every order. If $q \geq \Gamma_- > 0$ on $\overline{B}_{\ell'} \times I$, the explicit additional condition

$$\sum_{n=0}^{J_m} \sup_{\mathbb{S}^1 \times B_{\ell'} \times I} |V_n| \leq \eta\Gamma_- \qquad (0 \leq \eta < 1) \tag{5.15}$$

ensures $\Gamma_< + \gamma \geq (1-\eta)\Gamma_-$ on that transported tube. Positivity is not needed for the finite recursion or recovery.

Proof. The coefficients of $\mathcal{B}$ are smooth and bounded on the fixed compact domain. A linear ODE with these coefficients has a smooth propagator throughout I. Inductively the source at level n is smooth and is already determined. Formula (5.13) proves existence and uniqueness. Phase averaging its equation gives a homogeneous ODE with zero data, so both unknowns have zero mean. The same holds at level zero because F has zero mean. The normalized primitive gives zero mean for U_n. Outside $\operatorname{supp} g$, all earlier levels and their derivatives vanish; the sources therefore vanish there, and the pointwise ODE preserves this vanishing. Flatness at τ follows first from w at level zero and then by differentiating the inhomogeneous ODE: every time derivative of its source vanishes at τ.

The two principal expressions at level $n \geq 1$ are

$$V_{n,t} + NdU_{n,s} = \partial_t V_n + \frac{d}{\sigma\kappa}\widetilde{W}_n,$$
$$\partial_t(\partial_s W_n) - Nc\zeta_1\partial_s V_n = \frac{N}{\sigma}\partial_s(\partial_t\widetilde{W}_n - \sigma c\zeta_1 V_n).$$

They equal $-\mathsf{P}_{\neq 0}S_n^\Gamma$ and $-\mathsf{P}_{\neq 0}S_n^\xi$ by (5.12). The normalization has already included the derivative of κ in the second identity. At level zero the same calculation leaves precisely $\dot{w}TFg$ and $\dot{w}\Omega F'g$. Expanding (5.6) with (5.10) assigns every other term once to (5.11). The largest scalar grade is $J_m + J_m + 2 = 2J_m + 2$, and the largest vorticity grade is $J_m + (J_m + 2) + 2 = 2J_m + 4$. Summing proves (5.14), including when $J_m = 0$. The collar in (5.1) justifies smooth zero extension in physical space. Finally the triangle inequality proves the positivity conclusion under (5.15). The physical scalar is prescribed in circulation, so no square-root inversion occurs. □

Lemma 5.3 (Phase parity)**.** *At every constructed level,*

$$V_n(-s,a,t) = (-1)^{n+1}V_n(s,a,t), \quad W_n(-s,a,t) = (-1)^{n+1}W_n(s,a,t),$$
$$U_n(-s,a,t) = (-1)^n U_n(s,a,t).$$

Also Z_j and $Z_j^\flat$ have parity $(-1)^j$. Consequently $\langle S_n^\Gamma\rangle_s = 0$ for even n and $\langle S_n^\xi\rangle_s = 0$ for odd n.

Proof. The base parity follows from oddness of F and evenness of its mean-zero primitive. A phase derivative or its mean-zero inverse reverses parity; a label derivative preserves it. These rules applied to (5.10) give the asserted parity of Z_j. In (5.11) a fast bracket contributes one phase derivative and a slow bracket contributes none. Its index sums then give parity $(-1)^{n+1}$ for every scalar source and $(-1)^n$ for every vorticity source. The two older-vorticity terms have these same parities with their respective shifts by one and two; both quadratic circulation terms do as well. Projection removes only an even constant. The ODE and its uniqueness therefore give the induction. ☐

These identities keep the label a fixed. They give no spatial reflection symmetry, and the corrections can be nonzero where $g = 1$. For example, $N\partial_s U_0 \partial_\perp b$ and $-N\varpi\zeta_1\partial_s(V_0^2)$ occur already at grade one on that region.

Lemma 5.4 (The squared circulation)**.** *The squared-circulation increment is the polynomial expression*

$$\Theta_{\mathrm{new}} - \Theta_< = (2qV + V^2)^\natural, \qquad 2qV + V^2 = \sum_{n=0}^{2J_m+1} \Sigma_n, \qquad \Sigma_n = 2qV_n + \sum_{i+j=n-1} V_i V_j. \tag{5.16}$$

Here circulation indices outside $0, \dots, J_m$ are zero and sums have nonnegative indices. The parity of Σ_n is $(-1)^{n+1}$ and $\langle \Sigma_n \rangle_s = \sum_{i+j=n-1} \langle V_i V_j \rangle_s$. On a fixed phase interval where $F(s) = s$, the profiles V_n, W_n are polynomials in s of degree at most $n+1$, U_n has degree at most $n+2$, and Σ_n has degree at most $n+1$. Their coefficients are smooth functions of (a, t).

Proof. The identity for the squared increment and its grades follows by expanding the square and assigning each ordered product once, to $n = i + j + 1$. Phase parity and the mean formula follow from Lemma 5.3. On the stated phase interval, $P(s) = P(0) + s^2/2$, which proves the initial degree bounds. Label and time derivatives preserve degree; phase differentiation lowers it, phase projection subtracts a constant, and the normalized primitive raises the local degree by at most one. If the bounds hold below n, (5.10) has degree at most j at composite index j. Each scalar source in (5.11) then has degree at most $n+1$ and each vorticity source has degree at most n. This follows for the quadratic terms by adding the two input degrees, subtracting one for a fast derivative, and using their displayed index sums. The linear terms have the same bounds. The phase-independent ODE preserves these polynomial spaces, giving the induction and the asserted bound for Σ_n. Global periodic averages enter as smooth coefficients. ☐

This polynomial observation is local in the independent phase variable; it gives no estimate for the label-dependent coefficients. The complete squared residual difference retains the older force. With $F^\Theta = D_v\Theta$ for each total state,

$$F^\Theta_{\mathrm{new}} - F^\Theta_< = \left[2(q+V)\mathcal{E}_\Gamma(V, U) + 2V(F^\Gamma_< \circ X)\right]^\natural. \tag{5.17}$$

Indeed $F^\Theta = 2\Gamma F^\Gamma$ at each of the two total states, so subtraction proves the formula. The reduced-vorticity residual is unchanged because its scalar source is the derivative of the same square in (5.16).

5.3. **The phase averages and their primitives.** For periodic A_1, A_2, integration by parts in phase gives

$$\langle \mathcal{J}_s(A_1, A_2) \rangle_s = \partial_\perp \langle (A_1)_s A_2 \rangle_s, \qquad \langle \{A_1, A_2\}_a \rangle_s = \mathrm{div}_a \langle A_2 J \nabla_a A_1 \rangle_s. \tag{5.18}$$

All linear terms in the sources have zero phase mean. This includes $\partial_t Z_n^\flat$, since coefficients are independent of s; it also includes both older-vorticity terms and both linear circulation terms. The fast quadratic circulation mean vanishes as an exact phase derivative. Define the compact label vector fields

$$\begin{aligned}Q^\Gamma_n &= NJp\sum_{i+j=n-1}\langle U_{i,s}V_j\rangle_s+\sum_{i+j=n-2}\langle V_jJ\nabla_aU_i\rangle_s,\\ Q^\xi_n &= NJp\sum_{i+j=n-1}\langle U_{i,s}Z^\flat_j\rangle_s+\sum_{i+j=n-2}\langle Z^\flat_jJ\nabla_aU_i\rangle_s\\ &\qquad -\frac{1}{2N^2}\langle\sum_{i+j=n-2}W_iW_j\rangle_sJ\nabla_a(\kappa^{-1})-\varpi h\langle\sum_{i+j=n-2}V_iV_j\rangle_s.\end{aligned}\tag{5.19}$$

Then, with the same index conventions as before,

$$\langle S^\Gamma_n\rangle_s=\operatorname{div}_a Q^\Gamma_n,\qquad \langle S^\xi_n\rangle_s=\operatorname{div}_a Q^\xi_n.\tag{5.20}$$

In particular, the last two terms of Q^ξ_n retain the variable-symbol and quadratic-circulation slow means.

Here is the calculation for the carrier vorticity terms. The symmetric fast sum vanishes pointwise in label:

$$\sum_{i+j=k}\langle U_{i,s}\partial_sW_j\rangle_s=\frac{1}{2N^2\kappa}\langle\partial_s\sum_{i+j=k}W_iW_j\rangle_s=0.\tag{5.21}$$

Set $K=\kappa^{-1}$ and $Q_i=\partial_s^{-1}W_i$ temporarily. For the slow sum, the product rule gives

$$\sum_{i+j=k}\langle\{U_i,\partial_sW_j\}_a\rangle_s=\frac{K}{N^2}\sum_{i+j=k}\langle J\nabla_aQ_i\cdot\nabla_a\partial_sW_j\rangle_s+\frac{J\nabla_aK}{N^2}\cdot\sum_{i+j=k}\langle Q_i\nabla_a\partial_sW_j\rangle_s.$$

Phase integration by parts converts the first sum into $-\sum\langle J\nabla_aW_i\cdot\nabla_aW_j\rangle_s=0$ by interchange of i,j. It converts the vector in the second sum into $-\frac12\nabla_a\langle\sum W_iW_j\rangle_s$. Thus

$$\sum_{i+j=k}\langle\{U_i,\partial_sW_j\}_a\rangle_s=-\operatorname{div}_a\left(\frac{1}{2N^2}\langle\sum_{i+j=k}W_iW_j\rangle_sJ\nabla_a(\kappa^{-1})\right).\tag{5.22}$$

This calculation applies to the full symmetric index set, including its zero terms. Combining it with (5.18) and (5.4) proves (5.20). Label differentiation in these formulas differentiates κ^{-1}, ϖ, h, and every amplitude, including on the region where $g=1$.

If $Q(a,t)$ is a compact label vector field, Piola also gives the physical primitive

$$(\operatorname{div}_a Q)\circ A=\operatorname{div}_y\big[(D_aX\,Q)\circ A\big],\qquad \int(\operatorname{div}_a Q)\circ A\,dy=0.\tag{5.23}$$

These are formulas for phase-independent means. Phase averaging before evaluation is not spatial integration of an oscillatory profile; in particular zero phase mean of U_n does not assert $\int U^\natural_n\,dy=0$.

5.4. **Physical force and pressure.** The same fixed compact inverse curl as in Lemma 2.1 recovers the entire residual. Indeed, direct integration of (5.5) gives

$$\int_{\mathbb{R}^2}E^\xi_m\,dy=0,\qquad \int rE^\xi_m\,dz\,dr=0.\tag{5.24}$$

The first term has zero integral because $\int\xi'\,dy=\int L\psi\,dy=0$ and $v_<$ is divergence free. Each transport term involving v' is the divergence of the corresponding scalar times v'. The final term integrates to zero because $\partial_{y_1}W=0$. The second equality is the change of measure $dy=r\,dz\,dr$.

Set

$$f^\varphi_m=F^\Gamma_m/r,\qquad (f^z_m,f^r_m/r)=\mathcal{R}_yE^\xi_m.\tag{5.25}$$

The ordered pair in the second formula is in the y chart:

$$\operatorname{curl}_y(f^z_m,f^r_m/r)=\partial_{y_1}(f^r_m/r)-\partial_{y_2}f^z_m=(\partial_zf^r_m-\partial_rf^z_m)/r=E^\xi_m.$$

Let ϕ_0 denote the fixed unit-integral bump defining $\mathcal{R}_y$, supported in $\mathcal{D}$. Its support conclusion is $\mathrm{conv}(\operatorname{supp} E_m^\xi \cup \operatorname{supp} \phi_0) \subset \overline{\mathcal{D}}$. No parity hypothesis is needed for this curl identity. The divergence identity, support and derivative bounds in [1, Lemma 5.2] are precisely the properties used in Lemma 2.1.

Write the physically evaluated vorticity residual in (5.14) as $E^{\mathrm{act}} + E^{\mathrm{mean}} + E^{\mathrm{rem}}$. Each summand of E^{mean} has zero spatial integral by (5.23). Their complement $E^{\mathrm{act}} + E^{\mathrm{rem}}$ has zero integral by (5.24). Thus every force term is assigned by

$$\begin{aligned}(f_m^z, f_m^r/r) &= \mathcal{R}_y(E^{\mathrm{act}} + E^{\mathrm{rem}}) + \sum_{n=1}^{J_m} \mathcal{R}_y[(\langle S_n^\xi\rangle_s)^\natural], \\ f_m^\varphi &= r^{-1}\left[\left(\dot w TFg + \sum_{n=J_m+1}^{2J_m+2} S_n^\Gamma\right)^\natural + \sum_{n=1}^{J_m} (\langle S_n^\Gamma\rangle_s)^\natural\right].\end{aligned} \tag{5.26}$$

Linearity gives exactly (5.25). An isolated activation term need not have zero spatial integral; it is grouped with the terminal remainder for this reason.

For every integer $k \geq 0$, Lemma 2.1 gives

$$\|f_m\|_{C^k_{x,t}(I)} \leq C_{k,\mathcal{D}}\big(\|F_m^\Gamma\|_{C^k_{y,t}(I)} + \|E_m^\xi\|_{C^k_{y,t}(I)}\big) < \infty. \tag{5.27}$$

Here $x \in \mathbb{R}^3$, the norms take the maximum over all mixed derivatives of total order at most k, and the fixed geometry includes the distance from the axis. This bound follows because the recovery operator commutes with time derivatives and has no spatial derivative loss; the radial multipliers, chart change and cylindrical basis have bounded derivatives on the support.

Adding (5.25) to the older recovered force gives exactly the recovered force of the new total state, since the residual differences in Lemma 5.1 telescope and the same linear operator and radial multipliers are used at every step. Lemma 2.1 then gives a smooth pressure: the momentum residual has zero azimuthal component and a curl-free meridional component on the simply connected half-plane. Its potential is constant near the axis and extends smoothly to three dimensions.

5.5. **The complete values at the two centers.** The following formulas retain every term used in a center evaluation. Let $K_i(a,t) = (\nabla_y^2 A_i)(X(a,t),t)$ and, for a profile Q, put $q_s^* = H^T\nabla_a Q_s$. At $y = X(a,t)$, with $s = Np\cdot a$,

$$\begin{aligned}\nabla_y Q^\natural &= N\zeta Q_s + H^T\nabla_a Q, \\ \nabla_y^2 Q^\natural &= N^2 Q_{ss}\zeta\otimes\zeta + N(\zeta\otimes q_s^* + q_s^*\otimes\zeta) + H^T(\nabla_a^2 Q)H \\ &\quad + \sum_{i=1}^2 (Q_{a_i} + Np_i Q_s)K_i.\end{aligned} \tag{5.28}$$

The chain rule applied twice proves these formulas. They apply to the full finite sums $Q = V$ and $Q = U$. Add the corresponding older derivatives to obtain the total circulation gradient and total streamfunction Hessian, and multiply the latter on the left by J to obtain the complete velocity gradient.

At the older-flow center $Y(t) = X(0,t)$ these expressions use $(s,a) = (0,0)$ and $H(0,t) = M(t)^{-1}$. For example, at phase zero the principal stream profile has

$$\nabla_a U_0 = wP(0)\nabla_a(Bg), \qquad \nabla_a^2 U_0 = wP(0)\nabla_a^2(Bg).$$

These slow terms remain even in the principal Hessian. At the full-flow point $Z(t)$ they instead use

$$a_*(t) = A(Z(t),t), \qquad s_*(t) = Np\cdot a_*(t), \qquad \dot a_* = H(a_*,t)v'(Z(t),t)$$

for the present single increment, with the sum of all new velocities in place of v' on a longer finite continuation. The zero-extended physical formulas remain meaningful if the point is outside the tube. In the pure-phase part of the velocity gradient set $e = \zeta/|\zeta|$. Exactly,

$$N^2 U_{ss} J\zeta \otimes \zeta = \frac{|\zeta|^2}{\kappa}(N^2 \kappa U_{ss}) J e \otimes e. \tag{5.29}$$

The metric ratio is evaluated at the same label and time as the other terms in (5.28). For instance, along the full flow, the total gradient obeys the exact equation

$$\frac{d}{dt}\nabla_y \Gamma(Z(t),t) = \nabla_y F^{\Gamma}(Z(t),t) - (\nabla_y v(Z(t),t))^T \nabla_y \Gamma(Z(t),t).$$

Along Y, the right side has the additional term $-v' \cdot \nabla_y(\nabla_y \Gamma)$, evaluated at Y. These identities do not supply estimates for the displacement or the correction derivatives.

5.6. **Finite derivative dependence.** For $D = (\nabla_y v_<) \circ X$ and $\dot{\mathsf{A}}_X = ((v_< \cdot \nabla_y)\mathsf{A}) \circ X$, the material time derivatives of the coefficients satisfy

$$\begin{aligned} &H_t = -HD, \qquad \zeta_t = -D^T \zeta, \qquad h_t = -HDe_1, \\ &C_t = H\big(\dot{\mathsf{A}}_X - D\mathsf{A}(X) - \mathsf{A}(X)D^T\big)H^T, \qquad \kappa_t = p^T C_t p, \\ &q_t = F^{\Gamma}_< \circ X, \qquad b_t = ch \cdot \nabla_a q + E^{\xi}_< \circ X, \end{aligned} \tag{5.30}$$

where $F^{\Gamma}_<, E^{\xi}_<$ are the older total residuals. These formulas follow by differentiating the flow and using (2.4); the older scalars need not be constant along that flow.

We record the data needed when differentiating these formulas. For a profile define

$$|Q|_k = \max_{j+|\alpha|+b \le k} \sup_{\mathbb{S}^1 \times \overline{B}_{\ell'} \times I} |\partial_s^j \partial_a^\alpha \partial_t^b Q|,$$

and use the same norm without phase derivatives for coefficients. The coefficient list in the lifted equations and recursion is

$$C,\ \mathfrak{l},\ \kappa^{\pm 1},\ \zeta,\ h,\ \varpi,\ q,\ \nabla_a q,\ b,\ \nabla_a b,\ c,\ d,\ \mathcal{B}.$$

Its derivatives include derivatives of the older circulation and of $\xi_< = L\Psi_<$, all composed with the same X. Stream recovery also uses derivatives of κ^{-1}, explicitly

$$\nabla_a U_n = \frac{\kappa^{-1}}{N^2}\partial_s^{-1}\nabla_a W_n - U_n \nabla_a \log \kappa,$$

and repeated Leibniz derivatives of this identity. The two lower elliptic operators use $C, \mathfrak{l}$ and their full outer time derivatives as in (5.7). Nonlinear sources use products, one fast or slow bracket, and the exact composite Z_j; mean primitives additionally use $\nabla_a(\kappa^{-1})$ and ϖh.

The remaining operations are phase projection and the literal primitive, solution of the finite linear ODE, evaluation on the phase and inverse map, differentiation of the moving points, compact force recovery, and the radial lift. For physical differentiation, set $\widehat{H} = D_y A$ and $\nu = \partial_t A = -\widehat{H} v_<$. The exact rules are

$$\begin{aligned} &\partial_{y_i} Q^\natural = \Big(\sum_j H_{ji}(Np_j \partial_s + \partial_{a_j})Q\Big)^\natural, \\ &\partial_t Q^\natural = (Q_t)^\natural + \sum_j \nu_j\big(Np_j Q_s + Q_{a_j}\big)^\natural. \end{aligned} \tag{5.31}$$

Thus every further derivative differentiates H, ν, or a profile; the center formulas additionally use K_i, the trajectories and the complete older derivatives. These map quantities are determined by the supplied smooth older flow. Their relevant derivatives are finite on the compact collar. The radial lift uses derivatives of $r^{\pm 1}$, the fixed chart and the cylindrical basis on $\overline{\mathcal{D}}$.

For each fixed k and J_m, induction in n proves finiteness of all the displayed seminorms: at a level one uses finitely many derivatives of earlier levels, the Leibniz rule, the bounded phase operators, and the differentiated finite linear ODE. The derivative inventory is therefore finite at every prescribed depth and output order. The resulting constants may depend on J_m, N, σ, I, the geometry and the finitely many smooth-input seminorms actually differentiated, as well as the positive minimum of κ. No uniform bound over different older states or longer intervals follows from this finite argument. Sharp order allocations, frequency gains and lifetime bounds remain the estimates required in Section 6.

6. Mixed derivative estimates for the forces

All estimates concern the physical force. For a scalar or vector field h on the volume chart and an interval I, write

$$\|h\|_{C^k_{y,t}(I)} = \max_{|\alpha|+b\le k} \sup_{y\in\mathbb{R}^2,\, t\in I} |\partial_y^\alpha \partial_t^b h(y,t)|.$$

The recovery in Lemma 2.1 gives, for the lift to $\mathbb{R}^3$, a fixed constant $C_{k,\mathcal{D}}$ such that

$$\|f_m\|_{C^k_{x,t}(I)} \le C_{k,\mathcal{D}}\big(\|F^\Gamma_m\|_{C^k_{y,t}(I)} + \left\|E^\xi_m\right\|_{C^k_{y,t}(I)}\big). \tag{6.1}$$

Here $x\in\mathbb{R}^3$. This estimate also includes differentiation of the cylindrical basis vectors, which is bounded on the fixed off-axis support. Oscillation may improve particular estimates, but it does not remove the phase-independent contributions to either residual.

6.1. A quantitative finite estimate.

We give all scale conditions without fixing an eventual frequency sequence. Fix $k\ge 0$ and depth J_m, and set

$$r_* = k + 4J_m + 9, \qquad r_X = r_* + 2, \qquad r_F = r_* + 4. \tag{6.2}$$

Assume (3.5) through r_*, (4.5) through r_F, and the rigid-scalar, metric and small-deformation hypotheses preceding (4.11) through their stated orders for this r_F. The containing region has a transported collar satisfying (5.1). Their constants and sizes are part of the input. Use the profile norm (3.8) and $\mathcal{T} = \max(|I|, \Pi^{-1})$ from Lemma 3.2. All quantities below refer to this one layer.

Let $\chi, C_0, L_0, G_0, B_0, c_0, h_0, \varpi_*, z_0$ respectively be positive size bounds for

$$\kappa^{-1},\ C,\ \mathfrak{l},\ \nabla_a q,\ \nabla_a b, c,\ h,\ \varpi,\ \zeta_1.$$

They may be the corresponding sizes in (4.11). Let $C_*\ge 1$ bound all their normalized derivative constants through r_X after replacing their first-label rates by λ and absorbing any fixed number of rate summands. Let it also bound the corresponding constants for $D_aX, H, \mathfrak{v}$ through r_X and the relative amplitude data through r_*. These constants are calculated from the preceding finite rules, rather than inferred from smoothness. Define

$$H_{r_*} = 2^{16(r_*+1)}(1+C_*)^8, \qquad z_1^* = (2C_0\lambda + L_0)\chi/N, \qquad z_2^* = (C_0\lambda^2 + L_0\lambda)\chi/N^2.$$

The factor H_{r_*} bounds the at most five-factor Leibniz products, two-dimensional component sums and bounded phase operations in each source row. It also bounds one extra derivative in the mean formulas.

Choose $0 < \delta \le 1$. Require that each quotient in the following table is at most one. The column headed "shift" is its grade shift; the matrix equation, not an estimate, determines these shifts.

(6.3)

term	shift	required quotient
$Z^\flat$ first piece, normalized	1	z_1^*/δ
$Z^\flat$ second piece, normalized	2	z_2^*/δ^2
$\{U,q\}_a$	1	$\mathcal{T}\chi\lambda G_0/(N\sigma\delta)$
$\partial_t(N\mathcal{D}_1U),\ \partial_t\mathcal{D}_2U$	$1,2$	$\mathcal{T}\Pi z_1^*/\delta, \quad \mathcal{T}\Pi z_2^*/\delta^2$
$NU_s\partial_\perp b,\ \{U,b\}_a$	$1,2$	$\mathcal{T}\chi B_0/(N\delta), \quad \mathcal{T}\chi\lambda B_0/(N^2\delta^2)$
$ch\cdot\nabla_a V$	1	$\mathcal{T}\sigma c_0 h_0\lambda/(N\delta)$
$2\varpi Vh\cdot\nabla_a q$	1	$2\mathcal{T}\sigma\varpi_* h_0 G_0/(N\delta)$
fast transport products	1	$\mathcal{T}\chi\lambda Y_*/(\sigma\delta)$
slow transport products	2	$\mathcal{T}\chi\lambda^2 Y_*/(N\sigma\delta^2)$
$N\varpi\zeta_1\partial_s(V^2)$	1	$\mathcal{T}\sigma\varpi_* z_0 Y_*/\delta$
$\varpi h\cdot\nabla_a(V^2)$	2	$\mathcal{T}\sigma\varpi_* h_0\lambda Y_*/(N\delta^2)$

Every vorticity row in this table includes its factor σ/N. The first row denotes the normalized first lower-elliptic contribution $(\sigma/N)N\mathcal{D}_1U$, not an additional factor of N. In particular the fast circulation-square row has an amplitude-smallness condition; it is not assigned a nonexistent inverse-frequency gain.

For explicit profile constants put

$$\begin{aligned}
A_{r_*} &= \max_{|\alpha|+b\le r_*} \max(1, E_{\alpha,b}),\\
w_{r_*} &= \max_{b\le r_*+1} \Pi^{-b}\|\partial_t^b w\|_\infty,\\
g_{r_*} &= \max_{|\alpha|\le r_*+1} \ell^{|\alpha|}\|\partial_a^\alpha g\|_\infty, \qquad F_{r_*} = \|F\|_{C^{r_*+1}},\\
E_{r_*} &= 2^{3r_*+3}A_{r_*}\max(1,w_{r_*})\max(1,g_{r_*})\max(1,F_{r_*}).
\end{aligned}$$

Leibniz and Lemma 3.1 give $[V_0]_{r_*}, [\widetilde{W}_0]_{r_*} \le E_{r_*}Y_*$. Take $a_0 = E_{r_*}$. At each $1 \le n \le J_m$, first form b_j for $0 \le j < n$ from the known a_j, and then define s_n and a_n by

(6.4)

$$b_j = a_j + H_{r_*}(a_{j-1} + a_{j-2}),$$

$$s_n = H_{r_*}\left[6a_{n-1} + 3a_{n-2} + 3\sum_{i+j=n-1} a_i(a_j+b_j) + 3\sum_{i+j=n-2} a_i(a_j+b_j)\right], \qquad a_n = 4D_{r_*}s_n.$$

Use zero for negative indices. Every a_i, a_j, b_j used to form s_n has index less than n. After construction put $a_j = 0$ for $j > J_m$, keep the formula for b_j through J_m+2, and use the formula for s_n through $2J_m+4$. The D_{r_*} is the explicit constant of Lemma 3.2; all numbers in this recursion are fixed before testing the scale inequalities.

Proposition 6.1 (Finite correction estimates)**.** *Under these hypotheses and* (6.3)*, the actual finite construction in Section 5 satisfies*

$$[V_n]_r, [\widetilde{W}_n]_r \le a_n Y_*\delta^n, \qquad [U_n]_r \le H_{r_*}\frac{\chi}{N\sigma}a_n Y_*\delta^n, \qquad r+2n \le r_*. \tag{6.5}$$

For every retained composite and source,

$$\left[\frac{\sigma}{N}Z_j\right]_r \le b_j Y_*\delta^j, \quad r+2j+1 \le r_*, \qquad [S_g^\Gamma]_r, \left[\frac{\sigma}{N}S_g^\xi\right]_r \le s_g Y_*\delta^g/\mathcal{T}, \quad r+2g \le r_*. \tag{6.6}$$

The source bound includes all terminal grades, with the stated zero convention.

Proof. The level-zero estimate follows from the definition of E_{r_*}. Stream recovery is

$$U_n = (N\sigma)^{-1}\kappa^{-1}\partial_s^{-1}\widetilde{W}_n.$$

In particular,

$$N\partial_s U_n = (\sigma\kappa)^{-1}\widetilde{W}_n, \qquad \frac{\sigma}{N}Z_j^\flat = \sigma\mathcal{D}_1 U_{j-1} + \frac{\sigma}{N}\mathcal{D}_2 U_{j-2}.$$

Since $NU_{n,s} = (\sigma\kappa)^{-1}\widetilde{W}_n$, the first expression is

$$\frac{1}{N}\left[2Cp\cdot\nabla_a(\kappa^{-1}\widetilde{W}_{j-1}) + (p\cdot\mathfrak{l})\kappa^{-1}\widetilde{W}_{j-1}\right].$$

It has size at most $H_{r_*}z_1^* a_{j-1}Y_*\delta^{j-1}$ and uses the preceding pair at one additional order. This differentiates κ^{-1} as well as the pair. The second has size at most $H_{r_*}z_2^* a_{j-2}Y_*\delta^{j-2}$ and uses two additional orders. The first two scalar tests give the composite bound, with the one extra phase derivative on its principal part.

For completeness, the calculation of each source scale is as follows. A slow derivative of U_i costs $\chi\lambda/(N\sigma)$ times the pair size. Against the older gradient this gives the third row of (6.3). The full outer time derivative of either lower-elliptic expression costs Π, including differentiation of $C, \mathfrak{l}$ and κ^{-1}. This gives the fourth row. Fast and slow multiplication by $\nabla_a b$ give respectively $\chi B_0/N$ and $\chi\lambda B_0/N^2$ after vorticity normalization. The two linear circulation terms give the next two rows directly. The fast brackets have scale $\chi\lambda/\sigma$ times two pair sizes, and the slow brackets have scale $\chi\lambda^2/(N\sigma)$. For their vorticity versions write $(\sigma/N)Z_j = \partial_s\widetilde{W}_j + (\sigma/N)Z_j^\flat$ and use the already proved composite bound. Finally, the normalized fast and slow derivatives of V_iV_j have scales $\sigma\varpi_* z_0$ and $\sigma\varpi_* h_0\lambda/N$. This accounts for all the displayed terms in (5.11). Dividing by $Y_*\delta^g/\mathcal{T}$ gives exactly the table. The source contains five first-shift linear terms, two second-shift linear terms, and the stated transport and square products; the larger integer coefficients in (6.4) cover their sums. The projected two-component forcing has Euclidean norm at most four times this source majorant, since projection costs two and the mean-zero phase primitive costs $\pi/2$. Lemma 3.2 proves the next pair estimate.

Here is the derivative count behind that induction. Stream inversion does not increase the required pair order. The first lower-elliptic piece uses one label derivative of level $j-1$ and the second uses two of level $j-2$. Their outer time derivatives use respectively two and three total derivatives of those levels. A fast bracket with principal vorticity uses up to two derivatives of its second pair; a slow bracket does so as well. For a lower-elliptic second argument these counts are at most three at level $j-1$ and four at level $j-2$. With $i+j = g-1$ or $g-2$, these all fit the allowance $r+2g$. Every other row uses at most one extra derivative of level $g-1$ or $g-2$. Differentiating a coefficient rather than a pair requires the same total order and cannot increase the largest needed order. The Duhamel estimate requires no extra order. This proves the induction, including the composite indices J_m+1, J_m+2 and the terminal sources through $2J_m+4$. The largest source requirement at output order k is $k+2(2J_m+4) = r_*-1$. The additional order in r_* also covers phase differentiation in activation and mean-primitive differentiation. □

We now give a finite expression for the physical force. This also records which means are retained. Put $\mathsf{A}_i = a_iY_*\delta^i$ for $0\le i\le J_m$ and zero otherwise, and

$$\mathsf{B}_j^\flat = H_{r_*}(z_1^*\mathsf{A}_{j-1} + z_2^*\mathsf{A}_{j-2}) \quad (0\le j\le J_m+2).$$

Let Λ_K be the first-label rate of κ^{-1} from (4.11). Define

$$
\begin{aligned}
M_n^\Gamma &= H_{r_*}\left[\frac{\chi}{\sigma}\sum_{i+j=n-1}\mathsf{A}_i\mathsf{A}_j+\frac{\lambda\chi}{N\sigma}\sum_{i+j=n-2}\mathsf{A}_i\mathsf{A}_j\right],\\
M_n^\xi &= H_{r_*}\left[\frac{N\chi}{\sigma^2}\sum_{i+j=n-1}\mathsf{A}_i\mathsf{B}_j^\flat+\frac{\lambda\chi}{\sigma^2}\sum_{i+j=n-2}\mathsf{A}_i\mathsf{B}_j^\flat\right.\\
&\qquad\left.+\left(\frac{\chi\Lambda_K}{2\sigma^2}+\varpi_*h_0\right)\sum_{i+j=n-2}\mathsf{A}_i\mathsf{A}_j\right].
\end{aligned}
\tag{6.7}
$$

Then $[Q_n^\Gamma]_{k+1}\le M_n^\Gamma$ and $[Q_n^\xi]_{k+1}\le M_n^\xi$. Indeed these are, in order, the absolute values of the terms in (5.19), using $[W_i]=(N/\sigma)[\widetilde{W_i}]$ and the normalized lower-elliptic bound. The reciprocal-symbol term retains Λ_K and the circulation-square term retains ϖ_*h_0. Neither is omitted when the envelope is constant. Thus $[\langle S_n^\Gamma\rangle_s]_k\le 2\lambda M_n^\Gamma$ and $[\langle S_n^\xi\rangle_s]_k\le 2\lambda M_n^\xi$. The required profile orders in these formulas lie below r_*: for $n\le J_m$ their largest pair index is at most $n-1$, and the primitive plus divergence uses at most four additional derivatives.

Let $C_X\ge 1$ bound through order k the normalized coefficient constants of $H,\mathfrak{v}$ and their sizes divided by K,V_X. Define

$$
\Lambda_{\mathrm{ph}}=1+\Pi+(N+\lambda)(K+V_X),\qquad \Lambda_{\mathrm{slow}}=1+\Pi+\lambda(K+V_X),\qquad C_k^{\mathrm{ev}}=k!\,[16(1+C_X)]^{2k+2}.
$$

Repeated physical differentiation in (5.31) gives

$$
\|Q^\natural\|_{C^k_{y,t}}\le C_k^{\mathrm{ev}}\Lambda_{\mathrm{ph}}^k[Q]_k. \tag{6.8}
$$

For a phase-independent profile replace Λ_{ph} by Λ_{slow}. To see the count, each derivative is $H^T(Np\partial_s+\nabla_a)$ or $\partial_t-\mathfrak{v}\cdot(Np\partial_s+\nabla_a)$. At the next application it differentiates either an existing coefficient or the profile. At most $k!$ distributions occur after enlarging by the four coordinate choices per derivative. Coefficient derivatives cost λ or Π, and are bounded by the supplied normalized constants; these are absorbed by Λ_{ph} or Λ_{slow} and the displayed constant. Spatial derivatives of a phase-independent mean introduce no factor N.

The same calculation through order $k+2$ bounds the complete gradients and Hessians in (5.28), including inverse-map curvature. Here use the coefficient maximum C_X through order $k+2$, available from C_* through r_X; the pair orders suffice since $k+2+2i\le r_*$ for $i\le J_m$. Their values at either $Y(t)$ or a supplied curve $Z(t)$ are bounded by these domainwise estimates. At Y, evaluation corresponds to $a=0,s=0$; differentiation along that curve is at fixed label. Along Z, a profile is differentiated by

$$
\partial_t+\left[H(\dot Z-v_<\circ X)\right]\cdot(Np\partial_s+\nabla_a)\quad\text{at } a=A(Z,t).
$$

Given derivatives of $\dot Z$ through the requested order, Leibniz expansion of this operator and of the spatial operators proves the corresponding mixed curve estimates, with constants also depending on those supplied derivatives. This statement preserves (4.4); it does not identify the two centers.

Let $I_{\mathrm{act}}\subset I$ be a closed interval containing the supports of all positive derivatives of w, and suppose $\sup_{I_{\mathrm{act}}}|R(0,t)|\le Y_{\mathrm{act}}$ as in Section 3. Take $E_{k+1}^{\mathrm{act}}=E_{r_*}$. On this activation interval, the two normalized activation profiles are then bounded through k by $E_{k+1}^{\mathrm{act}}\Pi Y_{\mathrm{act}}$. Put $S_{\mathrm{rem}}=\sum_{g=J_m+1}^{2J_m+4}s_g\delta^g$.

Proposition 6.2 (Physical force estimate)**.** *Under the hypotheses of Proposition 6.1, the stated activation bound and the fixed support condition (5.1), the physical force satisfies*

$$\|f_m\|_{C^k_{x,t}(I)} \le C_{k,\mathcal{D}} C^{\mathrm{ev}}_k \Bigg[\Lambda^k_{\mathrm{ph}}(1+N/\sigma)\left(E^{\mathrm{act}}_{k+1}\Pi Y_{\mathrm{act}} + \frac{Y_*}{\mathcal{T}} S_{\mathrm{rem}}\right) + 2\lambda\Lambda^k_{\mathrm{slow}} \sum_{n=1}^{J_m} (M^\Gamma_n + M^\xi_n)\Bigg]. \tag{6.9}$$

The constant $C_{k,\mathcal{D}}$ depends only on k, the fixed off-axis chart and the fixed compact inverse. All other dependencies occur in the displayed numbers. The bound is uniform for supplied fields with these same numerical inputs.

Proof. Equations (5.14), (5.26), the mean estimates and (6.8) bound the complete circulation and reduced-vorticity residuals. The factor N/σ undoes the normalization of the vorticity source.

The inverse curl is applied to activation plus remainder together and to each mean separately, exactly as in (5.26). Their compatibility was proved in Section 5; the triangle inequality does not assert compatibility of either isolated oscillatory piece. The fixed inverse commutes with time derivatives. Its no-loss spatial estimate, the radial factors and the cylindrical basis give $C_{k,\mathcal{D}}$. This proves the physical mixed bound. □

The right-hand side defines the finite expression $\mathfrak{F}_{m,k}$. An application seeking a bound ϵ_k requires the displayed right-hand side to be at most ϵ_k, in addition to (6.3). This is a finite scalar test on inputs, not an assumed estimate on the unknown force. The terminal sum contains powers δ^g with $g \ge J_m + 1$; its depth-dependent constants and full finite sum must also be bounded. Activation is controlled by its small datum and the phase-independent terms by (6.7).

The derivative allocation and dependence are as follows:

quantity	orders supplied or used	scales retained
rigid gradients and non-affinity	$r_F = r_* + 4$	$K, \Pi, V_c, \eta(t), \mathfrak{d}, G_0, B_0, \mu_\phi, \Lambda_{\mathrm{met}}$
composed coefficients and reciprocals	$r_X = r_* + 2$	$\kappa_-^{-1}, K, \zeta_\pm, \Gamma_\pm, W_\pm$, first-label rates
amplitude family and profiles	$r_* = k + 4J_m + 9$	$C_R, c_*, \mathfrak{n}, c_{\mathrm{dat}}, Y_*, Y_{\mathrm{act}}, \Pi, \lambda$
pair at level n	$r + 2n \le r_*$	$Y_*\delta^n, (N\sigma)^{-1}$ for U_n
composite $Z_j, j \le J_m + 2$	$r + 2j + 1 \le r_*$	N/σ
terminal sources $g \le 2J_m + 4$	$r + 2g \le r_*$	$Y_*\delta^g/\mathcal{T}$
mean primitive and divergence	$k+1$ then k	$M^\Gamma_n, M^\xi_n, \lambda,\ \Lambda^k_{\mathrm{slow}}$
physical evaluation and force	k	$\Lambda^k_{\mathrm{ph}},\ N/\sigma,\ C_{k,\mathcal{D}}$

The implication proved here requires the actual older fields to supply: the central weighted and activation hypotheses; the integrated and instantaneous amplitude variation tests and datum variation; the central deformation, speed, non-affinity, scalar and metric bounds on the full containing regions; the transported support condition and positive older circulation when positivity is requested; and the displayed scalar scale tests. Proposition 11.1 returns these inputs for the same finite corrected family under its closed scalar conditions; its activation specialization retains the prescribed seed and short ramp. Section 7 supplies the local family used there. Proposition 12.3 realizes all the scalar conditions on one infinite selected history.

7. Finite controlled families

The angular velocity must be constructed together with the older corrected fields: those fields determine the angular velocity, and every material map subsequently uses it. At a fixed number of increments this dependence has an integral form in time. We first prove existence of that family, and then give sufficient quantitative conditions for a return.

Choose an open ball $\mathcal{P} \Subset \mathcal{D}_0$ about y_* on which $H = |y - y_*|^2/2$ and both base cutoffs equal one. All material supports considered in this section have containing collars transported into $\mathcal{P}$. The fixed coefficients are still

$$\mathsf{A}(y) = \operatorname{diag}((2y_2)^{-1}, 1), \qquad W(y) = (2y_2)^{-2}, \qquad y_* = (z_0, r_0^2/2).$$

No change of the physical coordinates is made.

Fix a finite collection of frequencies N_j, radii, depths J_j, positive constants σ_{j-1}, envelopes and activation functions. At the beginning t_0 of the interval under consideration, all existing profiles and maps have smooth data on collars of their material supports. The maps are restrictions of the global older flows. At a new insertion, $X_j(a, t_0) = Z(t_0) + a$ and the higher levels have zero data. The amplitude datum may be any smooth pair; when a growing datum is required, write $\widehat{a}_j, \widehat{b}_j$ for the two off-diagonal coefficients of (3.4). They must be strictly positive on its whole collar, and the datum is

$$-\epsilon_j \left(1, \sqrt{\widehat{b}_j/\widehat{a}_j}\right), \qquad \epsilon_j > 0.$$

The square root is evaluated at insertion. Its relative derivatives are separate quantitative requirements when weighted estimates are used.

The state consists of ϑ, all the X_j, the amplitude pairs, all pairs $(V_{j,n}, \widetilde{W}_{j,n})$ for $1 \le n \le J_j$, and the full-flow point Z. The physical increments are the complete sums

$$\gamma_j = \left(\sum_{n=0}^{J_j} V_{j,n}\right)^{\natural}, \qquad \psi_j = \left(\sum_{n=0}^{J_j} U_{j,n}\right)^{\natural}, \qquad v_j = J\nabla\psi_j.$$

Write $\alpha = \dot{\vartheta}$. The base streamfunction $\Psi^b = \alpha H$ and vorticity $\xi^b = \alpha L H$ are recovered from the current rate and are not integrated state variables. In particular a central zero of Ω_j leaves all these definitions in force. During a stage ending with increment m, Z is read only at the next insertion. The centers $Y_j = X_j(0, t)$, which enter the current control, are distinct from Z.

7.1. **The control and its finite derivative requirements.** At a center lying in $\mathcal{P}$, write

$$D_j = \alpha J + E_j, \qquad E_j = \sum_{i<j} \nabla v_i(Y_j, t), \qquad \mathcal{F}_j = -\frac{(E_j^T \zeta_j)_1}{\zeta_{j,2}}, \qquad \mathcal{F}_0 = 0, \tag{7.1}$$

where in center expressions $\zeta_j = M_j^{-T} p_j$ and $\alpha = \dot{\vartheta}$. The open domain of a control requires $|\zeta_{j,2}| > 0$ for every feedback that it uses, positive symbols, and the stated interior support conditions. The following control branches are smooth functions of spatial state values on this domain:

$$\alpha = \mathcal{F}_{m-1}, \qquad \alpha = (1-\eta_b)\mathcal{F}_{m-1} + \eta_b \mathcal{F}_m, \qquad \alpha = \mathcal{F}_m - \frac{\dot{z}_\mu(t)}{\zeta_{m,2}}, \qquad \alpha = \mathcal{F}_m. \tag{7.2}$$

The first branch keeps the parent's first covector component fixed (and is $\alpha = 0$ when $m = 1$). The next three branches interpolate to the current feedback, prescribe the current first component, and keep that component fixed after return, respectively; see Lemma 7.4. Here η_b is a prescribed smooth transition and z_μ is a prescribed covector component with its entry value frozen. The parameter μ is fixed during a trial. In particular $\dot{z}_\mu$ is prescribed data, not the time derivative of an evolving state evaluation.

The identity $D_j - \alpha J = E_j$ in (7.1) is an identity of the actual fields. Although $\xi^b = \alpha L H$ enters the level sources instantaneously, it is absent from the feedback except through the integrated profiles. This distinction is essential below.

Lemma 7.1 (Finite derivative allocation). *For fixed $m, J_1, \ldots, J_m$ and requested spatial output order k, a finite backward allocation of spatial derivatives supplies all the maps, coefficients, profiles, level rates, physical evaluations and feedbacks through order k, including locally Lipschitz dependence on a continuous trial rate. The allocation is obtained by the rules below; it is independent of any derivative of the trial rate. For requested time order b, a further finite allocation supplies the state* *through order $b+1$ from the trial rate through order b. For an integer $k \geq 3$, one sufficient spatial allocation is*

$$r_m^{\mathrm{loc}} = k+4, \qquad r_{j-1}^{\mathrm{loc}} = r_j^{\mathrm{loc}} + 4J_j + 12 \quad (j = m, \ldots, 1). \tag{7.3}$$

Initial data and fixed base profiles through r_0^{loc} suffice. For smaller requested orders use $k = 3$. These local orders are distinct from the derivative orders admitted in the infinite construction.

Proof. We give a finite algorithm, so that an asserted output order specifies its actual input derivatives. Count phase and label derivatives together. Start with all desired spatial outputs, including the two derivatives of each stream profile used in E_j. Enlarge the requested order by one when comparing evaluations at different moving points. Generate the following finite list of expressions in increasing layer order. For a layer generate X, H, C, κ, q, b, the other coefficients in (5.2), the principal pair and its rate, and then each level, its rate and its recovered stream profile in increasing level order. Generate the full-flow point last.

Before assigning orders, expand each outer derivative in (5.7). In a lower profile derivative use

$$(U_n)_t = (N\sigma)^{-1}\left\{(\kappa^{-1})_t \partial_s^{-1}\widetilde{W}_n + \kappa^{-1}\partial_s^{-1}(\widetilde{W}_n)_t\right\}.$$

Replace $(\widetilde{W}_n)_t$ by its already generated level equation, and use the principal equation when $n = 0$, including $\dot{w}$. Use $\mathfrak{l}_t = \operatorname{div}_a C_t$ and (5.30) for C_t, κ_t. No derivative of b in time is required in (5.11). In particular one must use $b = (\alpha LH + \sum_{i<j} L\psi_i) \circ X_j$ directly, rather than differentiating an older residual to compute it. Every substituted time derivative reduces either the level index, or the layer index when an older physical field is involved. The procedure terminates. Its resulting expressions contain $\alpha(t)$, but no $\dot{\alpha}(t)$.

On this finite expression list propagate derivative requests backward. A sum, product or smooth scalar function on a compact range requests the same order of every argument. A spatial derivative of order d requests $k + d$ to produce order k. Phase average and the literal primitive request order k. A composition $f \circ g$, including a physical oscillatory evaluation, requests order $k+1$ of both factors for an order-k Lipschitz estimate. An inverse map through order k requests the map through order $k + 1$ and a bound for its inverse Jacobian. A flow through order k requests the driving velocity through order $k+1$; a pointwise linear equation through order k requests its coefficients, source and data through order k. Whenever a request reaches a previously listed expression, increase its order to the maximum of all requests there and continue backward. The finite acyclic list makes this a terminating recursion in integers.

For completeness, the rules have explicit analytic justifications. For a derivative with position set S, the chain rule is

$$D^S(f \circ g) = \sum_{\pi \in \mathrm{Part}(S)} (D^{|\pi|}f) \circ g\, [D^{B_1}g, \ldots, D^{B_{|\pi|}}g].$$

Subtract two such expressions and telescope their factors. A mean-value estimate of $D^{|\pi|}f$ at two arguments costs one extra derivative. Differentiating $X \circ A = \mathrm{Id}$ first gives $DA = (DX \circ A)^{-1}$. For $k \geq 2$, it expresses $D^k A$ as $-(DX \circ A)^{-1}$ times a sum involving derivatives of X through k and lower derivatives of A. The same subtraction proves its Lipschitz rule. Differentiating the flow or the linear equation produces a linear equation for the highest derivative, with a finite sum of lower derivatives as source. Integrating it proves the indicated rule by induction in the derivative order. The inverse Jacobian bound comes from its own linear equation; globally the flow is one-to-one by uniqueness of characteristics.

These rules count the actual sources. For example, a slow bracket with Z_j requests one derivative of Z_j, whose lower elliptic part requests two of U_{j-2}. An outer derivative of that part requests two derivatives of the substituted lower rate. The older-vorticity terms request three spatial derivatives of an older streamfunction. Derivatives of κ^{-1}, the nonlinear inverse map, and the material metric are included as expressions, not treated as constants. Thus this allocation includes the extra Euler terms.

To check (7.3), at layer j allocate the level n pair and its substituted rate through $r_j^{\mathrm{loc}}+4(J_j-n)+4$. Allocate map differences through $r_j^{\mathrm{loc}}+4J_j+8$, with one further uniform map derivative. The four-derivative difference between successive level orders covers the three derivatives in a bracket containing a composite and the two derivatives of a substituted lower rate. At coefficient order a, $C_t, \mathfrak{l}_t$ need older primitive streamfunctions through at most $a+3$, with one further derivative for comparison; their map requirements are at most $a+2$. All these requests lie below the stated map and older-field allocations. The flow comparison through the allocated map order, including its uniform spare, needs older streamfunctions through at most $r_j^{\mathrm{loc}}+4J_j+10$, less than r_{j-1}^{loc}. The four extra profile orders at the last level cover physical evaluation and its comparison. This verifies the numerical allocation by the same backward rules.

To request time derivatives, differentiate the expanded rate equations b times. Their right sides involve the rate through b and the state through b; a term of highest state time order is obtained by differentiating previously generated layers or levels. Generate these expressions in time order, then layer order and level order, and apply the same spatial request rules. The expression for a time derivative of a composition is the same partition formula with time positions included. This gives a finite allocation for each b, without claiming one finite spatial order for all b. □

Proposition 7.2 (Local finite family)**.** *Suppose the smooth initial data belong to the open control domain, with all initial transported collars compactly contained in $\mathcal{P}$. Each branch of (7.2), and each smooth finite interpolation of these branches, has a unique local solution of the actual finite construction. The solution is smooth in space and time. It depends smoothly on a finite-dimensional parameter in the smooth data and prescribed control profiles, on a common local interval for parameters in a sufficiently small compact neighborhood. All intervals and constants here may depend on the fixed finite data.*

Proof. First prescribe any $\alpha \in C([t_0, t_0+h])$ with $\|\alpha\|_\infty \le A$. Set $\vartheta(t) = \vartheta(t_0) + \int_{t_0}^t \alpha$. Construct layer 1, then layer 2, and so on. Its older velocity is continuous in time, smooth and compactly supported in space, so its characteristic equation has a unique flow. The flow and its inverse are smooth in space and continuously differentiable in time. Then solve the pointwise principal linear equation and the successive linear level equations, using the substituted sources of Lemma 7.1. Extend each compact profile by zero after physical evaluation. Its support stays away from the boundary of its collar, so this is a smooth spatial extension. Finally solve the equation for Z with the full velocity.

All spatial derivatives of the resulting state exist, and are continuously differentiable in time. For any finite allocation they are bounded on a common sufficiently short interval. This follows in layer and level order from the differentiated flow equations and linear equations; their coefficients are bounded on the compact collars. The interval can first be made small enough to preserve the initial support and denominator margins.

Let $\mathcal{Y}[\alpha]$ denote the resulting state. With identical initial data, put $I_{\alpha,\alpha'}(t) = \int_{t_0}^t |\alpha - \alpha'|\, ds$. The same differentiated equations and the allocation just proved give

$$\|\mathcal{Y}[\alpha](t) - \mathcal{Y}[\alpha'](t)\|_{\mathrm{sp,k}} \le C_k I_{\alpha,\alpha'}(t), \tag{7.4}$$

for all requested spatial state quantities. Here the norm is the maximum of the finitely requested C^k phase-label, map and point norms, with the allocated larger input orders understood. Indeed differences of rates at one step are bounded by a constant times $|\alpha(t) - \alpha'(t)|$, earlier state differences and the unknown difference at that step. Gronwall and then induction give (7.4). Differences of instantaneous rates instead satisfy

$$\|\partial_t\mathcal{Y}[\alpha](t) - \partial_t\mathcal{Y}[\alpha'](t)\|_{\mathrm{sp,k}} \le C_k\big(|\alpha(t) - \alpha'(t)| + I_{\alpha,\alpha'}(t)\big). \tag{7.5}$$

This direct term is integrated when the rate is used at the next level.

Write the chosen right side of (7.2) as $u(t, \mathcal{Y}; \mu)$. It reads spatial values, and (7.1) cancels its instantaneous rigid rotation. Consequently

$$|u(t, \mathcal{Y}[\alpha]; \mu) - u(t, \mathcal{Y}[\alpha']; \mu)| \leq C I_{\alpha,\alpha'}(t).$$

On the closed unit ball about the constant initial rate $u(t_0, \mathcal{Y}(t_0); \mu)$, continuity and a smaller h make the map $\alpha \mapsto u(t, \mathcal{Y}[\alpha]; \mu)$ preserve the ball. Taking $Ch < 1$ makes it a contraction in the supremum norm on continuous rates. This proves existence and uniqueness. Any two local solutions can be compared on successive intervals of this length, which removes the restriction to that initial ball.

The spatial state derivatives are C^1 in time, hence the feedback gives $\alpha \in C^1$. The time allocation in Lemma 7.1 then gives C^2 state derivatives. Repeating yields smoothness of every requested mixed derivative. For parameter derivatives the triangular integral equations can be differentiated: the first derivative solves the corresponding linear variational equations, and the remainder quotient tends to zero by the mean-value bounds above. Repetition gives every order, with a finite enlarged spatial allocation at each order. The differentiated fixed-point equation has the form $(\mathrm{Id} - K)h = g$, $\|K\| < 1$, so its inverse is $\sum_{n \geq 0} K^n$. The same argument proves smooth parameter dependence of the fixed point, without applying a smooth inverse-map theorem on a single finite differentiability space. Flat transition profiles give matching time jets at the junctions; inserted physical profiles and their levels have zero jets there by the flat activation and their zero-data integral formulas. □

Proposition 7.3 (Finite continuation)**.** *Let a solution with a fixed finite number of increments have maximal interval $[t_0, T)$, $T < \infty$. Suppose its transported collars remain in a fixed compact subset of $\mathcal{P}$, the prescribed data and control profiles extend smoothly past T, α is bounded, and the feedback denominators are uniformly separated from zero. Then the finite solution extends past T. A positive circulation window used for an additional operation must also stay separated from zero. These conditions can be imposed uniformly for a compact family of trials, giving a common continuation interval.*

Proof. At layer 1 the bounded rate bounds every spatial derivative of the base velocity and base circulation on the fixed geometry. The differentiated flow equations bound its map and inverse derivatives on the finite interval. Ellipticity and these inverse bounds bound κ^{-1}. Its principal pair is bounded by the exponential of an integral of bounded coefficients. Each successive level is a linear equation with bounded coefficients and previously bounded source. Apply the backward allocation for each prescribed output order. This bounds all spatial derivatives of layer 1 and its first rate. Continue in layer order. Polynomial nonlinear sources in lower levels cause no differential inequality in an unknown highest level. This gives bounds for every fixed spatial derivative of every layer and for Z up to T.

The bounded spatial state and positive denominator margins bound $\partial_t \alpha$ through the feedback equation. The time allocation and induction give all higher mixed bounds. In each C^k norm the first rate is bounded, so the state has a limit as $t \uparrow T$; taking all k gives smooth limiting spatial data. The map and its inverse have limiting inverse Jacobians and remain inverse maps. The assumed support and denominator margins place these data in the open domain. Proposition 7.2 restarts the equations at T. The same bounds are uniform on a compact parameter set, giving the last assertion. □

7.2. **Exact center and covector identities.**

Lemma 7.4 (Center identities and prescribed covectors)**.** *For the finite solution, the center and covector equations below hold for $1 \le j \le m$. The older-center identity uses $1 \le i < j \le m$, and the full-point identity uses $1 \le i \le m$:*

$$\begin{aligned} &\dot{Y}_j = \alpha J(Y_j - y_*) + \sum_{i<j} v_i(Y_j), && \dot{M}_j = (\alpha J + E_j)M_j, \\ &\dot{\zeta}_j = \alpha J\zeta_j - E_j^T \zeta_j, && \dot{Z} = \alpha J \nabla H(Z) + \sum_{i\le m} v_i(Z), \\ &\frac{d}{dt}A_i(Y_j,t) = D_y A_i(Y_j,t) \sum_{i\le l<j} v_l(Y_j,t), && \frac{d}{dt}A_i(Z,t) = D_y A_i(Z,t) \sum_{i\le l\le m} v_l(Z,t). \end{aligned} \tag{7.6}$$

The older centers lie in $\mathcal{P}$. The displayed equation for Z uses the global base velocity; it reduces to the quadratic expression $\alpha J(Z - y_*)$ only when $Z \in \mathcal{P}$. Moreover $\det M_j = 1$ *and*

$$\dot{\zeta}_{j,1} = (\mathcal{F}_j - \alpha)\zeta_{j,2}. \tag{7.7}$$

Thus the third branch of (7.2) *gives $\zeta_{m,1} = z_\mu$ whenever their entry values agree. If the selected endpoint satisfies $z_\mu = \widehat{\Omega}_m(0,t) = 0$, then under the fourth branch both remain zero and $T_m(0,t)$ is constant for as long as that branch exists.*

Proof. The flow equations give the first line and the full-flow equation. Differentiating $M_j^{-T} p_j$ gives the covector equation. Its first component is $-\alpha\zeta_{j,2} - (E_j^T\zeta_j)_1$, proving (7.7). The determinant satisfies $(\det M_j)' = (\operatorname{tr} D_j)\det M_j = 0$. The equations for the older labels follow from $(\partial_t + v_{<i} \cdot \nabla)A_i = 0$ by subtracting velocities. Finally the central lower amplitude coefficient is $\sigma_{m-1}c_m(0,t)\zeta_{m,1}$. With $z_\mu = 0$, the second amplitude is constant and hence zero; the first equation then has zero right side. Uniqueness for these equations proves the claim. ∎

For clarity, the central coefficients are exactly

$$\widehat{a}_j = -\frac{(J\zeta_j)\cdot\nabla\Gamma_{<j}(Y_j,t)}{\sigma_{j-1}\kappa_j(0,t)}, \qquad \widehat{b}_j = \sigma_{j-1}\, 2W(Y_j)\Gamma_{<j}(Y_j,t)\zeta_{j,1}, \quad \kappa_j(0,t) = \frac{\zeta_{j,1}^2}{2Y_{j,2}} + \zeta_{j,2}^2. \tag{7.8}$$

Every older layer in these formulas is evaluated at $(s,a) = (N_i p_i \cdot A_i(Y_j,t), A_i(Y_j,t))$ using (5.28). Its pure-phase strain has the factor $|\zeta_i|^2/\kappa_i$; mixed derivatives, label Hessians and inverse-map Hessians contribute the rest. In particular the two covectors at their respective own centers do not generally satisfy the same linear equation. If $i < j$ and $a \wedge b$ denotes the determinant of two vectors, then

$$\frac{d}{dt}(\zeta_i \wedge \zeta_j) = \big[(D_j - D_i)^T\zeta_i\big] \wedge \zeta_j, \quad D_j - D_i = \nabla v_{<i}(Y_j) - \nabla v_{<i}(Y_i) + \sum_{i\le l<j} \nabla v_l(Y_j). \tag{7.9}$$

This follows by adding and subtracting $-D_j^T\zeta_i$; the trace-free common part cancels. The displacement term remains.

The coefficient rates also retain the physical weights. Put $G_j = \nabla_y\Gamma_{<j}(Y_j,t)$ and $F^{\Gamma}_{<j} = (\partial_t + v_{<j}\cdot\nabla_y)\Gamma_{<j}$. With every right side evaluated at Y_j, differentiation gives

$$\begin{aligned} \dot{G}_j &= -D_j^T G_j + \nabla_y F^{\Gamma}_{<j}, \\ \partial_t d_j(0,t) &= (J\zeta_j)\cdot\nabla_y F^{\Gamma}_{<j}, \\ \partial_t c_j(0,t) &= 2WF^{\Gamma}_{<j} + 2\Gamma_{<j} v_{<j}\cdot\nabla_y W, \\ \partial_t \kappa_j(0,t) &= \zeta_j^T\big[(v_{<j}\cdot\nabla_y)\mathsf{A} - D_j\mathsf{A} - \mathsf{A}D_j^T\big]\zeta_j. \end{aligned} \tag{7.10}$$

The first identity is the differentiated circulation equation. For the second, differentiate $d_j(0,t) = (J\zeta_j)\cdot G_j$ and use $JD_j^T = -D_jJ$, which follows from $\operatorname{tr} D_j = 0$; the two velocity-gradient terms cancel. The last two identities are the product rule applied to $2W(Y_j)\Gamma_{<j}(Y_j,t)$ and $\zeta_j^T \mathsf{A}(Y_j)\zeta_j$. No physical coefficient is replaced by a constant in these formulas.

The volume-coordinate gradient on the moving circle is

$$G(t) = \nabla\Gamma(Z(t),t), \qquad \dot G = -(\nabla v(Z,t))^T G + \nabla F^\Gamma(Z,t).$$

For $t_b \le t$ set $K(t) = \int_{t_b}^t \|\operatorname{sym}\nabla v(Z(s),s)\|\,ds$. Differentiating the squared norm of a solution of the homogeneous equation bounds its fundamental matrix and its inverse by $e^{K(t)}$, so variation of constants gives the complete-gradient bound

$$|G(t)| \ge e^{-K(t)}\left(|G(t_b)| - \int_{t_b}^t e^{K(s)}|\nabla F^\Gamma(Z(s),s)|\,ds\right). \tag{7.11}$$

The physical gradient satisfies $\nabla_{z,r}\Gamma = \operatorname{diag}(1,r)\nabla_y\Gamma$; the fixed off-axis region gives uniform norm equivalence. Thus quantitative bounds on the two displayed integrals preserve a positive fraction of the complete endpoint gradient. Alternatively, when the inverse-map evaluations give

$$|\nabla\Gamma(Z,t) - N_mT_m(0,t_b)\zeta_m(t)| \le \epsilon|N_mT_m(0,t_b)\zeta_m(t)|, \quad \epsilon<1,$$

the gradient is at least $(1-\epsilon)|N_mT_m(0,t_b)\zeta_m(t)|$. These are two sufficient quantitative conditions. Neither follows from $B_m(0,t)=0$: off-center principal amplitudes and all correction levels remain in the complete formulas.

7.3. **A return from quantitative bounds.** The following comparison uses only a positive bounded coupling, so its hypotheses do not require a unit physical metric. Fix a smooth nondecreasing step η with its usual flat endpoints and $\eta_2 \in C_c^\infty((0,1))$, $\eta_2 \ge 0$, $\int_0^1 \eta_2 = 1$. Let $\tau = \gamma(t-t^1)$ for a fixed $\gamma>0$ and put $\tau_b = 1+\Lambda_{\rm ret}^{-1}$. The entry component $z_e>0$ and a positive pulse scale z_p are frozen. Prescribe

$$z_\mu(t) = \begin{cases} z_e(1-\eta(\tau)), & 0\le\tau\le 1,\\ -\mu z_p\Lambda_{\rm ret}\eta_2(\Lambda_{\rm ret}(\tau-1)), & 1\le\tau\le\tau_b,\end{cases} \qquad 0\le\mu\le c_p. \tag{7.12}$$

After τ_b prescribe zero. This is the short negative pulse in [1, Lemma 2.7, Lemma 2.9], with its entry covector fixed for the whole trial family. For the prescribed construction, write $\theta_m^{\rm in}$ for the fixed insertion angle, with

$$p_m = (\sin\theta_m^{\rm in}, \cos\theta_m^{\rm in}), \qquad \sin\theta_m^{\rm in} > 0.$$

Thus this angle is measured from the positive second coordinate toward the positive first coordinate; the physical phase remains s_m in (3.2). Use $z_p = |\zeta_m(t^1)|\sin\theta_m^{\rm in}$. The coefficient of the pulse profile in $z_\mu/|\zeta_m(t^1)|$ is then $-\mu\Lambda_{\rm ret}\sin\theta_m^{\rm in}$. The bounded applications below have $z_p = z_e$.

For the central pair write $T_\tau = a\widehat\Omega$, $\widehat\Omega_\tau = bT$, where $a = \widehat a_m/\gamma$ and $b = \sigma_{m-1}c_mz_\mu/\gamma$. For a fixed constant $c_0>0$ put

$$\upsilon = c_0\widehat\Omega/T, \qquad \mathfrak{k} = a/c_0, \qquad c(\tau) = c_0\sigma_{m-1}c_m(0,t)z_p/\gamma, \qquad c_r(\tau) = (z_e/z_p)c(\tau). \tag{7.13}$$

Where $T\ne 0$, its exact equation is

$$\upsilon' = \begin{cases} c_r(\tau)(1-\eta(\tau)) - \mathfrak{k}(\tau)\upsilon^2, & 0\le\tau\le1,\\ -\mu c(\tau)\Lambda_{\rm ret}\eta_2(\Lambda_{\rm ret}(\tau-1)) - \mathfrak{k}(\tau)\upsilon^2, & 1\le\tau\le\tau_b.\end{cases} \tag{7.14}$$

The coefficient c may depend on both time and the trial. Its definition retains the actual $2W\Gamma_{<m}$.

Lemma 7.5 (Positive amplitude and opposite endpoint signs)**.** *Assume on every existing initial part of the interval that*

$$0 < c_- \le c \le c_+, \qquad 0 \le c_r \le c_{r,+}, \qquad 0 < \mathfrak{k}_- \le \mathfrak{k} \le \mathfrak{k}_+, \qquad \upsilon(0) \in [\upsilon_-, \upsilon_+], \quad \upsilon_- > 0, \tag{7.15}$$

and $T(0) < 0$. *Put*

$$\mathsf{V}_+ = \max(\upsilon_+, \sqrt{c_{r,+}/\mathfrak{k}_-}), \quad \mathsf{V}_- = \frac{\upsilon_-}{1 + \mathfrak{k}_+ \upsilon_-}, \quad B = 2(\mathsf{V}_+ + c_p c_+ + 1).$$

Choose $c_p c_- > \mathsf{V}_+$ *and* Λ_{ret} *so large that* $\mathfrak{k}_+ B^2/\Lambda_{\text{ret}} \le \min(1, \mathsf{V}_-/2)$. *Then* T *remains negative and* $|\upsilon| \le B$ *on every existing initial segment. For trials defined through* τ_b, *the endpoint signs are*

$$\upsilon(\tau_b; 0) > 0, \qquad \upsilon(\tau_b; c_p) < 0. \tag{7.16}$$

Every zero member satisfies

$$\frac{\mathsf{V}_-}{2c_+} \le \mu \le \frac{\mathsf{V}_+}{c_-}, \qquad \upsilon \ge 0 \quad \textit{on the negative pulse.}$$

The logarithmic gain of $-T$ *is bounded in absolute value by* $\mathfrak{k}_+ B(1 + \Lambda_{\text{ret}}^{-1})$ *on every trial; on a zero member it is at least* $\mathfrak{k}_- \mathsf{V}_-$.

Proof. Before a possible zero of T, scalar comparison on the first part gives $\upsilon \le \mathsf{V}_+$, since the right side at V_+ is nonpositive. While $\upsilon > 0$, $(1/\upsilon)' \le \mathfrak{k}_+$, giving $\upsilon \ge \upsilon_-/(1 + \mathfrak{k}_+ \upsilon_- \tau) \ge \mathsf{V}_-$ and excluding a first zero. On the negative pulse put $y = \Lambda_{\text{ret}}(\tau - 1)$ and $\mathsf{V}(y) = \upsilon(1 + y/\Lambda_{\text{ret}})$. The exact integral identity is

$$\mathsf{V}(y) = \upsilon(1) - \mu \int_0^y c(1 + r/\Lambda_{\text{ret}}) \eta_2(r)\, dr - \Lambda_{\text{ret}}^{-1} \int_0^y \mathfrak{k}(1 + r/\Lambda_{\text{ret}}) \mathsf{V}(r)^2\, dr. \tag{7.17}$$

It is nonincreasing. Under the temporary bound $|\mathsf{V}| \le B$, its lower bound is $-c_p c_+ - 1 > -B$, and its upper bound is $\mathsf{V}_+ < B$. Hence a first exit from that bound cannot occur. On either part, $(\log(-T))' = \mathfrak{k}\upsilon$ where $T < 0$, so

$$|T(\tau)| \ge |T(0)| \exp[-\mathfrak{k}_+ B(1 + \Lambda_{\text{ret}}^{-1})].$$

This excludes a first zero of T and extends the ratio estimates over the whole existing interval. At $\mu = 0$, the pulse equation has the positive solution $\upsilon(\tau) = \upsilon(1)/(1 + \upsilon(1) \int_1^\tau \mathfrak{k})$. At $\mu = c_p$, (7.17) gives $\upsilon(\tau_b) \le \mathsf{V}_+ - c_p c_- < 0$. If the endpoint is zero, the same identity gives

$$\mu \int_0^1 c\eta_2 = \upsilon(1) - \Lambda_{\text{ret}}^{-1} \int_0^1 \mathfrak{k} \mathsf{V}^2 \in [\mathsf{V}_-/2, \mathsf{V}_+].$$

This proves the bounds on μ. Monotonicity gives $\mathsf{V} \ge 0$ for that member. Integrating $\mathfrak{k}\upsilon$ proves the two gain statements. ☐

Proposition 7.6 (Return for one common finite family)**.** *Start every trial in* (7.12) *from the same smooth corrected history at* t^1, *with* $\zeta_{m,1} = z_e$ *and* $T_m < 0$. *Use the third control in* (7.2), *and construct the local solutions by Proposition 7.2. Suppose the following bounds hold uniformly in* $\mu \in [0, c_p]$ *on every existing initial interval before* $t^b = t^1 + \tau_b/\gamma$: *the transported collars lie in one compact subset of* $\mathcal{P}$; *all control denominators have a common positive lower bound; all the* E_j *used by the control are bounded; every required older circulation has a common positive lower bound; and* (7.15) *holds with the constants and strict inequalities of Lemma 7.5. Then the whole common smooth family exists through* t^b *and has a nonempty compact zero set*

$$\{\mu \in (0, c_p) : \Omega_m(0, t^b; \mu) = 0\}.$$

It therefore has a least member. On each zero member $\zeta_{m,1}(t^b) = \Omega_m(0, t^b) = 0$, *and the central holding conclusion of Lemma 7.4 applies. The selected root need not depend smoothly on the history.*

Proof. The local family is already supplied by a proved construction, not by a return assumption. On every existing segment the feedback satisfies $|\alpha| \le C\|E_m\||\zeta_m| + C|\dot{z}_\mu|$, where the denominator bound is included in C. The covector length obeys $|(\log|\zeta_m|)'| \le \|E_m\|$, since the skew rotation has zero quadratic form. Hence the bounded E_m and finite prescribed pulse bound α uniformly. The assumed support and denominator margins, together with Proposition 7.3, exclude an endpoint before t^b for any trial. The same proof gives uniform finite spatial and mixed bounds for the compact parameter set; local smooth dependence therefore patches along the full interval.

The amplitude lower bound in Lemma 7.5 makes the endpoint quotient continuous. It has opposite endpoint signs, so the intermediate value theorem supplies a zero. The zero set is closed in the compact trial interval and avoids both ends by the strict signs. Multiplication by the nonzero T_m and fixed positive constants identifies its zeros with those of Ω_m. The pulse vanishes near t^b, so the selected central pair is already constant there. The fourth branch continues its zero first covector component. Smoothness of an individual trial does not assert transversality of the parameter zero; this is why only existence of a least root is stated. □

Proposition 7.7 (The generic quantitative return conditions)**.** *For $m \ge 2$, the preceding return result applies to the prescribed generic pulse with $\gamma = \sigma_{m-1}\sin\theta_m^{\rm in}$, $z_p = \mathfrak{z}_e \sin\theta_m^{\rm in}$, $\mathfrak{z} = |\zeta_m|$, $\mathfrak{z}_e = \mathfrak{z}(t^1)$ and $c_0 = \mathfrak{z}_e^{-1}$, under the following sufficient conditions. They are conditions on the actual common finite trial solutions, on every existing initial interval, in addition to the support, denominator, complete E_j and circulation bounds of Proposition 7.6. Write $\mathcal{W} = \Omega_{m-1}(0,t)/\sigma_{m-1}$. Suppose there are continuous coefficients c_i, d_i such that*

(7.18)
$$\begin{aligned} &\mathfrak{z}' = c_1\mathcal{W} + d_1, && \mathcal{W}' = c_2\mathcal{G}/\mathfrak{z} + d_2, && \mathfrak{k} = c_5\mathfrak{z}_e/\mathfrak{z}^2 + d_4,\\ &\mathcal{G} = \eta + d_3(1-\eta) \quad (\tau \le 1), && \mathcal{G} = 1 + \mu c_3 \Lambda_{\rm ret}\mathfrak{z}_e \eta_2(\Lambda_{\rm ret}(\tau-1)) \quad (\tau \ge 1), &&\\ &c_r = c_4/\mathfrak{z}_e \quad (\tau \le 1), && c = c_6 \quad (\tau \ge 1). && \end{aligned}$$

Assume $|c_i - 1| \le \delta$ for $1 \le i \le 6$ and $|d_i| \le \delta$ for $1 \le i \le 4$ on their indicated intervals. Here c_4, c_6 may depend on time and on the trial. Assume also

$$\begin{gathered} |\mathfrak{z}_e - 1| \le \delta, \quad |\mathcal{W}(0)| \le \delta, \quad |\upsilon(0) - 1| \le \delta, \\ \tfrac{1}{2} \le \mathfrak{z}/\mathfrak{z}_e \le 2, \qquad -1 \le \mathcal{W} \le 2 + c_p, \qquad \Lambda_{\rm ret}^{-1} \le \delta. \end{gathered} \tag{7.19}$$

For fixed $c_p > \bar{\upsilon} > 3/2$, sufficiently small $\delta > 0$ gives strict recovery of these two geometric bounds, a common return family and a least return root. With C depending only on $c_p, \bar{\upsilon}$, each zero member has

$$\tfrac{1}{2} - C\delta \le \mu \le \frac{\bar{\upsilon}}{1-\delta}, \qquad \tfrac{1}{3} - C\delta \le \mathcal{W}(\tau_b) \le 1 + \bar{\upsilon} + C\delta.$$

Its logarithmic gain satisfies

$$(1 - C\delta)I_0 \le \log\frac{-T(\tau_b)}{-T(0)} \le (1 + C\delta)\bar{\upsilon}, \qquad I_0 = \int_0^1 \frac{d\tau}{(1+\tau^2/2)^2(1+\tau)}.$$

Proof. These are the variables and geometric hypotheses of [1, Lemma 2.9]. The Euler changes in the scalar equation are exactly the time-dependent c_4 and the additional positive factor c_6 on the pulse; we check them explicitly. First the geometric equations, which do not read υ, give $\mathcal{W}' \ge -C\delta$ and $\mathfrak{z}' \ge -C\delta$ on $[0,1]$. Thus $\mathcal{W} \ge -C\delta$ and $\mathfrak{z} \ge \mathfrak{z}_e - C\delta$. Inserting this lower bound in the equation for $\mathcal{W}$ and then the equation for $\mathfrak{z}$ gives

$$\mathcal{W} \le \tau + C\delta, \qquad 1 - C\delta \le \mathfrak{z}/\mathfrak{z}_e \le (1 + C\delta)(1 + \tau^2/2).$$

On the pulse, $\mathcal{G} \ge 1$ and the original box gives $\mathcal{W}' > 0$ for small δ. The lower bound for $\mathfrak{z}$ therefore persists. Integrating the pulse term first in $\mathcal{W}$ and then in $\mathfrak{z}$, using its unit mass and length $\Lambda_{\rm ret}^{-1} \le \delta$, gives

$$-C\delta \le \mathcal{W} \le 1 + c_p + C\delta, \qquad 1 - C\delta \le \mathfrak{z}/\mathfrak{z}_e \le \tfrac{3}{2} + C\delta.$$

These intervals lie strictly inside the initial geometric box. They imply $\mathfrak{k} > 0$, $\mathfrak{k} \le 1 + C\delta$ on the whole interval and $\mathfrak{k} \ge (1 - C\delta)(1 + \tau^2/2)^{-2}$ on its first part. In particular $\mathfrak{k} \ge 4/9 - C\delta$ on the whole interval after decreasing δ once more.

On the first part the forcing is nonnegative and at most $1 + C\delta$. The reciprocal comparison gives $\upsilon \ge (1 - C\delta)/(1 + \tau)$. At $\upsilon = \bar{\upsilon}$, its derivative is at most $1 - 4\bar{\upsilon}^2/9 + C\delta(1 + \bar{\upsilon}^2) < 0$; thus $\upsilon < \bar{\upsilon}$. These bounds exclude a zero of T exactly as in Lemma 7.5. That lemma applies with $c_- = 1 - \delta$, $c_+ = 1 + \delta$, and the just obtained bounds for $\mathfrak{k}, c_r$; choose δ small enough that $c_p(1 - \delta) > \bar{\upsilon}$ and its short-length inequality holds. It supplies the global quotient bound and the two endpoint signs. Its precise pulse identity now reads

$$\upsilon(\tau_b) = \upsilon(1) - \mu \int_0^1 c_6(1 + y/\Lambda_{\mathrm{ret}})\eta_2(y)\,dy - \Lambda_{\mathrm{ret}}^{-1} \int_0^1 \mathfrak{k}\mathsf{V}^2\,dy.$$

The last term is $O(\delta)$ uniformly in all trials. At a zero it gives $\mu \ge 1/2 - C\delta$ and $\mu \le \bar{\upsilon}/(1 - \delta)$. This calculation is valid for time-dependent c_6 and uses no derivative of that coefficient.

At a zero, the parent impulse in the second geometric equation is at least $(2/3 - C\delta)\mu - C\delta$, while its upper bound is $1 + \mu + C\delta$. These follow by bounding $c_2c_3\mathfrak{z}_e/\mathfrak{z}$ between $2/3 - C\delta$ and $1 + C\delta$ inside the pulse integral; the non-pulse integral on that short interval is $O(\delta)$ and $\mathcal{W}(1) \in [-C\delta, 1 + C\delta]$. The asserted endpoint bounds follow. On the first part, integrating $\mathfrak{k}\upsilon$ gives the lower bound $(1 - C\delta)I_0$ and the upper bound $(1 + C\delta)\bar{\upsilon}$. On a zero member the pulse quotient is nonnegative and bounded, so its contribution to that gain lies between zero and $C\delta$. This proves the gain statement. The common continuation and zero selection now follow from Proposition 7.6. The recovered geometric intervals exclude a first exit from their original box, so they need only be supplied as temporary bounds on existing segments. □

For Euler, the identities in (7.18) must be derived from (7.6) with the complete older correction and displaced-center terms. In particular $c_6 = c_m(0, t)$ in the displayed normalization, and $\mathfrak{k}$ is the exact metric expression in (7.8), multiplied by $\mathfrak{z}_e/\gamma$. The small bounds for the $c_i - 1, d_i$ are additional sufficient conditions, not consequences of smoothness. This proposition supplies the same quantitative return implication for the prescribed pulse. Proposition 11.1 supplies their actual-field bounds under its explicit scalar conditions.

For an application of the return proposition the required quantitative inputs are precisely the support margins, all active denominator margins, bounds for the complete E_j, positive older circulation, the entry ratio bounds, and positive bounds for the two exact coefficients in (7.13). A preceding amplification interval additionally requires a positive lower and upper bound for $d\log(-T_m)/dt$ up to its chosen threshold, and the transition must return these entry conditions. Suppose the finite solution continues from the beginning of growth through duration G/g_-, with bounds $0 < g_- \le d\log(-T_m)/dt \le g_+$. A prescribed logarithmic gain $G > 0$ is then reached between durations G/g_+ and G/g_-: integration proves existence of the crossing, strict monotonicity proves uniqueness, and the nonzero time derivative proves smooth dependence of that crossing. Any complete-gradient conclusion on the following interval also requires either of the displayed estimates after (7.11). The mixed amplitude, nonlinear map and force bounds of Sections 3, 4 and 6 have their own additional quantitative inputs. Proposition 11.1 obtains these bounds for every older corrected layer on the finite histories satisfying its scalar tests. Their simultaneous infinite realization is proved in Proposition 12.3.

7.4. **Initialization and bounded applications.**

Proposition 7.8 (A first and a second finite application)**.** *For every prescribed $r_0 > 0, z_0$, fixed finite depths and frequencies, and fixed finite positive logarithmic gains, there are sufficiently small positive angles, material radii and nonzero circulation seeds for which the construction initializes with zero meridional velocity and completes a first return and a second return. All corrected fields, their centers, compact forces and their derivatives of every fixed order exist on these finite intervals.*

The return roots in this bounded application are simple and can be chosen smoothly. The constants and allowable seeds can depend on all this finite data.

Proof. Choose $e_0 = (0, 1)$ in (2.7). Increase Γ_0 or decrease the base oscillation size so that $\Gamma^b \geq \Gamma_0/2$ on $\mathcal{P}$, for every angle used below. Take $\vartheta = 0$ before the first insertion. Then $\Psi^b = 0$ and all inserted physical profiles are flat and zero initially, giving zero initial meridional velocity and nonzero compact swirl.

For the first layer choose $p_1 = (\sin s, \cos s)$, $s > 0$ small. Its older center is exactly y_*, its older map is rotation on the containing ball, and $\mathcal{F}_1 = 0$. Along that center

$$\zeta_1 = (\sin(s - \vartheta), \cos(s - \vartheta)), \quad d_1 = -A_0 \sin s, \quad c_b^* = 2\Gamma_0/r_0^4,$$
$$\kappa(\vartheta) = r_0^{-2} \sin^2(s - \vartheta) + \cos^2(s - \vartheta), \qquad \kappa_s = r_0^{-2} \sin^2 s + \cos^2 s.$$

During growth $\alpha = 0$; with the growing datum the central pair grows exactly at the rate

$$\gamma_1^{\mathrm{gr}} = \sin s \sqrt{A_0 c_b^*/\kappa_s} > 0.$$

It reaches the prescribed finite gain, and a constant transition adds any chosen finite duration. Both statements are independent of the size of the nonzero seed.

For the first return use (7.12) with $z_e = \sin s$, $\gamma = \gamma_1^{\mathrm{gr}}$, $z_p = z_e$ and fixed $c_p > 1$ and

$$\Lambda_{\mathrm{ret}} \geq 2c_p \|\eta_2\|_\infty.$$

These are the first-pulse conditions of [1, Lemma 2.7]. Increase Λ_{ret} also to satisfy the short-length inequality of Lemma 7.5 for coefficients in a sufficiently small fixed neighborhood of one. The explicit angle is

$$\vartheta = s - \arcsin z_\mu, \qquad \zeta_{1,2} = \sqrt{1 - z_\mu^2}.$$

Choose s so small that $c_p \Lambda_{\mathrm{ret}} \|\eta_2\|_\infty \sin s < 1/4$. The denominator is at least $\sqrt{15}/4$ for every trial. In (7.13) take $c_0 = A_0 \sin s/(\sigma_0 \kappa_s \gamma_1^{\mathrm{gr}})$. Then $\upsilon(0) = 1$, $c \equiv 1$ and $\mathfrak{k} = \kappa_s/\kappa(\vartheta)$ exactly. As $s \downarrow 0$, this $\mathfrak{k}$ tends to 1 uniformly together with its first derivative in μ, because

$$\kappa(\vartheta) = 1 + (r_0^{-2} - 1) z_\mu^2.$$

Thus for sufficiently small s the bounds of Lemma 7.5 hold with strict room. The unperturbed quotient has strictly negative parameter derivative

$$-\int_0^1 \eta_2(y) \exp\left[-\frac{2}{\Lambda_{\mathrm{ret}}} \int_y^1 \upsilon(1 + r/\Lambda_{\mathrm{ret}}; \mu)\, dr\right] dy,$$

bounded away from zero uniformly in the compact trial interval. This is the differentiated scalar equation proved in [1, Lemma 2.7]; its linear variational equation also shows that the derivative remains negative for the displayed C^1-small metric perturbation. Therefore the actual first root is unique and simple. Its selected angle is s and its central second amplitude is zero near the endpoint. This calculation retains the metric for arbitrary r_0.

All first-layer profiles and their first rates are smooth functions of its seed ϵ_1 on this compact family. At $\epsilon_1 = 0$ every profile is zero: each source vanishes when all its own lower profiles vanish. Consequently, for any fixed finite derivative order q, the complete physical increments satisfy

$$\|\gamma_1\|_{C^q} + \|\psi_1\|_{C^q} \leq C_q \epsilon_1 \tag{7.20}$$

on the finite interval under consideration, including any fixed smooth continuation of its angle. The constant may depend on the fixed frequency, radius, depth and duration. These bounds follow by the mean-value formula in the seed, using the parameter dependence proved above. In particular the full point satisfies $Z - y_* = O(\epsilon_1)$; it is not set equal to the older center.

To prepare the second application, extend the first central pair divided by its seed to zero seed. Its growth and return times and angle remain those just computed. Set $\epsilon_1 = 0$ only as this reference family, keep the landed base angle s, and take $p_2 = (\sin s_2, \cos s_2)$ with $s_2 > 0$ small.

The older state is then the rotated base. Its central coefficients are $d_2 = -A_0 \sin(s + s_2)$, c_b^*, and $\kappa_{s_2} = r_0^{-2} \sin^2 s_2 + \cos^2 s_2$. The exact growth rate is

$$\gamma_2^{\mathrm{gr}} = \sqrt{A_0 c_b^* \sin(s + s_2) \sin s_2 / \kappa_{s_2}} > 0.$$

Here $\mathcal{F}_1 = \mathcal{F}_2 = 0$ on growth and transition. During return the increment of the base angle is $s_2 - \arcsin z_\mu$. With the corresponding growing-line normalization, the same quotient equation has $c = 1$ and $\mathfrak{k} = \kappa_{s_2}/[1 + (r_0^{-2} - 1)z_\mu^2]$. Taking s_2 small gives a second compact family with strict denominator and return margins and a simple root.

Choose the material collars so small that both reference families lie strictly inside the quadratic base region and inside the linear part of the base carrier wherever that identity is used. The centers of the reference families remain at y_*, and their maps are rotations, so these choices are possible. The central growth coefficients are strictly positive; reducing the collars makes the fibre coefficients positive as well. Their growing data are therefore smooth. Now restore $\epsilon_1 > 0$ sufficiently small and continue its complete first-layer fields while constructing the second layer with the actual controls in (7.2). On the reference family, compactness gives finitely many local existence neighborhoods covering the two finite intervals and all trial parameters. Successively apply Proposition 7.2 in these neighborhoods. The perturbed solution stays in them for small ϵ_1 by the same Gronwall estimates with the additional initial-data difference. Hence it exists on the common full interval and retains all the strict coefficient, support and denominator margins. This argument also applies across the transversal growth hitting time, using its nonzero time derivative.

The first layer is now nonzero and is evaluated at the displaced second center. Its entire finite correction sum enters E_2, d_2, c_2, and every level source. Smooth dependence preserves the two endpoint signs and the simple second root; the scalar implicit function theorem applies only to this simple root. Choose also $\epsilon_2 > 0$ small enough that the new full point and any positivity requirement after its insertion retain their margins. The new physical fields are again nonzero and compact. The finite force recovery of Lemma 2.1 supplies their smooth compatible force and pressure. Every fixed mixed derivative is finite by the continuation and allocation results. If the complete endpoint gradient is nonzero, continuity or (7.11) also gives a positive holding interval on which it retains a fixed positive fraction of its size. For these small seeds it is nonzero since it is close to the nonzero base gradient. □

The last proposition chooses seeds after fixing its finite gains and durations. Its constants provide no bound as those gains, depths, or the number of increments increase. The quantitative lifetime bounds at the prescribed amplifying sizes are established in Sections 9–12.

8. Low-order estimates on a finite continuation

The estimates below separate two matters: the exact cancellations in a complete finite layer, and the quantitative bounds for the differentiated profiles that enter those cancellations. In particular, a large velocity parallel to a layer's phase curves need not produce a large displacement of its phase. All intervals in this section are compact initial segments of the actual finite family of Section 7; no solution on a later part of an interval is assumed.

8.1. **Profile and inverse-map bounds.** Fix a compact initial interval $I = [t_0, t_1]$ of a finite trial family. All bounds are uniform over its compact parameter interval. Choose a convex open ball $\mathcal{P} \Subset \{y_2 > 0\}$ about $y_* = (z_0, r_0^2/2)$ on which the base velocity is $\alpha J(y - y_*)$ and its circulation is affine. Write $r_- \leq \sqrt{2y_2} \leq r_+$ on $\mathcal{P}$. Every label segment used below lies in a specified older label collar; the corresponding physical connecting segments lie in $\mathcal{P}$. Fix a number $0 < \rho_i^{\mathrm{core}} < \ell_i$ for which $g_i = 1$ on $B_{\rho_i^{\mathrm{core}}}$. All phase-cell estimates below are on that label core.

Put $\lambda_i = \ell_i^{-1}$ and $U_i = \sum_{n=0}^{J_i} U_{i,n}$, $V_i = \sum_{n=0}^{J_i} V_{i,n}$. For each layer suppose positive numerical functions $\mathcal{U}_{i,n}, \mathcal{V}_{i,n}$ and a rate $\Pi_i \geq 1$ satisfy

$$\begin{aligned} &|\partial_t^b \partial_s^u D_a^v U_{i,n}| \leq \mathcal{U}_{i,n} \Pi_i^b \lambda_i^v, \qquad |\partial_t^b \partial_s^u D_a^v V_{i,n}| \leq \mathcal{V}_{i,n} \Pi_i^b \lambda_i^v, \\ &b \leq 1, \quad u+v+b \leq 4, \qquad 0 \leq n \leq J_i. \end{aligned} \tag{8.1}$$

Spatial derivatives have their Euclidean multilinear operator norms; t is differentiated at fixed (s, a). Bounds used outside a phase cell are assumed for all phases there. The inverse maps satisfy

$$|D_y A_i|,\ |D_a X_i| \leq \mathcal{H}_i, \quad |D_y^2 A_i| \leq \mathcal{H}_{i,2}, \quad |D_y^3 A_i| \leq \mathcal{H}_{i,3}, \qquad \mathcal{H}_i \geq 1 \tag{8.2}$$

on the stated physical regions and label collars. For the principal amplitudes assume

$$|w_i B_i| \leq b_i, \quad |D_a^r(w_i B_i)| \leq b_i^{\mathrm{lab}} \lambda_i^r, \qquad |w_i T_i| \leq t_i, \quad |D_a^r(w_i T_i)| \leq t_i^{\mathrm{lab}} \lambda_i^r \quad (1 \leq r \leq 3).$$

Their first-time versions, when used, are supplied in the same convention.

These are explicit analytic hypotheses, not scalar parameter conditions. For example, Proposition 6.1, under all its hypotheses and $4 + 2n \leq r_*(i)$, supplies

$$\mathcal{V}_{i,n} = a_{i,n} Y_{*,i} \delta_i^n, \qquad \mathcal{U}_{i,n} = \frac{H_{r_*(i)} \chi_{\kappa,i}}{N_i \sigma_{i-1}} a_{i,n} Y_{*,i} \delta_i^n. \tag{8.3}$$

Here $\chi_{\kappa,i}$ is the layer-i reciprocal-symbol size denoted by χ in Section 6; the level constants depend on the specified order. Establishing these hypotheses with intrinsic first rates on all later intervals is proved in Proposition 4.3. The calculations below give the finite numerical implications used there.

Retain the four complete sums

$$\mathcal{U}_i^{\mathrm{e}} = \sum_{\substack{2 \leq n \leq J_i \\ n \text{ even}}} \mathcal{U}_{i,n}, \quad \mathcal{U}_i^{\mathrm{o}} = \sum_{\substack{1 \leq n \leq J_i \\ n \text{ odd}}} \mathcal{U}_{i,n}, \qquad \mathcal{V}_i^{\mathrm{e}} = \sum_{\substack{2 \leq n \leq J_i \\ n \text{ even}}} \mathcal{V}_{i,n}, \quad \mathcal{V}_i^{\mathrm{o}} = \sum_{\substack{1 \leq n \leq J_i \\ n \text{ odd}}} \mathcal{V}_{i,n}.$$

No maximum over levels is taken before their scale factors are included. Choose a phase cell $|s| \leq s_c$ on which $F(s) = s$, with $s_c \leq 1$, and put $P_c = |P(0)| + s_c^2/2$. For $0 \leq S \leq s_c$ define

$$\begin{aligned} U_s^i(S) &= b_i S + (\mathcal{U}_i^{\mathrm{o}} + S\mathcal{U}_i^{\mathrm{e}}), \\ U_a^i(S) &= P_c b_i^{\mathrm{lab}} \lambda_i + \lambda_i(\mathcal{U}_i^{\mathrm{e}} + S\mathcal{U}_i^{\mathrm{o}}), \\ U_{sa}^i(S) &= b_i^{\mathrm{lab}} \lambda_i S + \lambda_i(\mathcal{U}_i^{\mathrm{o}} + S\mathcal{U}_i^{\mathrm{e}}), \\ U_{aa}^i(S) &= P_c b_i^{\mathrm{lab}} \lambda_i^2 + \lambda_i^2(\mathcal{U}_i^{\mathrm{e}} + S\mathcal{U}_i^{\mathrm{o}}), \\ U_{ss}^{i,\mathrm{corr}}(S) &= (\mathcal{U}_i^{\mathrm{e}} + S\mathcal{U}_i^{\mathrm{o}}), \\ V_s^{i,\mathrm{corr}}(S) &= (\mathcal{V}_i^{\mathrm{e}} + S\mathcal{V}_i^{\mathrm{o}}), \\ V_a^{i,\mathrm{corr}}(S) &= \lambda_i(\mathcal{V}_i^{\mathrm{o}} + S\mathcal{V}_i^{\mathrm{e}}). \end{aligned} \tag{8.4}$$

All are nonnegative numerical functions. Phase parity and the fundamental theorem of calculus prove these bounds. For instance, $U_{i,n}$ is odd when n is odd, so its second phase derivative vanishes at zero and has size at most $S\mathcal{U}_{i,n}$ on this cell. When n is even its first phase derivative vanishes at zero. Label differentiation preserves these statements. The opposite phase parity of $V_{i,n}$ gives the last two lines. This argument uses no spatial reflection property of the physical fields.

8.2. **Complete gradients at displaced labels.** For a fixed layer let $a = A_i(y, t)$, $s = N_i p_i \cdot a$, and suppress i. The material covector is $\zeta(a, t) = (D_a X(a, t))^{-T} p$. In this subsection ζ means its value at $a = A(y, t)$, and $\zeta^0 = \zeta(0, t)$ is its own-label value. Assume $|\zeta(a, t)| \leq z_+$ on the label segment joining zero to a. Suppose also that

$$|D_a \zeta| \leq z_a, \qquad |a| \leq D, \qquad |s| \leq S.$$

The number z_a can be supplied as $\mathcal{H}\mathcal{H}_2$: indeed $D_a\zeta = (D_y\zeta)D_aX$ and in two dimensions $\|D_aX\| = \|D_yA\|$ when the determinant is one. Let $e = \zeta/|\zeta|$, and let

$$\mathcal{G}^0 = wNT(0,t)\zeta^0, \qquad \mathcal{S}^0 = wN^2B(0,t)J\zeta^0 \otimes \zeta^0.$$

The matrix $\mathcal{S}^0$ equals $w\Omega(0,t)|\zeta^0|^2\kappa(0,t)^{-1}Je^0 \otimes e^0$. It equals $w\Omega Je^0 \otimes e^0$ only when the metric ratio equals one; the exact difference is displayed below.

Lemma 8.1 (Complete displaced-center decomposition)**.** *The actual fields satisfy*

$$\nabla\gamma = \mathcal{G}^0 + g^{\mathrm{al}}e + g^{\mathrm{rem}}, \qquad \nabla v_i = \mathcal{S}^0 + x^{\mathrm{al}}Je \otimes e + D^{\mathrm{rem}}.$$

The following numerical bounds are valid:

$$\begin{aligned} |g^{\mathrm{al}}| &\le Nz_+\big(t^{\mathrm{lab}}\lambda D + V_s^{\mathrm{corr}}(S)\big), \\ |g^{\mathrm{rem}}| &\le Ntz_aD + \mathcal{H}\big(St^{\mathrm{lab}}\lambda + V_a^{\mathrm{corr}}(S)\big), \\ |x^{\mathrm{al}}| &\le N^2z_+^2\big(b^{\mathrm{lab}}\lambda D + U_{ss}^{\mathrm{corr}}(S)\big), \\ \|D^{\mathrm{rem}}\| &\le 2N^2bz_+z_aD + 2Nz_+\mathcal{H}U_{sa}(S) + \mathcal{H}^2U_{aa}(S) + \mathcal{H}_2\big(NU_s(S) + U_a(S)\big). \end{aligned} \tag{8.5}$$

The identities and bounds hold separately at Y_q and at Z, using their respective labels and phases. They do not identify these points.

Proof. Use (5.28) and split the phase-gradient term $N\zeta V_s$ into $wNT(0)\zeta^0$, $N\zeta[w(T(a) - T(0)) + \sum_{n\ge1} V_{n,s}]$, and $wNT(0)(\zeta - \zeta^0)$. The second term is parallel to e; the third is bounded by Ntz_aD. The remaining gradient term is $H^T[ws\nabla_aT + \sum \nabla_aV_n]$, and its bound is the second line.

For the Hessian split $N^2U_{ss}J\zeta \otimes \zeta$ in the same way, using $U_{0,ss} = wB$. The tensor difference satisfies

$$\|J\zeta \otimes \zeta - J\zeta^0 \otimes \zeta^0\| \le 2z_+|\zeta - \zeta^0|.$$

The remaining terms in (5.28) have bounds $2Nz_+\mathcal{H}U_{sa}$, $\mathcal{H}^2U_{aa}$ and $\mathcal{H}_2(NU_s + U_a)$. The last estimate uses the vector-valued Hessian norm of A, so it includes both inverse-map components. There is no vanishing assertion about $\nabla_a^2B(0,t)$. ∎

For comparison with an orthonormal-direction system, the difference between $\mathcal{S}^0$ and $w\Omega(0,t)Je^0\otimes e^0$ is exactly

$$w\Omega(0,t)\left(\frac{|\zeta^0|^2}{\kappa(0,t)} - 1\right)Je^0 \otimes e^0.$$

It is another aligned term. On a holding interval $B(0,t) = 0$; b^{lab} and the complete correction sums in (8.5) need not vanish there.

For use on a containing region, third spatial derivatives can also be bounded explicitly. Define

$$\begin{aligned} L_1^i(S) &= N_iU_s^i(S) + U_a^i(S), \\ L_2^i(S) &= N_i^2\big(b_i + (\mathcal{U}_i^{\mathrm{e}} + S\mathcal{U}_i^{\mathrm{o}})\big) + 2N_iU_{sa}^i(S) + U_{aa}^i(S), \\ L_3^i(S) &= 3N_i^2\lambda_ib_i^{\mathrm{lab}} + 3N_i\lambda_i^2Sb_i^{\mathrm{lab}} + P_c\lambda_i^3b_i^{\mathrm{lab}} \\ &\quad + N_i^3(\mathcal{U}_i^{\mathrm{o}} + S\mathcal{U}_i^{\mathrm{e}}) + 3N_i^2\lambda_i(\mathcal{U}_i^{\mathrm{e}} + S\mathcal{U}_i^{\mathrm{o}}) \\ &\quad + 3N_i\lambda_i^2(\mathcal{U}_i^{\mathrm{o}} + S\mathcal{U}_i^{\mathrm{e}}) + \lambda_i^3(\mathcal{U}_i^{\mathrm{e}} + S\mathcal{U}_i^{\mathrm{o}}), \\ V_{i,2}^{\mathrm{num}}(S) &= \mathcal{H}_i^3L_3^i(S) + 3\mathcal{H}_i\mathcal{H}_{i,2}L_2^i(S) + \mathcal{H}_{i,3}L_1^i(S). \end{aligned} \tag{8.6}$$

Then $|\nabla_y^2v_i| \le V_{i,2}^{\mathrm{num}}(S)$. To see this, first differentiate $U_i(N_ip_i \cdot a, a)$ three times in a. The principal third pure-phase term vanishes because $P''' = 0$ in the cell; its other terms are the three principal terms of L_3^i. The parity estimates give the remaining four terms. Compose with A_i: the third derivative has the partition types $1+1+1$, $2+1$ (three choices), and 3. This gives exactly the final line. Thus large third phase derivatives of the first correction are retained, while a nonexistent third phase derivative of the principal polynomial is not introduced.

Suppose on $Y_q + M_q B_{2\ell_q}$ that all old layers are in such cells and $\|M_q^{\pm 1}\| \le K_q(t)$. Put $V_2(t) = \sum_{i<q} V_{i,2}^{\mathrm{num}}(S_i)$; the base velocity has zero second spatial derivative on this region. Taylor's formula gives

$$|\widetilde{v}| \le 2K_q^3 V_2 \ell_q^2, \qquad |D_{a'}\widetilde{v}| \le 2K_q^3 V_2 \ell_q, \qquad |D_{a'}^2 \widetilde{v}| \le K_q^3 V_2.$$

Consequently

$$\eta_q = 4\ell_q \max\left\{ \int_{t_0}^{t_1} K_q^3 V_2 \, dt, \ \Pi_q^{-1} \sup_I K_q^3 V_2 \right\}, \qquad \eta_q \le \tfrac{1}{8} \tag{8.7}$$

is a sufficient scalar low-order non-affinity test. When $\eta_q > 0$, put $\mathfrak{d} = 4\ell_q K_q^3 V_2/\eta_q$. Then $\int_I \mathfrak{d} \le 1$ and $\mathfrak{d} \le \Pi_q$. The first two Taylor estimates and the second-label estimate meet the corresponding hypotheses in the proof of Lemma 4.1 with constants at most one. On the full interval beginning at insertion, this proves its order-zero and first-label conclusions. If $\eta_q = 0$, the nonlinear remainder vanishes on the region. Higher spatial or positive-time estimates still require their own inputs; none follows from this low-order Taylor calculation.

8.3. **Phase displacement and containment.** Let C_q denote either Y_q, moving with $v_{<q}$, or Z, moving with the full finite velocity. Let m be the largest active label, and write $q = m+1$ in the latter case. For $i < q$ put $a_i^q = A_i(C_q, t)$ and $x_i^q = p_i \cdot a_i^q$. The exact identities are

$$\begin{aligned} (a_i^q)' &= J\big(N_i p_i U_{i,s} + \nabla_a U_i\big) + H_i \sum_{i<j<q} v_j(C_q, t), \\ (x_i^q)' &= p_i \cdot J\nabla_a U_i + \zeta_i \cdot \sum_{i<j<q} v_j(C_q, t). \end{aligned} \tag{8.8}$$

Here the first term is the complete own layer. The identity $H_i J H_i^T = J$, a consequence of $\det H_i = 1$, proves the formula. In particular the entire own phase velocity, including every correction, is absent from the second equation.

Use scalar bounds V_j^{num} for the full velocity of each younger layer at the relevant points. They can, for example, be the following global bounds calculated from the prescribed profiles and (8.1):

$$V_j^{\mathrm{num}} = (N_j z_{+,j} + \mathcal{H}_j \lambda_j)(\mathcal{U}_{j,0} + \mathcal{U}_j^{\mathrm{e}} + \mathcal{U}_j^{\mathrm{o}}),$$

provided (8.1) holds on all phases and the entire relevant label collar, including the principal envelope. When the point is known to lie in the smaller cell, the sharper value $N_j z_{+,j} U_s^j(S_j) + \mathcal{H}_j U_a^j(S_j)$ is available.

Choose proposed phase bounds $0 < S_i \le s_c/4$. On an interval $[t_0, t]$ define, entirely in numbers,

$$\begin{aligned} \mathfrak{p}_i^q(t) &= \mathfrak{p}_i^q(t_0) + \int_{t_0}^t \left[U_a^i(S_i) + z_{+,i} \sum_{i<j<q} V_j^{\mathrm{num}} \right] ds, \\ \mathfrak{o}_i^q(t) &= \mathfrak{o}_i^q(t_0) + \int_{t_0}^t \left[N_i^2 b_i \mathfrak{p}_i^q(s) + N_i(\mathcal{U}_i^{\mathrm{o}} + S_i \mathcal{U}_i^{\mathrm{e}}) + U_a^i(S_i) + \mathcal{H}_i \sum_{i<j<q} V_j^{\mathrm{num}} \right] ds. \end{aligned} \tag{8.9}$$

The numerical functions on the right may vary with time but must have been specified from scalar bounds, not defined as suprema of the unknown offsets.

Lemma 8.2 (Displacement without an exponential in the own shear). *If initially $|x_i^q| \le \mathfrak{p}_i^q(t_0)$ and $|a_i^q| \le \mathfrak{o}_i^q(t_0)$, and the existing segment stays in the prescribed collars with the temporary bounds $|N_i x_i^q| \le S_i$ and $|a_i^q| \le \rho_i^{\mathrm{core}}/2$, then*

$$|x_i^q(t)| \le \mathfrak{p}_i^q(t), \qquad |a_i^q(t)| \le \mathfrak{o}_i^q(t).$$

Consequently the scalar tests

$$N_i\mathfrak{p}_i^q(t) \le S_i/2, \qquad \mathfrak{o}_i^q(t) \le \rho_i^{\mathrm{core}}/4 \tag{8.10}$$

strictly recover the corresponding relaxed bounds $|N_i x_i^q| \le S_i$ *and* $|a_i^q| \le \rho_i^{\mathrm{core}}/2$.

Proof. Integrate the second equation in (8.8) first. It gives $|x_i^q| \le \mathfrak{p}_i^q$ directly. On the cell, $NU_{0,s} = NwBs = N^2wBx_i^q$. Insert the already bounded phase in the first equation. This gives the second integral in (8.9). No unknown label displacement is multiplied by the large coefficient $N^2 b_i$. □

For an actual layer index $q \le m$, to include its tube, suppose $|D_a X_q| \le \mathcal{H}_q$ on its collar and $|D_y A_i| \le \mathcal{H}_i$, $|\zeta_i| \le z_{+,i}$ on the connecting region. For $y = X_q(a,t)$, $|a| \le \ell_q$,

$$\begin{aligned} |A_i(y,t)| &\le \mathfrak{o}_i^q + \mathcal{H}_i\mathcal{H}_q\ell_q, \\ |N_i p_i \cdot A_i(y,t)| &\le N_i\mathfrak{p}_i^q + N_i z_{+,i}\mathcal{H}_q\ell_q. \end{aligned} \tag{8.11}$$

Thus $\mathfrak{o}_i^q + \mathcal{H}_i\mathcal{H}_q\ell_q \le \rho_i^{\mathrm{core}}/2$ and $N_i\mathfrak{p}_i^q + N_i z_{+,i}\mathcal{H}_q\ell_q \le s_c/2$ are genuine scalar containment tests. The connecting segments must lie in the regions on which the displayed derivative bounds hold; it suffices to construct the bounds first on the containing ellipsoids of Section 4. Absolute containment also needs a numerical bound $\mathfrak{r}_q^{\mathrm{cen}}(t)$ for $|Y_q - y_*|$. On the quadratic base region it is supplied by

$$\mathfrak{r}_q^{\mathrm{cen}}(t) = \mathfrak{r}_q^{\mathrm{cen}}(t_0) + \int_{t_0}^t \sum_{j<q} V_j^{\mathrm{num}}(s)\, ds. \tag{8.12}$$

Indeed the base rotation makes no contribution to the derivative of $|Y_q - y_*|$. For Z the sum includes the current layer. If $\mathfrak{r}_q^{\mathrm{cen}} + \mathcal{H}_q\ell_q < \mathrm{dist}(y_*, \partial\mathcal{P})$, the tube stays in $\mathcal{P}$. For the containing rigid region $Y_q + M_q B_{2\ell_q}$ use its radius $2K_q\ell_q$ instead of $\mathcal{H}_q\ell_q$ in both nesting inequalities and in this absolute-distance test. The strict tests exclude the respective first exits while the derivative hypotheses hold.

8.4. **Positive physical coefficients.** Fix $r_0 > 0$ and choose

$$y_* = (z_0, r_0^2/2), \qquad \Gamma_0 = r_0^4/2.$$

This is a choice of the constant background circulation, not a change of the Euler coupling. On a fixed region $r_- \le r \le r_+$ put

$$W_+ = r_-^{-4}, \quad W_1 = 4r_-^{-6}, \quad A_1 = 2r_-^{-4}, \quad A_\Delta = \sup_{r_- \le r \le r_+} |r^{-2} - 1|.$$

Let $\mathcal{R}_q(t) \subset \mathcal{P}$ be the specified packet-containing region. Suppose $|y - y_*| \le d_c$ and $|s_i(y,t)| \le S_i$ for $y \in \mathcal{R}_q(t)$, with all corresponding label points in their cores. A uniform numerical bound g_c for $|\Gamma_{<q}(y,t) - \Gamma_0|$ is then available. Specifically, take

$$g_c = A_0 d_c + \sum_{i<q} \big(t_i S_i + (\mathcal{V}_i^{\mathrm{e}} + \mathcal{V}_i^{\mathrm{o}})\big).$$

The same formula at a single center is only a center estimate. Covectors and symbols at y mean their material values at $A_q(y,t)$. All positivity conclusions here concern $\mathcal{R}_q(t)$, not the entire support of a compactly supported circulation. Directly,

$$\begin{gathered} |2W(y)\Gamma_{<q}(y) - 1| \le 2W_+ g_c + 2\Gamma_0 W_1 d_c, \\ \Gamma_{<q} \ge \Gamma_0 - g_c, \\ \min(1, r_+^{-2})|\zeta_q|^2 \le \kappa_q \le \max(1, r_-^{-2})|\zeta_q|^2. \end{gathered} \tag{8.13}$$

In particular $g_c \le \Gamma_0/2$ gives a positive circulation window. For the own-center normalized metric $\widehat{\kappa}_q = \kappa_q(0,t)/|\zeta_q(0,t)|^2$ one has the more useful exact formula

$$\widehat{\kappa}_q = 1 + (r(Y_q)^{-2} - 1)e_{q,1}^2, \qquad |\widehat{\kappa}_q - 1| \le A_\Delta e_{q,1}^2. \tag{8.14}$$

Hence $A_\Delta \theta^2 \le \varepsilon$ suffices when $|e_{q,1}| \le \theta$. Smallness of $\|\mathsf{A} - I\|$ is not required and is not available for arbitrary r_0.

Let $V_c, \Phi_c, F_c, F_{c,1}$ bound respectively $|v_{<q}(Y_q)|$, $|\dot{\phi}_q|$, $|F^\Gamma_{<q}(Y_q)|$ and $|\nabla F^\Gamma_{<q}(Y_q)|$. These numerical residual bounds must include the activation, every terminal source and every phase-independent mean in Section 5. The exact central identities give

$$
\begin{aligned}
|\dot{c}_q| &\le 2W_+ F_c + 2(\Gamma_0 + g_c) W_1 V_c, \\
|\dot{\widehat{\kappa}}_q| &\le A_1 V_c \theta^2 + 2A_\Delta \theta \Phi_c, \\
|\dot{d}_q| &\le z_{+,q} F_{c,1}.
\end{aligned}
\tag{8.15}
$$

Here $e_q = (\sin\phi_q, \cos\phi_q)$ and $|\sin\phi_q| \le \theta$. Differentiation gives the exact metric rate

$$\dot{\widehat{\kappa}}_q = -2r(Y_q)^{-4} \dot{Y}_{q,2} \sin^2\phi_q + 2(r(Y_q)^{-2} - 1) \sin\phi_q \cos\phi_q \, \dot{\phi}_q.$$

An estimate of $|\dot{\phi}_q|$ should retain the actual angular equation; the generic norm bound by $|D_q|$ need not give the small rate required for long growth intervals. The first and third bounds are (7.10). If $c_q \ge c_- > 0$ and $\widehat{\kappa}_q \ge k_- > 0$, division by c_- and k_- gives the logarithmic first rates. No time derivative of α occurs.

8.5. **Coefficient variation and insertion data.** Here is a direct bound that does not use a conservation law for two covectors evaluated at different centers. On a containing region for layer q suppose numerical functions bound

$$
\begin{aligned}
&|D_a X_q| \le x_1, \quad |D_a \zeta_q| \le z_a, \quad 0 < z_- \le |\zeta_q| \le z_+, \\
&|\nabla \Gamma_{<q}| \le G_1, \quad |\nabla^2 \Gamma_{<q}| \le G_2, \quad \Gamma_- \le \Gamma_{<q} \le \Gamma_+.
\end{aligned}
$$

The covector windows are supplied analytic bounds. The other numbers can be calculated from the profile and map bounds; here $\sigma = \sigma_{q-1}$. For example, put $V_i^{\mathrm{corr}} = \mathcal{V}_i^{\mathrm{e}} + \mathcal{V}_i^{\mathrm{o}}$. On the containing phase cells one may use

$$
\begin{aligned}
G_1 &= A_0 + \sum_{i<q} \Big[N_i z_{+,i} t_i + \mathcal{H}_i \lambda_i S_i t_i^{\mathrm{lab}} + (N_i z_{+,i} + \mathcal{H}_i \lambda_i) V_i^{\mathrm{corr}} \Big], \\
G_2 &= \sum_{i<q} \big[N_i t_i \mathcal{H}_{i,2} + 2N_i z_{+,i} \mathcal{H}_i \lambda_i t_i^{\mathrm{lab}} \\
&\qquad + S_i (\mathcal{H}_i^2 \lambda_i^2 + \mathcal{H}_{i,2} \lambda_i) t_i^{\mathrm{lab}} \\
&\qquad + V_i^{\mathrm{corr}} \{ (N_i z_{+,i} + \mathcal{H}_i \lambda_i)^2 + \mathcal{H}_{i,2}(N_i + \lambda_i) \} \big].
\end{aligned}
\tag{8.16}
$$

Indeed $\gamma_{i,0} = w_i T_i(A_i) s_i$ has gradient $N_i w_i T_i \zeta_i + s_i (D_y A_i)^T D_a(w_i T_i)$. Differentiating once more gives the first three terms of G_2: the phase Hessian, the two cross terms, and the Hessian of $w_i T_i(A_i)$. The correction sum is a composition with $(N_i p_i \cdot A_i, A_i)$; its two chain-rule terms give the last line. The base circulation is affine. If these physical derivative bounds are used in a rigid coordinate $y = Y_q + M_q a'$, their first and second spatial bounds are $K_q G_1$ and $K_q^2 G_2$, respectively. Let $a_+ = \max(1, r_-^{-2})$, $\kappa_- = \min(1, r_+^{-2}) z_-^2$, and define

$$
\begin{aligned}
c_+ &= 2W_+ \Gamma_+, & c_a &= 2x_1 (W_1 \Gamma_+ + W_+ G_1), \\
d_+ &= z_+ G_1, & d_a &= z_a G_1 + z_+ x_1 G_2, \\
k_a &= 2a_+ z_+ z_a + A_1 x_1 z_+^2, \\
a_a &= \frac{d_a}{\sigma \kappa_-} + \frac{d_+ k_a}{\sigma \kappa_-^2}, & b_a &= \sigma (c_+ z_a + c_a z_+),
\end{aligned}
\tag{8.17}
$$

$$\mathfrak{m}_q(t) = \ell_q(a_a + b_a), \qquad c_{*,q} = \int_{t_0}^{t_1} \mathfrak{m}_q(t) \, dt.$$

The definitions are numerical expressions in the displayed bounds. The product rule and the reciprocal rule prove $\ell_q \|D_a \mathcal{B}_q\| \le \mathfrak{m}_q$. In particular a zero of $\zeta_{q,1}(0,t)$ causes no division in this estimate. For higher label derivatives the finite product/partition rules apply; the additional relative-rate bounds remain part of the higher-order lifetime assertion.

At insertion the map is a translation and $\zeta = p_q$ is constant. Suppose the central coefficients $\widehat{a}_0, \widehat{b}_0$ are positive and let $a_a^{\rm in}, b_a^{\rm in}$ be the numbers just calculated. The scalar conditions

$$\ell_q a_a^{\rm in} \le \widehat{a}_0/4, \qquad \ell_q b_a^{\rm in} \le \widehat{b}_0/4 \tag{8.18}$$

prove positive coefficients on the whole insertion ball. For $R^{\rm in}(a) = -\epsilon_q(1, \sqrt{\widehat{b}(a)/\widehat{a}(a)})$,

$$\ell_q |D_a R^{\rm in}| \le c_{{\rm dat},q} |R^{\rm in}(0)|, \qquad c_{{\rm dat},q} = 2\ell_q \left(\frac{a_a^{\rm in}}{\widehat{a}_0} + \frac{b_a^{\rm in}}{\widehat{b}_0} \right).$$

Indeed differentiate the logarithm of the square root and use the $[3/4, 5/4]$ windows. The bound for the square root relative to its central value is less than two, giving the stated constant. Thus

$$c_{*,q} + c_{{\rm dat},q} \le \frac{\log 2}{6C_R^2}$$

is an explicit scalar test once the central weighted constant C_R and the spatial numerical inputs have been established. The test by itself does not return the central weighted history.

8.6. **The coefficients in a generic return.** Consider a return interval of layer $m \ge 2$ on which all activations are one. Write $p = m-1$, $\sigma = \sigma_{m-1}$ and set $s = \theta_m^{\rm in}$ for the fixed insertion angle in this subsection. Assume $\sin s > 0$, set $\gamma = \sigma \sin s$, and use normalized time $\tau = \gamma(t - t^1)$. The length $\mathfrak{z} = |\zeta_m(0,t)|$ has entry value $\mathfrak{z}_e = \mathfrak{z}(t^1) > 0$; write $z_e = \zeta_{m,1}(0,t^1)$. These two entry values are distinct. All covectors and symbols in the following formulas are evaluated at the stated labels. Assume positive lower bounds for the covector norms and symbols, $T_m(0,t) \ne 0$, and positive cosines for the angles used below.

A superscript c indicates the parent's own center and a superscript l its label $A_p(Y_m,t)$ at the current center. Define

$$\mathscr{A}_p^c = -N_p T_p(0,t)|\zeta_p(0,t)|, \qquad \mathscr{A}_p^l = -N_p T_p(A_p(Y_m),t)|\zeta_p(A_p(Y_m),t)|,$$

$$\mathfrak{r}_p^l = \frac{|\zeta_p(A_p(Y_m),t)|^2}{\kappa_p(A_p(Y_m),t)}, \qquad \mathfrak{r}_m = \frac{\mathfrak{z}^2}{\kappa_m(0,t)}.$$

Let e_p^c, e_p^l, e_m be the corresponding unit covectors, written as $(\sin\phi, \cos\phi)$. Put

$$\theta^c = \phi_m - \phi_p^c, \quad \theta^l = \phi_m - \phi_p^l, \quad q_w = \frac{\mathfrak{z}\sin\theta^c}{\sin s}, \quad q_l = \frac{\mathfrak{z}\sin\theta^l}{\sin s}, \quad c_\phi = \cos\phi_m, \quad c_\theta = \cos\theta^c.$$

Require $q_w c_\phi > 0$. The physical couplings and amplitude variables are

$$\mathfrak{c}_p = 2W(Y_p)\Gamma_{<p}(Y_p), \qquad \mathfrak{c}_m = 2W(Y_m)\Gamma_{<m}(Y_m), \qquad \mathcal{W} = \frac{\Omega_p(0,t)}{\sigma}, \qquad \upsilon = \frac{\widehat{\Omega}_m(0,t)}{\mathfrak{z}_e T_m(0,t)}.$$

In particular $\mathfrak{c}_p$ is not the pulse parameter c_p in Proposition 7.7. Use the complete *older* velocity in

$$\mathcal{E}_D = \nabla v_{<m}(Y_m) - \alpha J - \mathfrak{r}_p^l \Omega_p(0,t) J e_p^l \otimes e_p^l, \qquad \mathcal{E}_G = \nabla\Gamma_{<m}(Y_m) + \mathscr{A}_p^l e_p^l.$$

Thus the difference of the parent's local and own-center principal vorticities belongs to $\mathcal{E}_D$.

Lemma 8.3 (Exact generic-return coefficients). *For the pulse* (7.12), *with* $z_p = \mathfrak{z}_e \sin s$, *the coefficients of* (7.18) *are exactly*

$$
\begin{aligned}
&c_1 = \mathfrak{r}_p^l q_l \cos\theta^l, && d_1 = -\mathfrak{z}\, e_m \cdot \mathcal{E}_D^T e_m/\gamma,\\
&c_2 = \mathfrak{c}_p(\mathscr{A}_p^c/\sigma^2) q_w c_\phi, && c_3 = c_\theta/(q_w c_\phi),\\
&d_3 = 1 - (z_e/\sin s) c_3, && d_2 = 0,\\
&c_5 = (\mathscr{A}_p^l/\sigma^2)\mathfrak{r}_m q_l, && d_4 = -\frac{\mathfrak{z}_e (J\zeta_m)\cdot \mathcal{E}_G}{\sigma\gamma\kappa_m(0,t)},\\
&c_4 = \mathfrak{c}_m z_e/\sin s, && c_6 = \mathfrak{c}_m.
\end{aligned}
\tag{8.19}
$$

Proof. The length equation is $\dot{\mathfrak{z}} = -\mathfrak{z} e_m \cdot (\nabla v_{<m})^T e_m$. The base rotation contributes zero. The parent shear contributes $\mathfrak{z}\mathfrak{r}_p^l \Omega_p(0) \sin\theta^l \cos\theta^l$, giving the first line after division by γ. The principal parent equation and the angle identity are

$$\dot\Omega_p(0) = -\mathfrak{c}_p \mathscr{A}_p^c \sin\phi_p^c, \qquad -\sin\phi_p^c = \sin\theta^c \cos\phi_m - \cos\theta^c\, z_\mu/\mathfrak{z}.$$

They give $\mathcal{W}' = c_2[1 - c_3 z_\mu/\sin s]/\mathfrak{z}$. On the first pulse part $z_\mu = z_e(1-\eta)$, so the bracket is $\eta + d_3(1-\eta)$. On the second part $z_\mu = -\mu\Lambda_{\mathrm{ret}}\mathfrak{z}_e \sin s\, \eta_2$, giving the second expression for $\mathcal{G}$ in (7.18). Finally the upper-right amplitude coefficient obeys

$$\frac{\widehat{a}_m \mathfrak{z}_e}{\gamma} = \frac{\mathscr{A}_p^l}{\sigma^2}\frac{\mathfrak{r}_m q_l \mathfrak{z}_e}{\mathfrak{z}^2} - \frac{\mathfrak{z}_e (J\zeta_m)\cdot \mathcal{E}_G}{\sigma\gamma\kappa_m(0,t)}.$$

The forcing in the quotient equation is $\mathfrak{c}_m z_\mu/(\mathfrak{z}_e \sin s)$. On the two pulse parts it is respectively $(c_4/\mathfrak{z}_e)(1-\eta)$ and $-\mu c_6 \Lambda_{\mathrm{ret}} \eta_2$. These identities also retain the time and trial dependence of c_4, c_6. $\square$

Here are numerical sufficient bounds for the errors in this lemma. For $i < m$ use the local complete-jet formula at Y_m, and put

$$\mathfrak{E}_i^D = 2N_i z_{+,i}\mathcal{H}_i U_{sa}^i(S_i) + \mathcal{H}_i^2 U_{aa}^i(S_i) + \mathcal{H}_{i,2}\big(N_i U_s^i(S_i) + U_a^i(S_i)\big),$$
$$\mathfrak{E}_i^G = \mathcal{H}_i\big(S_i t_i^{\mathrm{lab}}\lambda_i + V_a^{i,\mathrm{corr}}(S_i)\big).$$

These are the non-phase terms at the local label; a change to an own label is not made in these two expressions. Assume $|D_a\zeta_i| \le z_{a,i}$ and $0 < z_{-,i} \le |\zeta_i| \le z_{+,i}$ on the connecting label segments. Write

$$k_i^{\mathrm{lab}} = 2a_+ z_{+,i} z_{a,i} + A_1 \mathcal{H}_i z_{+,i}^2, \qquad \kappa_{-,i} = \min(1, r_+^{-2}) z_{-,i}^2.$$

Thus $|\kappa_i(a) - \kappa_i(0)| \le k_i^{\mathrm{lab}}|a|$. Bounds for the remaining aligned coefficients are

$$
\begin{aligned}
&\mathfrak{S}_i = N_i^2 z_{+,i}^2 (b_i + U_{ss}^{i,\mathrm{corr}}(S_i)), && i < p,\\
&\mathfrak{S}_p = N_p^2 z_{+,p}^2 \left[b_p^{\mathrm{lab}} \lambda_p \mathfrak{o}_p^m + b_p k_p^{\mathrm{lab}} \mathfrak{o}_p^m/\kappa_{-,p} + U_{ss}^{p,\mathrm{corr}}(S_p)\right], &&\\
&\mathfrak{A}_i = N_i z_{+,i}(t_i + V_s^{i,\mathrm{corr}}(S_i)), && i < p,\\
&\mathfrak{A}_p = N_p z_{+,p} V_s^{p,\mathrm{corr}}(S_p). &&
\end{aligned}
\tag{8.20}
$$

For the parent shear, the additional symbol term follows by writing

$$B_p(a) - \frac{\kappa_p(0)}{\kappa_p(a)} B_p(0) = [B_p(a) - B_p(0)] + B_p(0)\frac{\kappa_p(a) - \kappa_p(0)}{\kappa_p(a)}.$$

The base gradient is $-A_0 e_b(t)$, with $e_b = e(t)$ from (2.7). Define all local angles by

$$e_i^l = \frac{\zeta_i(A_i(Y_m,t),t)}{|\zeta_i(A_i(Y_m,t),t)|} = (\sin\phi_i^l, \cos\phi_i^l) \quad (i < m), \qquad e_b = (\sin\phi_b, \cos\phi_b).$$

The base gradient is treated separately.

Let $\Theta_i^D, \Theta_i^G, \Theta_b^G$ be specified numerical bounds for $|\sin(\phi_m - \phi_i^l)\cos(\phi_m - \phi_i^l)|$, $|\sin(\phi_m - \phi_i^l)|$, and $|\sin(\phi_m - \phi_b)|$, respectively. Let $\delta > 0$ be the tolerance in Proposition 7.7. Then the explicit tests

$$
\begin{aligned}
\frac{z_{+,m}}{\gamma}\left[\sum_{i<m}\mathfrak{S}_i\Theta_i^D + \sum_{i<m}\mathfrak{E}_i^D\right] &\le \delta, \\
\frac{\mathfrak{z}_e z_{+,m}}{\sigma\gamma\kappa_{-,m}}\left[A_0\Theta_b^G + \sum_{i<m}\mathfrak{A}_i\Theta_i^G + \sum_{i<m}\mathfrak{E}_i^G\right] &\le \delta
\end{aligned}
\tag{8.21}
$$

give $|d_1|, |d_4| \le \delta$. An upper numerical bound for $\mathfrak{z}_e$ can be used in the second test. In particular each aligned error keeps its angle factor.

For $0 < \varepsilon \le 1/100$ suppose all the factors

$$\mathfrak{r}_p^l,\ \mathfrak{r}_m,\ \mathscr{A}_p^c/\sigma^2,\ \mathscr{A}_p^l/\sigma^2,\ q_w,\ q_l,\ \mathfrak{c}_p,\ \mathfrak{c}_m,\ \mathfrak{z}_e,\ z_e/\sin s,\ c_\phi,\ c_\theta,\ \cos\theta^l$$

are within ε of one. Products of at most four factors and $q_w c_\phi \ge (1-\varepsilon)^2$ give $|c_i - 1|, |d_3| \le 8\varepsilon$. Hence $8\varepsilon \le \delta$ and (8.21) return the coefficient bounds of Proposition 7.7. Their realization, and the entry and geometric bounds (7.19), remain separate quantitative hypotheses until estimated on the finite trials.

8.7. **First rates and the order of differentiation.** The first material time derivative of the complete correction equations is obtained without differentiating their coefficients in time a second time. At each level substitute

$$\partial_t(V_n, \widetilde{W}_n) = \mathcal{B}(V_n, \widetilde{W}_n) - (\mathsf{P}_{\neq 0}S_n^\Gamma, (\sigma/N)\partial_s^{-1}\mathsf{P}_{\neq 0}S_n^\xi)$$

with the normalization of (5.12). Stream recovery is $U_n = (N\sigma\kappa)^{-1}\partial_s^{-1}\widetilde{W}_n$. Each lower-level rate in a new source is replaced by its already formed equation. The rate of κ uses $D = \nabla v_<$ and $v_<$, and the rate of C uses the same quantities. The reduced-vorticity coefficient may contain the instantaneous base term αLH; its time derivative is not an input to these sources. This is the same triangular elimination as in the finite causal construction.

There is also a distinction between a material rate and a physical rate. For a curve C_q the chain rule gives

$$\frac{d}{dt}Q^\natural(C_q(t), t) = Q_t + (a_i^q)' \cdot \nabla_a Q + N_i p_i \cdot (a_i^q)' Q_s,$$

where the right side is evaluated at its phase and label. A bound for this first rate costs

$$\Pi_i + \lambda_i|(a_i^q)'| + N_i|p_i \cdot (a_i^q)'|, \tag{8.22}$$

not $\Pi_i + N_i|(a_i^q)'|$. The latter estimate would erase the phase cancellation. At a fixed physical point the exact rate instead is $Q_t - H_i v_{<i} \cdot \nabla_a Q - N_i\zeta_i \cdot v_{<i} Q_s$; its corresponding physical rate includes these two explicit transport terms. No assertion identifies it with the intrinsic material rate Π_i.

The differentiated gradient and Hessian have only spatial additions. For example, the first identity below is at fixed material label, with $D = (\nabla v_<) \circ X$; in the second, Eulerian identity all coefficients are evaluated at y:

$$
\begin{aligned}
\partial_t H &= -HD, \\
(\partial_t + v_< \cdot \nabla)D_y^2 A &= -D_y A\, D_y^2 v_< \\
&\quad - D_y^2 A[\nabla v_< \cdot, \cdot] - D_y^2 A[\cdot, \nabla v_< \cdot].
\end{aligned}
$$

Consequently the second expression has norm at most $\mathcal{H}\|D_y^2 v_<\| + 2\mathcal{H}_2\|D\|$. Along C_q it has the additional term $(\sum_{i\le j<q} v_j) \cdot \nabla D_y^2 A$, of size at most $\mathcal{H}_3 \sum_{i\le j<q}|v_j|$. Applying the product rule to (5.28) now proves first-rate versions of (8.5) from profile orders at most four and inverse-map orders

at most three. It uses no second material time derivative of a profile and no derivative of α. Bounds for the normalized numbers on an entire lifetime, however, require the spatial assertion (8.1)–(8.2).

8.8. **Initialization in an amplifying regime.** The first central pair can be treated before any correction estimates. Choose $e_0 = (0,1)$, $p_1 = (\sin s, \cos s)$ with $0 < s < \pi/2$, and $\sigma_0 = \sqrt{A_0}$. Take $\tau_1 = 0$ for this calculation. During first growth $\alpha = 0$, $Y_1 = y_*$, and

$$\kappa_s = 1 + (r_0^{-2} - 1)\sin^2 s, \quad \widehat{a}_1 = \sigma_0 \sin s/\kappa_s, \quad \widehat{b}_1 = \sigma_0 \sin s, \quad \gamma_1^{\mathrm{gr}} = \sigma_0 \sin s/\sqrt{\kappa_s}.$$

The negative growing datum is $(-\epsilon_1, -\epsilon_1\sqrt{\kappa_s})$. Therefore

$$T_1(0,t) = -\epsilon_1 e^{\gamma_1^{\mathrm{gr}}(t-\tau_1)}.$$

For any prescribed $0 < \epsilon_1 < T_1^*$, the threshold $-T_1 = T_1^*$ is met at $\tau_1 + (\gamma_1^{\mathrm{gr}})^{-1}\log(T_1^*/\epsilon_1)$, with a strictly positive crossing rate. One may take $T_1^* = N_1^{-1+\beta}$ and $\epsilon_1 = N_1^{-d}$ with $N_1 > 1$, $0 < \beta < 1$ and fixed $d > 1 - \beta$; this is genuine large gradient amplification $N_1 T_1^* = N_1^{\beta}$, not a fixed finite gain followed by a freely shrinking seed.

There is also an explicit first-label variation bound on this growth interval. Choose a fixed closed ball $B_{R_0}(y_*) \Subset \mathcal{P}$ on which $\Gamma_- \le \Gamma_0 - A_0 a_2 \le \Gamma_+$ with $\Gamma_- > 0$, and let $r_\pm$ be its radial bounds. On this ball put $a_- = \min(1, r_+^{-2})$. Since $X_1(a,t) = y_* + a$, the two matrix entries are

$$\widehat{a}_1(a) = \frac{\sigma_0 \sin s}{(r_0^2 + 2a_2)^{-1}\sin^2 s + \cos^2 s}, \qquad \widehat{b}_1(a) = 2\sigma_0 \sin s\,(r_0^2 + 2a_2)^{-2}(\Gamma_0 - A_0 a_2).$$

Define

$$C_1 = \frac{2r_-^{-4}}{a_-^2} + 8r_-^{-6}\Gamma_+ + 2r_-^{-4}A_0, \qquad C_{\mathrm{in}} = \frac{2r_-^{-4}}{a_-} + 4r_-^{-2} + A_0/\Gamma_-.$$

Differentiating these entries gives $\|D_a\mathcal{B}_1\| \le \sigma_0 \sin s\, C_1$. On this first growth interval use this sharper admissible majorant:

$$\mathfrak{m}_1 = \ell_1 \sigma_0 \sin s\, C_1, \qquad c_{*,1} = \int_0^{(\gamma_1^{\mathrm{gr}})^{-1}\log(T_1^*/\epsilon_1)} \mathfrak{m}_1(t)\, dt.$$

If $f(a) = \sqrt{\widehat{b}_1(a)/\widehat{a}_1(a)}$, then $|D_a \log f| \le C_{\mathrm{in}}/2$. Therefore, when $\ell_1 C_{\mathrm{in}} \le 1$,

$$\ell_1 |D_a[-\epsilon_1(1, f(a))]| \le \ell_1 C_{\mathrm{in}} |R_1^{\mathrm{in}}(0)|.$$

Here $f(a)/f(0) \le e^{\ell_1 C_{\mathrm{in}}/2} < 2$, which explains the relative normalization of this estimate. On this interval choose $c_{\mathrm{dat},1} = \ell_1 C_{\mathrm{in}}$ in place of the general datum bound.

For the central matrix use $S = \mathrm{diag}(1, \sqrt{\kappa_s})$. Then $S^{-1}\mathcal{B}_1(0)S = \gamma_1^{\mathrm{gr}} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus

$$C_R = \max(\sqrt{\kappa_s}, 1/\sqrt{\kappa_s}), \qquad \|\mathcal{P}_1(0;t,u)\| \le C_R e^{\gamma_1^{\mathrm{gr}}(t-u)} = C_R \frac{|R_1(0,t)|}{|R_1(0,u)|}.$$

Put $G = \log(T_1^*/\epsilon_1) > 0$ and $c_{\mathrm{pk}} = \log 2/(6C_R^2)$. The scalar conditions

$$\begin{gathered} 0 < 2\ell_1 \le R_0, \qquad \ell_1(C_1\sqrt{\kappa_s}G + C_{\mathrm{in}}) \le c_{\mathrm{pk}}/2, \\ \Pi_1 \ge \max\{1, \ell_1^{-1}, \sigma_0 \sin s \max(a_-^{-1}, 2r_-^{-4}\Gamma_+), \ell_1\sigma_0 \sin s\, C_1\}. \end{gathered} \tag{8.23}$$

imply $\ell_1 C_{\mathrm{in}} < 1$ and

$$c_{*,1} + c_{\mathrm{dat},1} \le \ell_1(C_1\sqrt{\kappa_s}G + C_{\mathrm{in}}) < \frac{\log 2}{6C_R^2}.$$

This is the first-label and integrated-variation part of (3.5). For the displayed amplifying seeds, $G = (d - 1 + \beta)\log N_1$; the bound permits a radius of order $(1 + \log N_1)^{-1}$ on this fixed first collar.

The first finite construction exists through this growth time whenever the prescribed smooth activation and other fixed data extend there. Its older flow and coefficients are those just displayed. Their denominators stay positive on the fixed collar, and the finite Duhamel recursion has smooth bounded coefficients on each such finite interval. It supplies all correction levels successively, with finite derivatives. This continuation argument proves finiteness, not a frequency-uniform bound for the complete corrections.

The first return uses the physical metric factor throughout. With the prescribed first covector component, $|\zeta_1| = 1$ and $\kappa_1 = 1 + (r_0^{-2} - 1)z_\mu^2$. For fixed pulse compression and compact trial set, $|z_\mu| \le C \sin s$. Hence the normalized coefficient in the scalar return equation differs from the constant-metric one by at most $C|r_0^{-2} - 1| \sin^2 s$. The same bound holds for its first trial derivative. The positive-amplitude and endpoint-sign argument in Proposition 7.8 therefore applies for sufficiently small s, independently of ϵ_1 and T_1^*, by linear homogeneity. The return gain and duration are bounded in normalized time. This establishes first central initialization at the displayed amplifying scale. It does not establish bounds for the first complete corrected field at that scale or provide the second-stage input.

8.9. **Scope of the estimates.**

Proposition 8.4 (Low-order consequences of numerical bounds)**.** *On an existing finite trial interval suppose* (8.1)–(8.2)*, the principal derivative bounds and the stated covector windows hold on the specified cores and containing regions. Each conclusion below assumes its associated numerical tests. Then the complete displaced gradients obey* (8.5)*, and the integral tests* (8.10)*, together with the nesting and absolute distance tests, exclude exits from their relaxed regions. On an interval beginning at insertion,* (8.7) *gives the order-zero and first-label nonlinear-map conclusions. The uniform circulation estimate* (8.13)*, when $g_c \le \Gamma_0/2$, returns positive circulation; the covector and radial windows give the positive symbol window.*

The numerical coefficient and datum bounds (8.17)–(8.18) *return their first-label amplitude inputs. The return factors and error tests following Lemma 8.3 return the coefficient bounds in* (7.18)*. The first evaluation rates are bounded with* (8.22) *and the displayed inverse-map identities. None of these conclusions requires a second material time derivative of a profile or a time derivative of α.*

Proof. Combine Lemmas 8.1, 8.2 and 8.3 with the preceding non-affinity, positivity, coefficient, datum and first-rate estimates. All comparisons are on existing initial segments; strict numerical margins exclude first exits from the corresponding relaxed conditions. □

The intrinsic complete profile and map estimates, their later-window orders and constants, the central weighted histories and strict control windows are proved in Sections 9–11. Proposition 11.1 returns this section's analytic inputs for the actual finite family under its scalar conditions, including selected complete-gradient holding and the next insertion. The first calculation above supplies the central growth and first-label variation bounds; complete correction bounds are obtained in Section 9. Proposition 12.3 gives one simultaneous infinite realization of the scalar conditions.

9. Continuation with the complete corrections

We prove the low-order estimates for the finite family constructed in Proposition 7.2. The scalar conditions in this section are imposed on the prescribed parameters and on finite recursions of numbers. The recursions are given below, including the coefficient recursion. Consequently none of the conditions is an assumption on an unknown derivative of a constructed field.

We use the magnified unit-ring coordinates of Lemma 2.3, suppressing primes. Write

$$c_g = 1 + 2\varepsilon_g x_2, \quad \mathsf{A} = \mathrm{diag}(c_g^{-1}, 1), \quad L = \partial_2^2 + c_g^{-1}\partial_1^2, \quad W = \varepsilon_g c_g^{-2}.$$

Here $\varepsilon_g = \varepsilon_{\mathrm{geom}}$ is fixed. On a fixed convex containing region $\mathcal{P} \Subset \{c_g > 0\}$, choose $0 < c_- \le c_g \le c_+$. All connecting segments below are required to lie in this region and in the indicated older core. The equations are

$$v = J\nabla\Psi, \quad \xi = L\Psi, \quad D_v\Gamma = F^\Gamma, \qquad D_v\xi - W\partial_1(\Gamma^2) = E^\xi, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{9.1}$$

In particular $\Theta = \Gamma^2$, whereas T_i is a principal circulation amplitude. On $\mathcal{P}$ the base is

$$v^b = \alpha Jx, \quad \Gamma^b = \Gamma_0 - A_0 e_b(t)\cdot x, \quad \Gamma_0 = (2\varepsilon_g)^{-1}, \quad \dot e_b = \alpha J e_b, \quad \xi^b = \alpha(1 + c_g^{-1}).$$

The constant circulation is combined with the radial weight before estimating:

$$2W\Gamma_{<i} = c_g^{-2}[1 + 2\varepsilon_g(\Gamma_{<i} - \Gamma_0)]. \tag{9.2}$$

Fix the carriers N_i, clocks σ_{i-1}, radii ℓ_i, depths $J_i \geq 2$, negative seeds, and smooth phase, label and time profiles of the finite construction. The envelopes have support in B_{ℓ_i}, are defined on $B_{2\ell_i}$, and are one on $B_{\rho_i^{\mathrm{core}}}$. Material estimates below concern initial labels in B_{ℓ_i}. The larger ball is an insertion-data collar; its all-time transport is not required. The odd phase profile F equals s on $|s| \leq s_c$, and its mean-zero primitive is P. The activation w_i is flat at insertion, lies in $[0,1]$, and is one after its prescribed ramp. Put

$$\delta_i = \Pi_i^{\mathsf{d}} N_i^{-\mathsf{h}}, \quad \bar\beta = 1 - \beta, \quad Y_{*,i} \leq C_Y C_\Theta N_i^{-\bar\beta}, \qquad 0 < \delta_i \leq \tfrac{1}{2}.$$

The older-flow point is $Y_i(t) = X_i(0,t)$. The pair $R_i = (T_i, \widehat\Omega_i)$ has instantaneous magnitude $a_i(t) = |R_i(0,t)| = Y_{*,i}\rho_i(t) > 0$, where the scalar amplitude comparison gives $0 < \rho_i \leq 1$.

For this finite implication, adopt the scalar schedule (11.6), the first-pulse conditions immediately following it, the fixed margins (11.7), and the tolerance rule (11.8). The first-rate conditions are (11.11) and the order-one comparisons in (11.21); their numerical realization is deferred to Section 12. In particular $C_S > 1$ and $C_T = 2C_S$ are the fixed duration constants there.

The scalar comparison constants are chosen with the strict margins of [1, Theorem 2.2]: in particular

$$0 < c_W < \tfrac{1}{3}, \qquad C_W > 1 + \bar\upsilon, \qquad C_W^{\mathrm{all}} > 1 + c_p, \qquad c_p > \bar\upsilon > \tfrac{3}{2},$$
$$0 < c_\theta < \tfrac{2}{3} < 1 < C_\vartheta, \qquad 0 < z_- < 1 < \tfrac{3}{2} < z_+.$$

The positive lower bound on the controlled second component, the angular cone, the gain bounds and the scalar duration margins are chosen there as well. We denote the resulting small tolerance by ε_s. Its choice incorporates the finite perturbation constants in Lemma 9.2, including $16C_\vartheta\varepsilon_s \leq 1/2$ and $8\varepsilon_s \leq c_\theta$. These constants are fixed before the field estimates. In particular $\varepsilon_s \leq 1/4$. When bounding a physical shear sum, use the fixed comparison letters

$$\overline{C}_W = \tfrac{4}{3}C_W + \tfrac{1}{8}c_W, \qquad \underline{c}_W = \tfrac{3}{20}c_W. \tag{9.3}$$

They bound the upper and lower physical coefficients in (9.43) below; they do not redefine the principal vorticity amplitude or its prescribed interval. The scalar constants for shear sums and their time integrals are evaluated with these letters. The return parameter ranges over the same compact interval $0 \leq \mu \leq c_p$ for every trial.

9.1. **A finite scalar recursion.** Fix a layer i at which the estimate is required; time orders are $b = 0, 1$. Its label derivative allowance is $L_i = 2J_i + 9$, and for its ancestors set

$$L_j = L_{j+1} + 2J_j + 6 \quad (j < i), \qquad \iota_j = 3J_j + 4. \tag{9.4}$$

At level n, use phase order at most $\iota_j - 2n$ and label order at most $L_j - 2n - 1$; the lower elliptic expressions at level q use label order $L_j - 2q$. The map is supplied two label derivatives further. These orders include all the third spatial derivatives and first material rates used below. Indeed a level at label order r consumes coefficients through $r + 2J_j + 1$, the map through $r + 2J_j + 3$, and one conversion consumes one further ancestor derivative. Also $n + 2 \leq \iota_j - 2n$ for $n \leq J_j$. Thus the polynomial degree $n + 2$ on the phase cell controls every phase derivative requested by an ancestor composition. Means and phase primitives are first evaluated on the entire phase circle.

Here are the scalar operations we use. We use Euclidean tensor norms. For a profile or pulled-back coefficient f_i, the array of numerical upper bounds means

$$\sup_{\substack{s\in\mathbb{T}\\ a\in B_{\ell_i}}} \ell_i^{|\alpha|}\left|\partial_t^b\partial_s^p D_a^\alpha f_i(s,a,t)\right| \le \mathfrak{w}_{f_i}(t) f^{\mathrm{bd}}_{i;b,p,\alpha}(t). \tag{9.5}$$

Time is differentiated at fixed (s,a). The displayed weight is 1 for coefficients, $a_i(t)$ for the principal pair and normalized correction pair, and $a_i(t)/(N_i\sigma_{i-1})$ for the recovered streamfunction; the factor χ_i remains in its coefficient array. Prescribed profiles use weight one. After means and primitives on $\mathbb{T}$ are formed, cell estimates restrict the phase to the declared $|s|\le S_i$. The map-remainder estimates are instead taken on their stated rigid containing ball. A scalar derivative number with a single spatial order, such as d_1^Γ, $X^{\mathrm{bd}}_{i,r}$ or $A^{\mathrm{bd}}_{i,r}$, has unscaled units: restore $\ell_i^{-|\alpha|}$ and its displayed weight from the array entry before using a radius in a mean-value estimate. The insertion datum numbers on $B_{2\ell_i}$ are calculated separately at insertion, where the map is affine. Addition and tensor contraction sum the entries. For products use

$$(u\star v)_\nu=\sum_{\lambda\le\nu}\binom{\nu}{\lambda}u_\lambda v_{\nu-\lambda}.$$

A derivative shifts the corresponding index, restoring ℓ_i^{-1} for a label derivative. For composition use the finite, component-labelled set-partition formula

$$\mathcal{P}_r(f;g)=\sum_{\pi\in\mathrm{Part}(r)} f_{|\pi|}\prod_{B\in\pi} g_{|B|},\qquad \mathcal{P}_0(f;g)=f_0. \tag{9.6}$$

The colours of the components distinguish a phase derivative from a label derivative. For a positive scalar $k\ge k_->0$, the reciprocal array is

$$j_0=k_-^{-1},\qquad j_\nu=k_-^{-1}\sum_{0<\lambda\le\nu}\binom{\nu}{\lambda}k_\lambda j_{\nu-\lambda}. \tag{9.7}$$

The mean, projection $\mathsf{P}=1-\langle\cdot\rangle_s$, and literal mean-zero primitive $\mathcal{I}$ have constants $1,2,\pi/2$. These rules follow from the product and chain rules and from $kk^{-1}=1$. They specify scalar polynomials, not maxima of unknown field norms. First time derivatives are formed by differentiating the displayed finite expressions once.

We retain a term's carrier, radius, amplitude and rate factors until it has been divided by its required output size. An entry $u@X$ means the positive coefficient u at the positive rate X. At first order a product retains the size of its undifferentiated factors. An integrated source entry uX^b becomes $(u/X)X^{b+1}$ in the differentiated equation; after division by an output size Π_iA, its coefficient is $u/(\Pi_iX)$. An unintegrated term receives no such division.

Fix the current finite horizon t_{end}. For each inserted layer, τ_i is its insertion time, t_i^a its growth hit, and t_i^b the end of its final return pulse. Write

$$I_i=[\tau_i,t_{\mathrm{end}}],\qquad G_i=[\tau_i,t_i^a]\cap I_i,\qquad W_i=[\tau_i,t_i^b]\cap I_i. \tag{9.8}$$

Thus W_i consists of growth, transition and both return pulses, whereas I_i includes all elapsed later evolution of layer i. Until a hit or return endpoint has been reached, the corresponding interval denotes only its existing initial part. This convention does not assert continuation to that endpoint. On the selected history $t_i^b=\tau_{i+1}$; the hold through the next growth and the remaining tail are the separate pieces of (9.46). If an infinite selected history is subsequently constructed with terminal time T_*, its lifetime notation is $I_i=[\tau_i,T_*)$, and every finite use is truncated as above.

Here is the first-rate list used in these rules. On the scalar window W_l, put

$$\gamma_l^{\mathrm{gr}}=\sigma_{l-1}\sin\theta_l^{\mathrm{in}},\qquad X_l^{\mathrm{rate}}=\Lambda_l\gamma_l^{\mathrm{gr}},\qquad \lambda_l^*=\Lambda_l\theta_l^{\mathrm{in}}\quad(l\ge2).$$

Here Λ_l is the prescribed logarithmic growth length; λ_l^* is the angle-amplitude bound for the rotation on that window, distinct from its return parameter. At layer one set $\gamma_1^{\mathrm{gr}} = \sigma_0 \sin\theta_1^{\mathrm{in}}$; with the fixed first-pulse compression $\Lambda_{\mathrm{ret},1}$, use

$$X_1^{\mathrm{rate}} = \Lambda_{\mathrm{ret},1}\gamma_1^{\mathrm{gr}}, \qquad \lambda_1^* = c_p \Lambda_{\mathrm{ret},1} \sin\theta_1^{\mathrm{in}}.$$

This fixed compression is distinct from its logarithmic growth length. Put

$$r_i = C_\psi(R_0+2) + 2(2\overline{C}_W + c_{\mathcal{B}})\sigma_{i-1}, \qquad \widehat{X}_{i,l} = \max(r_i, X_l^{\mathrm{rate}}), \qquad \varkappa_{i,l}^+ = \begin{cases} 1, & l = i, \\ \sigma_i/\sigma_{l-1}, & l \geq i+1. \end{cases}$$

Here $C_\psi, R_0, c_{\mathcal{B}}$ are respectively the fixed scalar rotation bound, containing radius, and central matrix-size constant; take $2c_{\mathcal{B}}\sigma_{i-1}$ to dominate both central off-diagonal coefficients on their scalar intervals. Define

$$\mathcal{L}_{i,l}[1] = r_i + \sum_{i \leq q \leq l} \varkappa_{i,q}^+ \widehat{X}_{i,q} + \sum_{1 \leq q \leq l} \lambda_q^* \widehat{X}_{i,q}. \tag{9.9}$$

The scalar schedule comparisons are $r_i \leq \Pi_i$, $\sum \lambda_q^* \leq 2$, the own-window rates $X_q^{\mathrm{rate}} \leq \Pi_i$ for $q \leq i+1$, and the two first-mass comparisons for the remaining sums in (11.21). Its X_q is X_q^{rate} here, and its $\mathscr{R}_i = \max(1, r_i)$ dominates the present floor. Adding these comparisons gives $\mathcal{L}_{i,l}[1] \leq C_1 \Pi_i$, with the same C_1 for every later window. These are scalar comparisons in the parameters, not assertions about fields. An invariant metric coefficient multiplies the last sum by its anisotropy factor, as shown in (9.17); an axis-facing coefficient does not. Only this first-mass assertion, not an arbitrary-order rate assertion, is used here.

The map and coefficients. Suppose layers $j < i$ have been calculated. Put

$$X_i = Y_i + M_i(a + \Xi_i), \quad A_i = X_i^{-1}, \quad H_i = (D_a X_i)^{-1}, \quad \zeta_i = H_i^T p_i, \quad C_i = H_i \mathsf{A}(X_i) H_i^T,$$

and $\kappa_i = p_i^T C_i p_i$, $\chi_i = \kappa_i^{-1}$. The scalar bounds for M_i, its first rate and its inverse come from the scalar comparisons below. On the rigid tube define

$$\widetilde{v}_i(a', t) = M_i^{-1}[v_{<i}(Y_i + M_i a', t) - v_{<i}(Y_i, t) - Dv_{<i}(Y_i, t) M_i a'].$$

To calculate its derivatives, apply (9.6) to each of the already calculated complete streamfunctions, using the identity

$$M_i^{-1} J \nabla_x (U_j^\natural)(Y_i + M_i a') = J \nabla_{a'} (U_j^\natural (Y_i + M_i a')). \tag{9.10}$$

Subtract the affine Taylor polynomial. If $\mathcal{V}_i(t)$ is the resulting positive sum for the second derivative, then

$$|\widetilde{v}_i| \leq 2\ell_i^2 \mathcal{V}_i, \quad |D\widetilde{v}_i| \leq 2\ell_i \mathcal{V}_i, \quad |D^2 \widetilde{v}_i| \leq \mathcal{V}_i, \qquad \eta_i(t) = 4\ell_i \int_{\tau_i}^t \mathcal{V}_i(s)\, ds. \tag{9.11}$$

The base contributes zero. The directional calculation in Lemma 9.1 is used before taking absolute values in (9.10). All further derivatives are the same partition sums with their additional label and curvature factors.

For clarity, the finite flow recursion can be written without implicit constants. Scale $a = \ell_i z$, write $h = \Xi_i/\ell_i$, and let F_r be the calculated bound such that $\|D^r f\| \leq \eta_i d_i(t) F_r$ in $h_t = f(z+h, t)$, where $\int d_i \leq 1$. Choose the positive scalar envelope $\mathcal{V}_i$ from the full second-derivative expressions, retaining each of their positive scalar amplitude factors. Every higher derivative is one of those expressions times additional calculated derivative factors. Thus F_r is their finite relative sum and is defined also when an amplitude factor vanishes. One may take $d_i = 4\ell_i \mathcal{V}_i/\eta_i$, with the final value of

η_i in the denominator. An identically zero remainder is assigned $h = 0$. Set

$$\begin{aligned} &E = e^{\eta_i F_1}, \qquad B_1 = E, \qquad D_1 = F_1 \int_0^1 e^{s\eta_i F_1}\, ds, \\ &D_r = E \sum_{\substack{\pi \in \mathrm{Part}(r) \\ |\pi| \geq 2}} F_{|\pi|} \prod_{B \in \pi} B_{|B|}, \qquad B_r = \eta_i D_r \quad (r \geq 2). \end{aligned} \tag{9.12}$$

Then $|D^r h| \leq \eta_i D_r$, $|h| \leq \eta_i F_0$, and

$$|\partial_t D^r h| \leq \eta_i d_i(t) \mathcal{P}_r(F; B). \tag{9.13}$$

To prove these assertions, isolate $Df(z+h)D^r h$ in the differentiated equation and use Gronwall, whose exponent is at most $\eta_i F_1$. Every other block has order less than r. For $r = 1$, $E - 1 = \eta_i D_1$; the first-time formula follows directly from the equation and reads no time derivative of f. The same proof with truncated integrals retains the factor $\eta_i(t)$ at time order zero.

The inequalities $\eta_i F_0 \leq 1/8$, $\eta_i F_1 \leq 1/8$ keep the image of B_{ℓ_i} in $B_{2\ell_i}$ and give $\|D\Xi_i\| < 1/4$. Determinant one gives the more precise formula

$$H_i = \mathrm{adj}(I + D\Xi_i) M_i^{-1}. \tag{9.14}$$

The adjugate is linear in dimension two, so every positive label derivative of the nonconstant part retains $\eta_i(t)$. For Eulerian inverse derivatives, the scalar recursion obtained from $X_i(A_i(x)) = x$ is

$$A^{\mathrm{bd}}_{i,1} = H^{\mathrm{bd}}_{i,0}, \qquad A^{\mathrm{bd}}_{i,r} = H^{\mathrm{bd}}_{i,0} \sum_{\substack{\pi \in \mathrm{Part}(r) \\ |\pi| \geq 2}} X^{\mathrm{bd}}_{i,|\pi|} \prod_{B \in \pi} A^{\mathrm{bd}}_{i,|B|}.$$

Thus inverse curvature, as well as the forward map, is calculated.

The metric derivative bounds are explicit:

$$|\partial_2^r c_g^{-1}| \leq r!(2\varepsilon_g)^r c_-^{-r-1}, \qquad |\partial_2^r W| \leq \varepsilon_g (r+1)!(2\varepsilon_g)^r c_-^{-r-2}.$$

Compose them and the complete older scalar arrays with X_i, and form, in this order,

$$\begin{aligned} &q_i = \Gamma_{<i} \circ X_i, \quad b_i = \xi_{<i} \circ X_i, \quad \varpi_i = W \circ X_i, \quad h_i = H_i e_1, \\ &\mathfrak{l}_{i,k} = (LA_{i,k}) \circ X_i, \quad d_i^\Gamma = Jp_i \cdot D_a q_i, \quad c_i = 2\varpi_i q_i, \qquad \mathcal{B}_i = \begin{pmatrix} 0 & -d_i^\Gamma \chi_i/\sigma_{i-1} \\ \sigma_{i-1} c_i \zeta_{i,1} & 0 \end{pmatrix}. \end{aligned} \tag{9.15}$$

The vector $\mathfrak{l}_i$ equals $\mathrm{div}_a\, C_i$. Indeed the volume-preserving change of variables in $\mathrm{div}_x(\mathsf{A}\nabla_x)$ gives this identity, and $\mathrm{div}_x \mathsf{A} = 0$. Its rigid part is zero. Using the curvature expression before taking absolute values avoids an extraneous metric-divergence term. Also $D_a b_i = (D_a X_i)^T (\nabla \xi_{<i}) \circ X_i$, with the full nonlinear derivative.

There are two useful first-time rules. At a rigid young point,

$$\dot{a}_j = H_j \left(\sum_{j \leq k < i} v_k - M_i \widetilde{v}_i \right), \quad \dot{s}_j = N_j p_j \cdot \dot{a}_j, \qquad \frac{d}{dt}\Phi = \Phi_t + \dot{s}_j \Phi_s + \dot{a}_j \cdot D_a \Phi. \tag{9.16}$$

The own fast term in $\dot{s}_j$ cancels. All velocities on the right are at time order zero. The pointwise bound on $d_i(t)$, as well as its integral, is retained in the remainder term. For invariant coefficients write $Dv_{<i} = \alpha J + D_i^p$. Then

$$\dot{C}_i = H_i \{ (v_{<i} \cdot \nabla)\mathsf{A} - D_i^p \mathsf{A} - \mathsf{A}(D_i^p)^T + \alpha(\mathsf{A}J - J\mathsf{A}) \} H_i^T. \tag{9.17}$$

The rotation coefficient has the factor $\|\mathsf{A} - I\|$. The same identity differentiated in labels, and the curvature formula for $\mathfrak{l}_i$, give its rates. In a co-rotating frame the isotropic term has no rotation rate. Axis-facing factors, including $\zeta_{i,1}$ and h_i, retain that rate. Split $b_i = b_i^{\mathrm{nr}} + \alpha b_i^{\mathrm{rot}}$, where $b_i^{\mathrm{rot}} = (1 + c_g^{-1}) \circ X_i$. Only its time-zero value is read in a first-time correction equation. In particular this calculation never requires $\dot{\alpha}$.

The principal pair and every correction. The central scalar comparisons give $\widehat{a}_i, \widehat{b}_i > 0$ at insertion and their positive ratio interval. The full-collar positivity test is written in (9.31) below. Differentiate the literal datum $-\epsilon_i(1, \sqrt{\widehat{b}_i/\widehat{a}_i})$ by (9.7) and the scalar chain rule. Let d_α^{in} be the resulting relative datum numbers. If M_β is the integral of the computed array for $\ell_i^{|\beta|} D_a^\beta \mathcal{B}_i$, set

$$P_\alpha = 2C_R \left[d_\alpha^{\mathrm{in}} + \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} M_\beta P_{\alpha-\beta} \right]. \tag{9.18}$$

Weighted Duhamel, starting with $\alpha = 0$, proves $\ell_i^{|\alpha|} |D_a^\alpha R_i| \le a_i(t) P_\alpha$. Its first rate is the product array for $\mathcal{B}_i R_i$. The propagator used here is justified below; its constant is $2C_R$, and its weight is the instantaneous $a_i(t)$.

We recall the full equations needed for the correction recursion. Suppress i, write $\sigma = \sigma_{i-1}$, and put

$$Q_n = (V_n, \widetilde{W}_n), \quad Q_0 = wRFg, \quad U_n = (N\sigma)^{-1}\chi\mathcal{I}\widetilde{W}_n, \quad W_n = (N/\sigma)\widetilde{W}_n.$$

With $\partial_\perp = Jp \cdot D_a$, set

$$\mathcal{D}_1 U = 2(Cp)\cdot D_a U_s + (p\cdot \mathfrak{l})U_s, \qquad \mathcal{D}_2 U = C : D_a^2 U + \mathfrak{l}\cdot D_a U,$$

$$Z_n = \partial_s W_n + Z_n^\flat, \qquad Z_n^\flat = N\mathcal{D}_1 U_{n-1} + \mathcal{D}_2 U_{n-2}. \tag{9.19}$$

Negative indices vanish. Although $Q_n = U_n = 0$ for $n > J$, Z_{J+1} and Z_{J+2} are retained. Write $\{U, V\} = JD_a U \cdot D_a V$ and $\mathcal{J}_s(U,V) = U_s \partial_\perp V - V_s \partial_\perp U$. The two source rows, apart from the outer time derivative of $Z_n^\flat$, are exactly

$$\begin{aligned} S_n^\Gamma &= \{U_{n-1}, q\} + N \sum_{u+v=n-1} \mathcal{J}_s(U_u, V_v) + \sum_{u+v=n-2} \{U_u, V_v\}, \\ S_n^{\xi,0} &= NU_{n-1,s}\partial_\perp b + \{U_{n-2}, b\} - ch\cdot D_a V_{n-1} - 2\varpi V_{n-1} h\cdot D_a q \\ &\quad + N \sum_{u+v=n-1} \mathcal{J}_s(U_u, Z_v) + \sum_{u+v=n-2} \{U_u, Z_v\} \\ &\quad - N\varpi\zeta_1 \partial_s \sum_{u+v=n-1} V_u V_v - \varpi h\cdot D_a \sum_{u+v=n-2} V_u V_v. \end{aligned} \tag{9.20}$$

In particular both circulation-square terms have their negative sign. Let

$$h_n = \begin{pmatrix} 0 \\ -(\sigma/N)\mathcal{I}\mathsf{P}Z_n^\flat \end{pmatrix}, \qquad f_n = \begin{pmatrix} -\mathsf{P}S_n^\Gamma \\ -(\sigma/N)\mathcal{I}\mathsf{P}S_n^{\xi,0} \end{pmatrix}.$$

Integration by parts gives the exact expression

$$Q_n = h_n + \int_{\tau_i}^t \mathcal{P}_i(a; t, s)(\mathcal{B}_i h_n + f_n)(s)\, ds. \tag{9.21}$$

On a later interval beginning at t_0, add $\mathcal{P}_i(a; t, t_0)(Q_n - h_n)(t_0)$. The original expression (9.21) is always available, so the recursion is never restarted with zero data.

Apply the positive rules to h_n and $g_n = \mathcal{B}_i h_n + f_n$, dividing their common amplitude by $a_i(t)$. Call the resulting arrays $H_{n,b,p,\alpha}$ and $G_{n,b,p,\alpha}$; products of two profiles retain the known factor $Y_{*,i}$, since $a_i^2 \le Y_{*,i} a_i$. In increasing label order form

$$
\begin{aligned}
E_{n,p,\alpha}(t) &= 2C_R \int_{\tau_i}^{t} \left[G_{n,0,p,\alpha} + \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} B^{\mathrm{bd}}_{i,0,\beta} E_{n,p,\alpha-\beta} \right](s)\, ds, \\
Q^{\mathrm{bd}}_{n,0,p,\alpha} &= H_{n,0,p,\alpha} + E_{n,p,\alpha}, \\
Q^{\mathrm{bd}}_{n,1,p,\alpha} &= H_{n,1,p,\alpha} + G_{n,0,p,\alpha} + \sum_{\beta\le\alpha} \binom{\alpha}{\beta} B^{\mathrm{bd}}_{i,0,\beta} E_{n,p,\alpha-\beta}.
\end{aligned}
\tag{9.22}
$$

Here $B^{\mathrm{bd}}_{i,0,\beta}$ is the calculated coefficient array of weight one, with the label scaling of (9.5). Weighted Duhamel applied to $Q_n - h_n$ proves (9.22): the factor $a_i(t)/a_i(s)$ cancels the source weight exactly. Induction in n and then $|\alpha|$ proves every estimate. The last line is the equation differentiated once, not a derivative of its upper bound. It reads h_n at first time order and g_n at time order zero. The rotational part of f_n is αK_n; thus its contribution uses α, not $\dot{\alpha}$. The first-rate entries from the integral receive the division described above; (9.17) preserves the small anisotropy factor in the unintegrated entries.

The recursion starts from the prescribed profile norms and moments, scalar interval and truncated history bounds, explicit metric derivatives, and already computed older-layer bounds. All central rates used by a first-time field are of order zero. The map precedes the coefficients, the coefficients precede the pair, and lower levels precede the next level. Each bound for the current layer is obtained in this order. The base begins this recursion with $\Xi_1 = 0$, $K_1 = 1$, and $\mathfrak{l}_1 = 0$. The same rules and constants are used when a fixed layer is read on a subsequent window.

Normalization of the complete sums. For each consumed derivative and first-rate entry, calculate the full coefficient $a^{\mathcal{R}}_{i,n}$ after extracting $Y_{*,i}\delta_i^n$, and also $(N_i\sigma_{i-1})^{-1}$ for a streamfunction. The superscript specifies its actual use, including all subsequent multiplication and conversion factors. Its smallness test is made after division by the required output size. For example, the pure-phase third derivative on a young tube has size

$$
\frac{N_i^2 Y_{*,i}\delta_i}{\sigma_{i-1}} \sum_{n=1}^{J_i} a^{\mathrm{tube}}_{i,n} \delta_i^{n-1}. \tag{9.23}
$$

Thus its relative level factor is δ_i^{n-1}. Positive low-order numbers $A^{\mathcal{R}}_{i,1}$, $A^{\mathcal{R}}_{i,2}$ are calculated from levels one and two, enlarged to at least one. Sufficient finite tests are

$$
\begin{aligned}
a^{\mathrm{tube}}_{i,1} &\le A^{\mathrm{tube}}_{i,1}, \qquad a^{\mathrm{tube}}_{i,n}\delta_i^{n-1} \le A^{\mathrm{tube}}_{i,1} 2^{-(n-1)} \quad (n \ge 2), \\
a^{\mathcal{R}}_{i,2} &\le A^{\mathcal{R}}_{i,2}, \qquad a^{\mathcal{R}}_{i,n}\delta_i^{n-2} \le A^{\mathcal{R}}_{i,2} 2^{-(n-2)} \quad (n \ge 3).
\end{aligned}
\tag{9.24}
$$

Equivalently one can test the finite sums directly. The tube sum is at most $2A^{\mathrm{tube}}_{i,1}$. The even sum is at most $\frac{4}{3}A^{\mathcal{R}}_{i,2}\delta_i^2$, and the odd sum above level one is at most $\frac{2}{3}A^{\mathcal{R}}_{i,2}\delta_i^2$. Its leading term remains $a^{\mathcal{R}}_{i,1}\delta_i$. Every aligned odd contribution at phase $|s| \le S_i$ gains a factor S_i; the comparison retains

$$
\frac{S_i}{\delta_i} \le C \Pi_i^{2-\mathsf{d}} N_i^{-(\bar{\beta}-\mathsf{h})}.
$$

In particular $\mathsf{d} \ge 2$, $\mathsf{h} \le \bar{\beta}$ give the required bounded ratio. These are full-depth tests, separately for each first-rate entry; they do not follow merely from $\delta_i \le 1/2$.

The four types of scalar comparisons are now completely specified: the map numbers multiplied by η_j; differentiated older scalar coefficients multiplied by their label variation factors; the complete center errors in their stated scalar units below; and $a^{\mathcal{R}}_{i,n}\delta_i^n$ multiplied by the size ratio of its use. In each case the inequality is the computed finite sum

$$\sum_{i,n,\nu} a_{i,n,\nu}\delta_i^n \frac{S_{i,n,\nu}}{S_{\mathrm{out}}} \le A_{\mathrm{out}}. \tag{9.25}$$

Both $S_{i,n,\nu}$ and $S_{\mathrm{out}} > 0$ are explicit products of the prescribed carriers, radii, clocks and scalar comparison amplitudes, obtained by the preceding rules. No division by a realized field occurs. Old complete levels are tested at their own index before use at the next index; later windows use those same numbers. The low partition condition is

$$\eta_i \sum_{s=2}^{9} \frac{18^s}{s!} \max_{q\le s} F_q \le (16e^2)^{-1}.$$

It follows from the own-layer refined condition $4^R C_i^F(R)\eta_i \le 1$, when that condition is available at $R \ge 47$, with $C_i^F(R) \ge \max_{q\le R} F_q$, because

$$16e^2 \sum_{s=2}^{9} 18^s/s! < 16e^{20} < 4^{47}.$$

The first-time map bound is read directly from (9.13), so it uses the same time-zero F_q. Higher label orders still use the finite general recursion (9.12); they require no new refinement condition.

9.2. **Containing tubes, coefficient variation, and displacement.**

Lemma 9.1 (Directional estimate on the containing tube)**.** *For a parent $i = j-1$, all the scalar derivatives needed in (9.11) follow from the preceding complete arrays and the following two scalar calculations. At $Y_j(t)$, put $z = \zeta_i(Y_j,t)$, and let $U_i = \sum_{n=0}^{J_i} U_{i,n}$. Then*

$$\frac{d}{dt}(M_j^T z) = M_j^T \nabla(v_i \cdot z) = -M_j^T \nabla(\partial_\perp U_i)^\natural. \tag{9.26}$$

With B_i^{dir} and E_i^{dir} the scalar bounds for $N_i|U_{i,sa}\cdot Jp_i|$ and $|H_i^T D_a \partial_\perp U_i|$ on the prescribed cell,

$$|M_j^T z - z(\tau_j)| \le e^{\int_{\tau_j}^t B_i^{\mathrm{dir}}(s)\,ds} \int_{\tau_j}^t [B_i^{\mathrm{dir}}|z(\tau_j)| + K_j E_i^{\mathrm{dir}}]\,ds =: \mathcal{D}_{ij}(t). \tag{9.27}$$

If the inherited local covector bound is $|z(\tau_j)| \le z_+$, $\int_{\tau_j}^t B_i^{\mathrm{dir}}(s)\,ds \le 1/2$, $\mathcal{D}_{ij} \le z_+/2$, and $2\ell_j K_j^2 A_{i,2}^{\mathrm{bd}} \le z_+/2$, the parent phase gradient in rigid young coordinates is bounded by $2N_i z_+$ throughout $Y_j + M_j B_{2\ell_j}$.

Proof. Covector transport gives $(Dv_i)^T z + (v_i\cdot\nabla)z$ inside M_j^T. Since $z = \nabla(p_i \cdot A_i)$, this equals $\nabla(v_i \cdot z)$. Determinant one gives $v_i \cdot z = -\partial_\perp U_i$, proving (9.26). Its gradient has a multiple of z of size B_i^{dir} and a remainder of size E_i^{dir}; Gronwall gives (9.27). This combines, rather than drops, transport of the covector and differentiation of that covector in the velocity. The phase-offset principal term, the slow principal Hessian, and every correction label derivative remain in $B_i^{\mathrm{dir}}, E_i^{\mathrm{dir}}$. The mean-value formula on the rigid segments adds $2\ell_j K_j^2 A_{i,2}^{\mathrm{bd}}$. Finally use (9.10). Its third derivative has the partition types $U^{[3]}q_1^3 + 3U^{[2]}q_1q_2 + U^{[1]}q_3$, where $q_r = |D^r s_i| + \ell_i^{-1}|D^r a_i|$. The principal pure-phase third derivative is zero because $P''' = 0$ on the cell. All label and inverse-curvature terms remain. The pure correction term is (9.23), bounded by (9.24). This produces $\mathcal{V}_j$ on the entire containing tube. Ancestors before the parent use their own partition arrays and ordinary deformation factors. ∎

The coefficient variation is a separate calculation. If the subscripts 0 and 1 denote scalar coefficient bounds and first label bounds, on B_{ℓ_i}, the product rule gives

$$\mathfrak{m}_i(t) = 2\ell_i \left[\frac{d_1^\Gamma \chi_0 + d_0^\Gamma \chi_1}{\sigma_{i-1}} + \sigma_{i-1}(c_1 z_0 + c_0 z_1) \right], \qquad c_{*,i} = \int_{I_i} \mathfrak{m}_i(t)\, dt. \tag{9.28}$$

Here z_0, z_1 concern $\zeta_{i,1}$; the factor two is a harmless enlargement on the initial support ball B_{ℓ_i}, not a claim about all-time trajectories from $B_{2\ell_i}$. Evaluate the factors by the directional formulas before taking their norms. Put $R_i^{\mathcal{B}} = 2c_{\mathcal{B}}\sigma_{i-1}$. The number $\mathfrak{n}_i$ is the larger of the calculated first-label arrays at time orders zero and one, divided respectively by $R_i^{\mathcal{B}}$ and $R_i^{\mathcal{B}}\mathcal{L}_{i,l}[1]$; the array uses $2\ell_i D_a$, with its full differential norm. It is a separate number; it is never obtained by division by a possibly vanishing central matrix. Higher positive label derivatives multiply the same factors by the finite numbers already calculated in (9.15).

For example the absolute frame term in $c_{*,j}$ is

$$4C\sigma_{j-1}\ell_j \int_{G_j} \int_{\tau_j}^t \mathcal{V}_j(s)\, ds\, dt. \tag{9.29}$$

Its parent pure-phase term is at most

$$CA^{\text{tube}}_{j-1,1}\Pi^{\mathsf{d}}_{j-1}\sigma_{j-2}^{-3}N_{j-1}^{-(\mathsf{a}+\mathsf{h}-1-3\beta/2)},$$

using $\ell_j = N_{j-1}^{-\mathsf{a}}$, the amplitude bound, $|G_j| \le C/\sigma_{j-2}$, and $\sigma_{j-1} \le CN_{j-1}^{\beta/2}$. The transition uses its own shorter length; the hold and remaining tail use $\sigma_{j-1}|I_j \setminus W_j| \le 2(1+\epsilon) + C_T$. Metric and scalar-curvature terms use their explicit first-label factors and the corresponding truncated scalar integrals. This preserves the sufficient exponent $\mathsf{a}+\mathsf{h}-1-3\beta/2 > 0$, equal to $\mathsf{a}-5\beta/2$ when $\mathsf{h} = \bar{\beta}$. All grandparent and logarithmic factors remain in the finite tests.

We also name the Eulerian curvature rate used in the later coefficient tests. Let $G^{\text{bd}}_{2,<i}$ be the numerical upper bound obtained from the complete Hessian expression G_2 in (8.16), with $q = i$. Use the complete principal and correction arrays (9.18), (9.22), the unscaled inverse-map arrays, and their prescribed scalar interval bounds on the young containing region $Y_i + M_i B_{2\ell_i}$ throughout I_i. All its older labels and phase cells are those required by the containing tube tests. In particular the bound includes every correction level with its own factor δ_j^n, both principal cross derivatives and the inverse-map Hessians. Define

$$g_{-,i} = (1-\varepsilon_s)\sigma_{i-1}^2, \qquad \mu_{<i} = \frac{G^{\text{bd}}_{2,<i}}{g_{-,i}}, \qquad \mu_{<1} = 0. \tag{9.30}$$

Here $g_{-,i} > 0$ is the prescribed gradient comparison size. The numerator is a finite scalar expression, not a supremum of an unknown field or a quotient by its realized gradient. The existing Hessian estimate is $|\nabla_x^2 \Gamma_{<i}| \le g_{-,i}\mu_{<i}$ on this region. The base is affine there, which gives the first-layer value zero. In rigid coordinates the Hessian bound is $K_i^2 g_{-,i}\mu_{<i}$; measured against the rigid gradient size $K_i g_{-,i}$, its first-label rate is $K_i\mu_{<i}$. Thus the factor K_i has not already been included in $\mu_{<i}$.

At insertion let $a_0^{\text{in}}, b_0^{\text{in}} > 0$ be the scalar central lower bounds and $a_1^{\text{in}}, b_1^{\text{in}}$ the calculated first-label upper bounds for $\widehat{a}_i, \widehat{b}_i$. Impose

$$\begin{gathered} 2\ell_i a_1^{\text{in}} \le a_0^{\text{in}}/4, \quad 2\ell_i b_1^{\text{in}} \le b_0^{\text{in}}/4, \qquad c_{\text{dat},i} = 4\ell_i(a_1^{\text{in}}/a_0^{\text{in}} + b_1^{\text{in}}/b_0^{\text{in}}), \\ c_{*,i} + c_{\text{dat},i} \le \frac{\log 2}{6C_R^2}, \qquad \mathfrak{n}_i \le 1. \end{gathered} \tag{9.31}$$

The mean-value formula proves positivity on $B_{2\ell_i}$. The logarithmic derivative of the square root proves the stated datum bound there. At insertion $M_i = \text{Id}$, and the central transverse numerator is of size $\sigma_{i-1}^2 \sin\theta_i^{\text{in}}$. Its curvature contribution is therefore $C\ell_i\mu_{<i}/\sin\theta_i^{\text{in}}$, with $\mu_{<i}$ as in (9.30); it is retained. Apply only the propagator and fibre-comparison conclusion of Lemma 3.1, on the initial ball B_{ℓ_i}. Its mean-value and Volterra proof uses the central weighted bound, $c_{*,i}$ and the

datum difference; these have just been supplied on that ball. This gives the weighted comparison in (9.18) and (9.22). The larger collar is used only for the insertion positivity just proved. The higher finite label derivatives follow from our own triangular recursion (9.18), and the first time derivative from $R_{i,t} = \mathcal{B}_i R_i$. Fix a base linear-phase half-width $0 < s_0 \le s_c$, and a fixed constant C_ℓ for the first-collar containment and variation tests. For layer one choose ℓ_1 after N_1, with $\ell_1 = s_0/(C_\ell N_0 \Lambda_\ell)$, where $\Lambda_\ell \ge (\log N_1)^2$ and its term linear in $\log N_1$ ensures (9.31) for the explicit rigid base. The scalar tests include $\ell_1^{-1} \le \Pi_1$ and containment of the next tube in this smaller core.

For the two centers write $C_q = Y_q$, or $C_q = Z$ and $q = m+1$, and put $x_i^q = p_i \cdot A_i(C_q)$. The exact equations are

$$\begin{aligned} \frac{d}{dt} A_i(C_q) &= J(N_i p_i U_{i,s} + D_a U_i) + H_i \sum_{i<j<q} v_j(C_q), \\ \dot{x}_i^q &= p_i \cdot J D_a U_i + \zeta_i \cdot \sum_{i<j<q} v_j(C_q). \end{aligned} \tag{9.32}$$

Using the complete cell arrays, first set the velocity bound

$$v_i^{\mathrm{bd}} = N_i z_{+,i} U_{s,i}(S_i) + H_{i,0}^{\mathrm{bd}} U_{a,i}(S_i). \tag{9.33}$$

Its two terms are the phase and label parts of $J\nabla\psi_i$. The symbol V_i^{bd} denotes the circulation profile bound. Define scalar integrals

$$\begin{aligned} \mathfrak{p}_i^q(t) &= \mathfrak{p}_i^q(t_0) + \int_{t_0}^t [U_{a,i}(S_i) + z_{+,i} \sum_{i<j<q} v_j^{\mathrm{bd}}]\, ds, \\ \mathfrak{o}_i^q(t) &= \mathfrak{o}_i^q(t_0) + \int_{t_0}^t [N_i^2 b_i^{\mathrm{pr}} \mathfrak{p}_i^q + N_i U_{s,i}^{\mathrm{corr}}(S_i) + U_{a,i}(S_i) + H_{i,0}^{\mathrm{bd}} \sum_{i<j<q} v_j^{\mathrm{bd}}]\, ds. \end{aligned} \tag{9.34}$$

Here b_i^{pr} bounds $|w_i B_i|$. The second integral uses the first one, since $U_{i,0,s} = w_i B_i N_i x_i^q$. It follows that $|x_i^q| \le \mathfrak{p}_i^q$ and $|A_i(C_q)| \le \mathfrak{o}_i^q$. For the full-flow point the data at τ_i are zero. At τ_q the data for Y_q are the previously obtained full-flow data. In particular they are not set to zero at a subsequent insertion. The integrals are accumulated from their original datum; one does not add a fresh lifetime bound at each new window. The scalar duration sums $\sum_{q>i} |W_q| \le 2C_T/\sigma_{i-1}$, and the actual younger velocity sums in (9.34), give the same constants on every later window. For the stated powers their sufficient comparison is $Q_{i+1}(\bar\beta + \mathsf{h}) - C_o \ge 2 - \beta$, with the older factors retained in C_o.

The strict containment tests are

$$\mathfrak{o}_i^q + 2H_{i,0}^{\mathrm{bd}} K_q \ell_q < \rho_i^{\mathrm{core}}, \qquad N_i \mathfrak{p}_i^q + 2N_i \mathfrak{z}_{iq} \ell_q < s_c, \tag{9.35}$$

where $\mathfrak{z}_{iq}$ bounds $|M_q^T \zeta_i|$ on the rigid segments; for the parent it is given by Lemma 9.1. The base-distance inequality is imposed too. These tests are evaluated with the relaxed displacement bounds when forming a tube, and with (9.34) when recovering its center. They close by alternating containment and layer production.

9.3. **Complete center estimates and scalar recovery.** All center quantities are evaluated using the two exact identities

$$\begin{aligned} \nabla\gamma_i &= N_i \zeta_i V_{i,s} + H_i^T D_a V_i, \\ Dv_i &= J\big[N_i^2 U_{i,ss} \zeta_i \otimes \zeta_i + N_i(\zeta_i \otimes H_i^T D_a U_{i,s} + H_i^T D_a U_{i,s} \otimes \zeta_i) \\ &\qquad + H_i^T D_a^2 U_i H_i + \sum_k (N_i p_{i,k} U_{i,s} + U_{i,a_k}) D_x^2 A_{i,k}\big]. \end{aligned} \tag{9.36}$$

The principal local shear coefficient is

$$w_i\mathfrak{q}_i(a,t)\Omega_i(a,t), \qquad \mathfrak{q}_i = \frac{|\zeta_i|^2}{\kappa_i} = \widehat{\kappa}_i^{-1}. \tag{9.37}$$

The symbol $\mathfrak{q}_i$ is distinct from the pulled-back circulation q_i in (9.15). The last two curvature terms are retained at both centers. First rates are calculated by (9.16), using the appropriate center velocity. Common rotation cancels when the components are measured in the local rotating frame. Differences between two points are calculated by integrating the full spatial differential on their connecting segment.

Here is an explicit specification of the center numbers used in the scalar tests. For $|s| \le S_i$, the even and odd sums of the calculated arrays give $U^{\mathrm{corr}}_{ss,i}, U^{\mathrm{corr}}_{sa,i}, U^{\mathrm{corr}}_{aa,i}, V^{\mathrm{corr}}_{s,i}, V^{\mathrm{corr}}_{a,i}$, and their first rates. Add the displaced principal differences from (9.18), with $|a| \le \mathfrak{o}^q_i$. The non-phase strain and gradient bounds are

$$\begin{aligned} D^{\mathrm{mis,bd}}_i &= 2N_i z_{+,i}H^{\mathrm{bd}}_{i,0}U_{sa,i} + (H^{\mathrm{bd}}_{i,0})^2 U_{aa,i} + A^{\mathrm{bd}}_{i,2}(N_iU_{s,i} + U_{a,i}),\\ G^{\mathrm{mis,bd}}_i &= H^{\mathrm{bd}}_{i,0}V_{a,i}, \qquad X^{\mathrm{bd}}_i = N_i^2 z_{+,i}^2 U^{\mathrm{corr}}_{ss,i}, \qquad G^{\mathrm{al,bd}}_i = N_i z_{+,i}V^{\mathrm{corr}}_{s,i}. \end{aligned} \tag{9.38}$$

The center value numbers needed for coupling and displacement are

$$|\gamma_i(C_q)| \le V^{\mathrm{bd}}_i(S_i), \qquad |v_i(C_q)| \le N_i z_{+,i}U_{s,i}(S_i) + H^{\mathrm{bd}}_{i,0}U_{a,i}(S_i).$$

The second right-hand side is v^{bd}_i in (9.33). Principal slow derivatives are included in $U_{sa,i}, U_{aa,i}, V_{a,i}$. For strain, the displaced principal value is the full product $w_i\mathfrak{q}_i\Omega_i$. Thus its additional scalar bound is

$$w_i\mathfrak{o}^q_i\left[\mathfrak{q}^{\mathrm{bd}}_{i,0}\Omega^{\mathrm{bd}}_{i,1} + \mathfrak{q}^{\mathrm{bd}}_{i,1}\Omega^{\mathrm{bd}}_{i,0}\right]. \tag{9.39}$$

Add this to X^{bd}_i; the coefficient derivatives have unscaled units and are calculated by (9.7). At a specified center C_q, suppressing its index and writing $a = A_i(C_q,t)$, $s = N_ip_i\cdot a$, let the signed aligned correction be, with $U^{\mathrm{corr}}_{i,ss} = \sum_{n=1}^{J_i}\partial_s^2 U_{i,n}$,

$$\mathcal{X}_i = N_i^2|\zeta_i(a,t)|^2U^{\mathrm{corr}}_{i,ss}(s,a,t) + w_i\big[\mathfrak{q}_i(a,t)\Omega_i(a,t) - \mathfrak{q}_i(0,t)\Omega_i(0,t)\big].$$

Its calculated bound is $|\mathcal{X}_i| \le X^{\mathrm{bd}}_i$, including (9.39); the relaxed comparison uses $2X^{\mathrm{bd}}_i$. The scalar $\mathcal{X}_i$ is distinct from the nonlinear flow X_i. In the local direction $e_i = \zeta_i(a,t)/|\zeta_i(a,t)|$, the complete phase strain is $(w_i\mathfrak{q}_i(0)\Omega_i(0) + \mathcal{X}_i)Je_i\otimes e_i$. Thus the remaining principal aligned coefficient at label zero is $w_i\mathfrak{q}_i(0)\Omega_i(0)$, not $w_i\Omega_i(0)$. For circulation add the first-label amplitude bound times $\mathfrak{o}^q_i$, as before. When a direction is moved, add $2|D_a\zeta_i|^{\mathrm{bd}}\mathfrak{o}^q_i/z_{-,i}$. These rules completely specify the additional center-difference and local-direction numbers. Their first rates follow by one product rule and (9.16). In particular the complete derivative of $\mathcal{G}^{\mathrm{corr}}$ includes the phase drift of the principal amplitude, not just the derivative of a correction profile.

Use continuous angles measured from the positive second coordinate:

$$e^0_i = \frac{\zeta_i(0,t)}{|\zeta_i(0,t)|} = (\sin\phi_i, \cos\phi_i), \qquad \theta_k = \phi_k - \phi_{k-1} \quad (k \ge 2).$$

The value θ^{in}_k is its prescribed insertion opening; the first opening θ^{in}_1 is specified separately. At a displaced label, write $e_i = (\sin\phi^{\mathrm{loc}}_i, \cos\phi^{\mathrm{loc}}_i)$ for the corresponding local direction. In Lemma 8.3, $\theta^c = \theta_m$, whereas $\theta^l = \phi_m - \phi^{\mathrm{loc}}_{m-1}$ uses the parent at $A_{m-1}(Y_m,t)$. The two-point terms retain this distinction.

For an angular scalar use its angular factor before summing. A fixed trace-free matrix $D = \left(\begin{smallmatrix} a & b\\ c & -a\end{smallmatrix}\right)$ contributes $T(\phi) = w + u\sin 2\phi + v\cos 2\phi$ to the angular equation, with $w = (b-c)/2$, $u = -a$, $v = -(b+c)/2$. Hence

$$T(\phi+\theta) - T(\phi) = 2\sin\theta[u\cos(2\phi+\theta) - v\sin(2\phi+\theta)]. \tag{9.40}$$

The unmatched parent term and the two-point terms are added separately. Let $d_i^{\mathrm{bd}}, d_i^{0,\mathrm{bd}}, t_k^{\mathrm{bd}}$ denote, respectively, the coefficients obtained by applying this formula to (9.38), its unmatched parent part, and its two-point part. For precision, write the non-phase Hessian in (9.36) as S_i^{mis}, so that its velocity matrix is JS_i^{mis}. Its unmatched value in its own local direction $e_i = \zeta_i/|\zeta_i|$ is

$$T_i(\phi_i^{\mathrm{loc}}) = -(Je_i)^T S_i^{\mathrm{mis}}(Je_i).$$

The two mixed tensors containing ζ_i therefore vanish in this expression. The scalar value $d_i^{0,\mathrm{bd}}$ uses only the slow Hessian and inverse-curvature tensors in this contraction, together with the already specified difference of local and central directions. The value d_i^{bd} bounds $|u_i|+|v_i|$. For the two-point term, subtract the exact matrices and covectors first and integrate their contracted differential on the connecting segment. This defines t_k^{bd}, retaining both the direction difference and transport by the parent. Thus the angular perturbation is bounded by $2d_{k-1}^{\mathrm{bd}}\sin\theta_k + d_{k-1}^{0,\mathrm{bd}} + t_k^{\mathrm{bd}} + 2\theta_k\sum_{i<k-1} D_i^{\mathrm{mis},\mathrm{bd}}$. This definition is an explicit linear contraction of the displayed matrix arrays and segment integrals; it requires no new field estimate.

Likewise, with $e_k = \zeta_k(0)/|\zeta_k(0)|$, the transverse coefficient is $\mathcal{G}_k = -(Je_k)\cdot\nabla\Gamma_{<k}(Y_k)$. Subtract the central principal sum to define $\mathcal{G}_k^{\mathrm{corr}}$. Its scalar bound is

$$G_k^{\mathrm{corr},\mathrm{bd}} = \sum_{i<k}\Big[G_i^{\mathrm{al},\mathrm{bd}}|\sin(\phi_k-\phi_i)| + G_i^{\mathrm{mis},\mathrm{bd}}\Big] + G_k^{\mathrm{point},\mathrm{bd}},$$

where $G_k^{\mathrm{point},\mathrm{bd}}$ is the sum of the principal amplitude and direction differences just specified. Differentiate the signed summands before estimating to obtain the first-rate array; in particular the derivative of each sine is multiplied by its undifferentiated aligned amplitude. A common rotation cancels from $\phi_k-\phi_i$. The base is evaluated exactly.

The center coupling and metric numbers are obtained from (9.2) and

$$\widehat{\kappa}_i = \kappa_i/|\zeta_i|^2 = 1 + (c_g^{-1}-1)\frac{\zeta_{i,1}^2}{|\zeta_i|^2}.$$

In particular, on the central metric interval,

$$\frac{1}{1+\varepsilon_s} \le \mathfrak{q}_i(0,t) \le \frac{1}{1-\varepsilon_s}, \qquad |\mathfrak{q}_i(0,t)-1| \le \frac{\varepsilon_s}{1-\varepsilon_s}, \qquad (\log\mathfrak{q}_i)' = -(\log\widehat{\kappa}_i)'. \tag{9.41}$$

At a displaced label the same reciprocal array and the calculated label difference supply its interval. No principal term proportional to $\mathfrak{q}_i(0)-1$ is put into the additive-correction bound. Their first logarithmic rates use the positive lower bounds and (9.7). Equivalently the exact central formulas are

$$\dot{c}_i = 2WF_{<i}^{\Gamma} + 2\Gamma_{<i}v_{<i}\cdot\nabla W, \quad \dot{\kappa}_i = \zeta_i^T[(v_{<i}\cdot\nabla)\mathsf{A} - Dv_{<i}\mathsf{A} - \mathsf{A}Dv_{<i}^T]\zeta_i.$$

The residual in this calculation is obtained from (9.20), including its phase means. Its center bound is an older-carrier sum and is not assigned a negative power of the newest carrier.

Lemma 9.2 (Scalar windows under the complete errors)**.** *Suppose the finite scalar schedule (11.6) and first-rate comparisons specified before (9.4) hold with the preceding strict margins. Suppose, on an existing initial segment, each complete error is bounded by twice its prescribed positive scalar bound. Impose the following scalar inequalities at that factor two. In the generic case $m\ge 2$, the k-indexed angular and transverse tests have $2\le k\le m$; the unqualified i-indexed coupling and metric tests have $1\le i\le m$, and sums over correction layers begin at $i=1$. The parent test in the first line has $m\ge 2$. At $m=1$, use the separate first-growth and first-return comparison: there are* *no older correction errors, and no σ_{-1} is read.*

$$
\begin{gathered}
2X_i^{\mathrm{bd}} \le \tfrac14 c_W\sigma_i \ (i\le m-2), \qquad 4K_\theta^X C_S X_{m-1}^{\mathrm{bd}}\Lambda_m/\sigma_{m-1}\le\varepsilon_s,\\
2\max(8,C_T)\sum_{i<m}(d_i^{\mathrm{bd}}+d_i^{0,\mathrm{bd}})\le\varepsilon_s,\\
16C_S(d_{k-1}^{0,\mathrm{bd}}+t_k^{\mathrm{bd}})\le c_\theta\varepsilon_s\theta_k^{\mathrm{in}}\sigma_{k-2},\\
2G_k^{\mathrm{corr,bd}}\le\varepsilon_s\sigma_{k-1}^2\theta_k^{\mathrm{in}},\quad 2\dot G_m^{\mathrm{corr,bd}}\le\varepsilon_s\mathfrak{s}_m\sigma_{m-1}^2\theta_m^{\mathrm{in}},\\
2(c_i-1)^{\mathrm{bd}}+2(\widehat\kappa_i-1)^{\mathrm{bd}}\le\varepsilon_s,\quad 2(\log c_m)'^{\mathrm{bd}}+2(\log\widehat\kappa_m)'^{\mathrm{bd}}\le\varepsilon_s\gamma_m^{\mathrm{gr}}/\Lambda_m.
\end{gathered}
\tag{9.42}
$$

Here $K_\theta^X=\max(6C_\vartheta,4C_\vartheta^2)$, $\gamma_m^{\mathrm{gr}}=\sigma_{m-1}\sin\theta_m^{\mathrm{in}}$, Λ_m is the prescribed logarithmic growth length, and $\mathfrak{s}_m$ is the numerical old-gradient first-rate sum defined immediately before (11.11). That test is $\mathfrak{s}_m\le C_s\gamma_m^{\mathrm{gr}}/\Lambda_m$. Its definition retains the first-parent term $C_{\mathrm{in}}\gamma_1^{\mathrm{gr}}\mathbf{1}_{m=2}$, and uses $\mathfrak{s}_1=0$. Aligned off-label terms are included in $X_i^{\mathrm{bd}},G_k^{\mathrm{corr,bd}}$. Then the prescribed amplitude, covector-length and positive-component intervals strictly improve on their respective growth, transition and return pieces. The adjacent angles satisfy $c_\theta\theta_k^{\mathrm{in}}\le\theta_k\le C_\vartheta\theta_k^{\mathrm{in}}$ for $2\le k\le m$. The return intervals are those of Proposition 7.7, uniformly in the current trial, and the duration bounds use C_S,C_T of (11.7), with the first duration retained separately in (11.6). The resulting four-part histories are (9.46), with C_R chosen by (9.48) from these scalar intervals.

Proof. We check the changes to the scalar proof of [1, Theorem 2.2]. The exact equations for the covectors and central pair are

$$\dot\zeta_k=\alpha J\zeta_k-E_k^T\zeta_k,\quad \dot T_k=\widehat a_k\widehat\Omega_k,\quad \dot{\widehat\Omega}_k=\sigma_{k-1}c_k\zeta_{k,1}T_k.$$

Insert (9.36). The aligned old coefficient, after the displaced-label differences have been included in $\mathcal{X}_i$, is $\mathfrak{q}_i(0,t)\Omega_i(0,t)+\mathcal{X}_i$; the old activations are one. On the relaxed old interval $(c_W/2)\sigma_i\le\Omega_i\le 2C_W\sigma_i$, (9.41) and the additive bound give

$$
\begin{aligned}
\mathfrak{q}_i\Omega_i+\mathcal{X}_i&\ge c_W\left[\frac{1}{2(1+\varepsilon_s)}-\frac14\right]\sigma_i\ge \underline{c}_W\sigma_i>0,\\
|\mathfrak{q}_i\Omega_i+\mathcal{X}_i|&\le\left[\frac{2C_W}{1-\varepsilon_s}+\frac{c_W}{4}\right]\sigma_i\le 2\overline{C}_W\sigma_i\qquad(i\le m-2).
\end{aligned}
\tag{9.43}
$$

Every nonnegative older shear sum in the angle and length equations therefore has its same sign and its same trigonometric factors. Its scalar upper estimates use $2\overline{C}_W$, and any reciprocal positive-shear estimate uses $\underline{c}_W$. These substitutions into the finite products and sums defining the scalar constants specify their enlargement before the tests are made. For example the strain bound uses $C_\psi+2\overline{C}_W\sum_{i<k}\sigma_i+\sum_{i<k}D_i^{\mathrm{mis,bd}}$, giving the first-rate bound chosen above, using the prescribed $\sum_{j<i}\sigma_j\le 2\sigma_{i-1}$. When the same shear is evaluated at two points, its product $\mathfrak{q}_i\Omega_i$ is differentiated before estimation; (9.39) and the stated direction difference give the two-point error. The remaining angular terms are exactly (9.40) and its stated unmatched and two-point parts. Their integrated changes are at most ε_s in modulus and at most $\varepsilon_s\theta_k^{\mathrm{in}}$ in angle, by the second and third lines of (9.42) and the prescribed duration sums. Thus the old modulus lower bound may decrease by ε_s, and the old angle upper bound may increase by $C\varepsilon_s\theta_k^{\mathrm{in}}$. Exact monotonicity is not used in the remaining two-sided comparisons.

On growth the transverse gradient is the parent positive amplitude times the adjacent sine, plus the old gradient and the calculated correction. The fourth line of (9.42), against its lower bound $c_\theta\sigma_{m-1}^2\theta_m^{\mathrm{in}}/5$, preserves positivity. Differentiating this expression gives the four terms of the growth-ratio proof in [1, Theorem 2.2]: parent amplitude, adjacent angle, modulus, and older gradient. The changes are the first correction-gradient rate, the angular error above, and the two logarithmic rates in the last line of (9.42). Each is bounded at the required scale $C\varepsilon_s\gamma_m^{\mathrm{gr}}/\Lambda_m$. Hence, with $u=\widehat{\Omega}_m/T_m$ and $u_*=(\widehat{b}_m/\widehat{a}_m)^{1/2}$,

$$(u/u_*)'=\sqrt{\widehat{a}_m\widehat{b}_m}[1-(u/u_*)^2]-(u/u_*)(\log u_*)'.$$

The vector field points inward at $1\pm C\varepsilon_s$. The bounded quotient excludes a zero of T_m, and gives the strictly positive logarithmic growth rate $(1+O(\varepsilon_s))\gamma_m^{\mathrm{gr}}$. It proves the prescribed hit and the growth duration without assuming either endpoint exists.

During the child's growth the parent's principal vorticity amplitude at its own center is zero. Thus $\mathfrak{q}_{m-1}(0)\Omega_{m-1}(0)=0$ exactly on that hold; its displaced value is already in (9.39). The additional parent rate is $K_\theta^X\theta_m^{\mathrm{in}}|\mathcal{X}_{m-1}|$, rather than $|\mathcal{X}_{m-1}|$. This follows directly from the factors $\sin^2\theta_m$ and $\sin\theta_m\cos\theta_m$ in the angle and modulus equations. Its logarithmic eigenline contribution is at most $6C_\vartheta\theta_m^{\mathrm{in}}|\mathcal{X}_{m-1}|$. The first line of (9.42) puts this contribution in the same growth-rate tolerance.

On the transition, the exact blend identity is unchanged because $\dot{\zeta}_{k,1}=(\mathcal{F}_k-\alpha)\zeta_{k,2}$. The principal shear terms keep their positive factors $\mathfrak{q}_i$. Their upper and lower comparison constants are those in (9.43); their relative deviations from the unit-metric coefficients are at most $\varepsilon_s/(1-\varepsilon_s)\le 2\varepsilon_s$. The additional inhomogeneity is the angular difference above, together with the $\mathcal{X}_i$ parts of the old shear sums. Its contribution to $\sup|\phi_{m-1}|/\theta_m^{\mathrm{in}}$ is $C\varepsilon_s/\Lambda_m$. The term containing that same supremum has coefficient at most $16C_\vartheta\varepsilon_s\le 1/2$, so it is absorbed. More explicitly, the extra contribution to the inhomogeneous constant is bounded by

$$K_F^{\mathrm{corr}}=\frac{2C_\vartheta+1}{C_T}+\frac{c_\theta}{32C_S},\qquad \sup\frac{|\phi_{m-1}|}{\theta_m^{\mathrm{in}}}\le 8(K_F+K_F^{\mathrm{corr}})\varepsilon_s/\Lambda_m.$$

Here K_F is the scalar transition constant with the positive shear substitutions (9.43) and the metric coefficient tolerance $2\varepsilon_s$. The first term comes from the angular-factor and tangent terms; the second comes from the unmatched and two-point terms. This yields the transition and entry-ratio bounds with enlarged fixed constants. No absolute value of an aligned parent shear is substituted for its factored angular contribution.

For the return, use the exact coefficients of Lemma 8.3, with their magnified metric. The local and own-parent directions and amplitudes there are distinct. The first and fifth coefficients in (8.19) retain, respectively, $\mathfrak{r}_p^l=\mathfrak{q}_p(A_p(Y_m),t)$ and $\mathfrak{r}_m=\mathfrak{q}_m(0,t)$. They are positive multiplicative coefficients, tested by (9.41) and its label differences. In the first error the removed parent term is $\mathfrak{q}_p(A_p(Y_m),t)\Omega_p(0,t)Je_p^l\otimes e_p^l$. Thus the remaining aligned difference contains $\mathfrak{q}_p(A_p(Y_m),t)[\Omega_p(A_p(Y_m),t)-\Omega_p(0,t)]$, not a principal multiple of $\mathfrak{q}_p-1$. The first and fourth errors obey, directly from (9.36),

$$|d_1|\le\frac{z_{+,m}}{\gamma_m^{\mathrm{gr}}}\Big(\sum_{i<m}\mathfrak{S}_i\Theta_i^D+\sum_{i<m}D_i^{\mathrm{mis,bd}}\Big),$$

$$|d_4|\le\frac{\mathfrak{z}_e z_{+,m}}{\sigma_{m-1}\gamma_m^{\mathrm{gr}}\kappa_{-,m}}\Big(A_0\Theta_b^G+\sum_{i<m}\mathfrak{A}_i\Theta_i^G+\sum_{i<m}G_i^{\mathrm{mis,bd}}\Big).$$

Here $\mathfrak{S}_i,\mathfrak{A}_i$ are the complete aligned arrays with the parent principal term removed, and the Θ's are the sine or sine-cosine factors of the recovered angular intervals. These are precisely the scalar sums in (8.21), now calculated from the complete arrays. Impose these two sums $\le\delta$, and impose the scalar factor tests following that equation, $8\varepsilon\le\delta$. They yield all six coefficient bounds $|c_r-1|\le\delta$ and four error bounds $|d_r|\le\delta$. In particular $c_4(t)=c_m(0,t)\zeta_{m,1}(t^1)/\sin\theta_m^{\mathrm{in}}$ and $c_6(t)=c_m(0,t)$ retain their time and trial dependence. Proposition 7.7 therefore improves $1/2\le\mathfrak{z}/\mathfrak{z}_e\le 2$ and $-1\le\mathcal{W}\le 2+c_p$ to

$$1-C\delta\le\mathfrak{z}/\mathfrak{z}_e\le\tfrac{3}{2}+C\delta,\qquad -C\delta\le\mathcal{W}\le 1+c_p+C\delta.$$

The same fixed margins give all the old scalar recoveries, by integrating from their original anchors. This is the cumulative, rather than window-by-window additive, estimate in the cited scalar proof.

The central deformation generated by the trace-free matrix has determinant one. Applying [1, Lemma 2.11] with the recovered covector interval gives

$$\|M_i^{\pm 1}(t)\| \le z_+ \left(1 + 2z_-^{-2} \int_{\tau_i}^t \|(D_i)_{\rm sym}\|\right) + z_-^{-1}.$$

For a unit vector e, $\|(Je \otimes e)_{\rm sym}\| = 1/2$. Consequently on any elapsed interval the complete strain integral is bounded by

$$\int \|(D_i)_{\rm sym}\|\, dt \le \frac{1}{2(1-\varepsilon_s)} \sum_{j<i} \int |w_j \Omega_j(0,t)|\, dt + \frac{1}{2} \sum_{j<i} \int |\mathcal{X}_j|\, dt + \sum_{j<i} \int D_j^{\rm mis,bd}\, dt. \tag{9.44}$$

The displaced values are included in $\mathcal{X}_j$, and all integrals have the same elapsed endpoints. The first term is the existing principal history multiplied by at most $(1-\varepsilon_s)^{-1}$; it is not assigned a correction's decay size. The additive terms are bounded by (9.42). Thus $K_i \le C_K/\theta_i^{\rm in}$, with the scalar margin in C_K. There is no exponential in the inverse angle.

Finally split each central history, in order, into growth G_i, the middle interval $I_i^{\rm mid} = I_i^{\rm tr} \cup I_i^+ \cup I_i^-$ (transition and both return pulses), holding $I_i^{\rm hold}$, and the entire remaining tail $I_i^{\rm tail}$. Every interval is intersected with the elapsed history under consideration. On growth define

$$\mathfrak{r}_i^{\rm eig} = \sqrt{\widehat{a}_i/\widehat{b}_i}, \qquad g_i^{\rm eig} = \sqrt{\widehat{a}_i \widehat{b}_i}, \qquad \left|\frac{(\mathfrak{r}_i^{\rm eig})'}{2\mathfrak{r}_i^{\rm eig}}\right| \le e_i^{\rm bd} := \tfrac{1}{4}(L_{a,i} + L_{b,i}). \tag{9.45}$$

Here $L_{a,i}, L_{b,i}$ are the logarithmic derivative bounds just obtained from the scalar growth equations, including the $\log \mathfrak{q}_i$ and $\log c_i$ rates. Positive scalar intervals $a_{-,i} \le \widehat{a}_i \le a_{+,i}$, $b_{-,i} \le \widehat{b}_i \le b_{+,i}$ give the prescribed bound

$$K_i^{\rm eig} \ge \max\{\sqrt{a_{+,i}/b_{-,i}}, \sqrt{b_{+,i}/a_{-,i}}\} \quad \text{throughout growth.}$$

Choose $0 < \rho_i^{\rm eig} < 1$ within the fixed scalar margins so that

$$2\sqrt{a_{-,i} b_{-,i}}\, \rho_i^{\rm eig} > e_i^{\rm bd}[1 + (\rho_i^{\rm eig})^2].$$

The growth drift bound $e_i^{\rm bd} \le C\varepsilon_s \gamma_i^{\rm gr}/\Lambda_i$ and the positive growth interval verify this choice. The datum is the growing eigenline, so it has zero decaying component in the companion lemma's eigenbasis. The names $K_i^{\rm eig}, \rho_i^{\rm eig}$ are comparison constants, distinct from physical radii and derivative orders.

On each piece let $A_i^{\rm hist}(t), B_i^{\rm hist}(t)$ be the positive scalar bounds for $|\widehat{a}_i|, |\widehat{b}_i|$ obtained by substituting the recovered intervals into

$$\widehat{a}_i = \frac{\mathfrak{q}_i \mathcal{G}_i}{\sigma_{i-1}|\zeta_i|}, \qquad \widehat{b}_i = \sigma_{i-1} c_i \zeta_{i,1}.$$

The complete transverse errors are included in $\mathcal{G}_i$. Define the following scalar history numbers using these bounds:

$$\begin{aligned} E_i^{\rm hist} &= \int_{G_i} e_i^{\rm bd}(t)\, dt, & S_i^{\rm hist} &= \int_{I_i^{\rm mid}} [A_i^{\rm hist}(t) + B_i^{\rm hist}(t)]\, dt, \\ H_i^{\rm hist} &= \int_{I_i^{\rm hold}} A_i^{\rm hist}(t)\, dt, & D_i^{\rm hist} &= \int_{I_i^{\rm tail}} [A_i^{\rm hist}(t) + B_i^{\rm hist}(t)]\, dt. \end{aligned} \tag{9.46}$$

These integrals are finite expressions in scalar intervals, durations and profiles, not additional norms of the constructed fields. The operator norm of the off-diagonal pair matrix is bounded by the sum of its two absolute entries, so $S_i^{\rm hist}$ bounds the middle-interval norm integral. The transition and the first return pulse each contribute their recovered coefficient-size times duration; the unit-metric generic comparison gives one in each row on each of those pieces. Their Euler upper-row bounds retain $(1-\varepsilon_s)^{-1}$, and their lower-row bounds retain $1+\varepsilon_s$, together with the calculated centre errors.

In particular the negative pulse is not discarded. With $y = \Lambda_{\mathrm{ret,i}}\gamma_i^{\mathrm{gr}}(t - t_i^+)$ and $\mathfrak{z}_{e,i} = |\zeta_i(0, t_i^1)|$, its exact prescribed covector on the complete negative pulse gives

$$\begin{aligned} &\int_{I_i^-} |\widehat{b}_i|\, dt = \mu \mathfrak{z}_{e,i} \int_0^1 c_i(y)\eta_2(y)\, dy \le (1 + \varepsilon_s) c_p \mathfrak{z}_{e,i}, \\ &\int_{I_i^-} |\widehat{a}_i|\, dt \le \frac{\sup_{I_i^-} A_i^{\mathrm{hist}}}{\Lambda_{\mathrm{ret,i}}\gamma_i^{\mathrm{gr}}}. \end{aligned} \tag{9.47}$$

Here t_i^+ is the end of the first return pulse; a partial pulse uses the corresponding partial integrals. The supremum of the explicit scalar interval bound in the second line is calculated from the same return boxes. At the first layer the fixed compression and its exact first-return bounds are used. Thus both pulse costs enter S_i^{hist}.

On a selected hold $\widehat{b}_i = \widehat{\Omega}_i(0) = 0$ exactly. Its first-layer upper-row integral is retained as a fixed positive term in H_1^{hist}, not required to tend to zero. The tail includes every later transition and pulse. Hence [1, Lemma 2.13] gives the central weighted estimate with the scalar choice

$$C_R \ge (K_i^{\mathrm{eig}})^2 \sqrt{1 + (\rho_i^{\mathrm{eig}})^2}(1 + H_i^{\mathrm{hist}}) \exp\big((1 + \rho_i^{\mathrm{eig}})E_i^{\mathrm{hist}} + 2S_i^{\mathrm{hist}} + 2D_i^{\mathrm{hist}}\big) \tag{9.48}$$

for each history used in the finite recursion. The scalar whole-history upper bounds fix this choice before the field estimates. Truncating the nonnegative integrals improves it. In a current trial, intervals not yet reached are omitted. Holds of older layers belong to their already selected histories; the current layer's hold is used only after selecting its return. No derivative of the return selector is taken. □

9.4. **Strict continuation and the next insertion.** For each center or displacement error, let $\beta_\nu(t)$ be its numerical upper expression from (9.38) or (9.34), including the specified center differences and inherited datum. These numbers are formed on the fixed relaxed phase and label regions. Denote the corresponding actual error magnitude by $R_\nu(t)$.

Proposition 9.3 (Low-order continuation under scalar conditions)**.** *Assume the scalar schedule inequalities in Lemma 9.2, the finite scalar tests* (9.24), (9.25), (9.31), (9.35), (9.42)*, the history choice* (9.48)*, and the two generic-return sums and factor tests in its proof. Impose the low map and directional smallness tests at their displayed orders. These tests are made with the finite scalar recursion above, independently of the realized trial, and with the scalar full-history bounds in place of later truncations. Then the complete finite family continues through the growth, transition and return of layer m, uniformly for $0 \le \mu \le c_p$. The older layers during these trials and all layers on the selected elapsed history satisfy the complete order-zero and order-one estimates with the calculated intrinsic constants. Each relaxed center and displacement bound $2\beta_\nu(t)$ improves to $\beta_\nu(t)$. The return zero set is nonempty and compact, and its least member selects the next finite history. Subject to the explicit cone and ramp tests below and the next-index transverse and datum scalar tests, that history also supplies the complete-gradient and positive insertion data for the next layer.*

Proof. Start with the common finite family of Proposition 7.2. The carried estimates at τ_m and the already proved full-flow offsets give the new initial center data. For $m = 1$ the central construction is exactly that of [1, Lemma 2.7, Lemma 2.8]: the base is affine and the central metric and coupling equal one. For $m = 2$, its growth comparison includes the already calculated first-layer errors; transition and return are generic, with no earlier-layer strain sums. Fix scalar phase and label comparison radii first, and form all arrays on the corresponding relaxed cells. They depend on the scalar parameters and these fixed radii, not on a resulting field or on a trial. Use the numerical bounds β_ν just defined, including their prescribed data. Only the centers Y_q, $q \le m$, and their older-layer offsets are included in this current comparison. The full-flow offsets up to τ_m are carried data, and their continuation is proved after the current layer is estimated below. All coefficients in these expressions are fixed before the trial is run. The conditions on these expressions are exactly the tests in the statement. In particular the integral expressions are tested to lie within the chosen phase and label radii. They are not used to redefine those radii.

On any initial segment satisfying the relaxed bounds, first apply Lemma 9.2. It uses those bounds at factor two and does not use the field conclusions of the present step. This returns the strict scalar intervals, bounded α, central deformation, and the truncated weighted histories. Next proceed in increasing layer order $i < m$. Form the rigid containing tube using the relaxed old offsets; apply Lemma 9.1, (9.12) and (9.15); apply (9.31); and then calculate the principal pair and all J_i levels by (9.18) and (9.22). After layer $m-1$ this gives the containing tube and map of layer m, which depend only on older fields. The four comparisons (9.25) and the complete tail tests give each declared output size. In particular the complete tube sum (9.23) is bounded by $2A_{i,1}^{\text{tube}} N_i^2 Y_{*,i}\delta_i/\sigma_{i-1}$; the time-one version uses the same test for each retained rate entry. There is no use of only two levels in this step.

Substitution in (9.36) proves the complete center bounds $R_\nu \le \beta_\nu$. The displaced odd aligned contribution uses S_i/δ_i; the displaced principal contribution uses the calculated label derivative and the actual offset. Applying (9.32) first to the phase and then to the full label proves the two integrals (9.34), again with coefficient one. The inherited datum was already part of each integral. This proves $R_\nu \le \beta_\nu < 2\beta_\nu$ for every center and displacement quantity. At a zero initial integral the undivided integral inequality applies up to and from that endpoint; no division by zero or positive lower bound for a seed is used.

The recovered scalar intervals keep every feedback denominator away from zero. The strict containing-tube inequalities give a common compact subset of $\mathcal{P}$; the prescribed data extend smoothly beyond each finite horizon. Proposition 7.3 excludes a first terminal time there. Continuity excludes a first exit through any relaxed bound. Thus growth, transition and each trial return reach their respective common horizons with the improved estimates.

Only now evaluate the endpoints. On the negative pulse,

$$\mathsf{V}(y) = \upsilon(1) - \mu \int_0^y c_6 \eta_2 - \Lambda_{\text{ret}}^{-1} \int_0^y \mathfrak{k}\mathsf{V}^2.$$

The zero trial is positive by reciprocal comparison. For $\mu = c_p$, $\delta \le (c_p - \bar{\upsilon})/(2c_p)$ gives $\mathsf{V}(1) \le -(c_p - \bar{\upsilon})/2 < 0$. Continuous dependence of the common trial family yields a nonempty compact zero set and hence its least member. The exact covector law then gives $\zeta_{m,1} = \widehat{\Omega}_m(0) = 0$ on the hold.

After selecting this member, calculate the complete youngest field on its elapsed window by the same recursion. The full-flow version of (9.34) now includes that field. Its outputs at the endpoint are the actual inherited data for Y_{m+1}. The complete center estimates at that point follow from (9.36). This closes the induction on the finite window. All constants for a fixed layer and consumed order were calculated using its own full scalar history bounds; none is recomputed from the number or identity of the later observing window. □

We finish with the complete-gradient assertion used at the next insertion. Let

$$\mathscr{A}_i = -N_i T_i(0,t)|\zeta_i(0,t)| > 0.$$

The scalar growth-ratio, return and subsequent amplitude intervals give $|R_i(0,t)| \le C_{T,i}(-T_i(0,t))$. This is a component-ratio bound: on growth it follows from the invariant interval for u/u_*, on return from the bounded quotient, on a hold from its zero second component, and subsequently from the positive amplitude and old vorticity intervals. It is independent of the seed.

Let $P_{i,s}$, $P_{i,a}$ be the complete correction arrays, including their parity factors at the recovered phase, divided by $a_i(t)$ and $a_i(t)\ell_i^{-1}$, respectively. Let $\lambda_{T,i}(-T_i(0,t))$ bound $|D_a T_i|$, as supplied by (9.18) and this component comparison, and let $z_{a,i}$ be the calculated first-label covector bound.

Then (9.36) gives at the full-flow point

$$\left|\nabla\gamma_i(Z) + w_i\mathscr{A}_i e_i^0\right| \le \epsilon_i^G \mathscr{A}_i, \quad e_i^0 = \frac{\zeta_i(0,t)}{|\zeta_i(0,t)|}, \tag{9.49}$$

where the entirely scalar value

$$\epsilon_i^G = \frac{(z_{a,i} + z_{+,i}\lambda_{T,i})\mathfrak{o}_i^{m+1}}{z_{-,i}} + \frac{H_{i,0}^{\mathrm{bd}} S_i \lambda_{T,i}}{N_i z_{-,i}} + \frac{C_{T,i}}{z_{-,i}}\left(z_{+,i}P_{i,s} + \frac{H_{i,0}^{\mathrm{bd}}}{N_i \ell_i} P_{i,a}\right)$$

retains the principal displacement, slow principal derivative, even phase derivatives, odd label derivatives and displaced odd phase terms. Impose $\epsilon_i^G \le \epsilon_G$ for the computed arrays. These are relative gradient tests with the instantaneous weight supplied by (9.22).

On a ramp, the central coefficient bound b_i^{ramp} is already a scalar output of Lemma 9.2. Direct integration of the central pair equation gives

$$\mathscr{A}_i(t) \le \mathscr{A}_i^{\mathrm{ramp,bd}} := N_i z_{+,i}\epsilon_i\sqrt{1 + (u_i^{\mathrm{in}})^2}\exp\left(\int_{I_i^{\mathrm{ramp}}} b_i^{\mathrm{ramp}}(t)\,dt\right).$$

Impose the scalar test

$$\sum_{i:\,w_i(t)<1} \mathscr{A}_i^{\mathrm{ramp,bd}} \le A_0 \quad \text{for the alive layers at each time.} \tag{9.50}$$

All simultaneously ramping indices are included. Summing (9.49) and using (9.50) proves

$$\left|\nabla\Gamma(Z) + A_0 e_b + \sum_i w_i \mathscr{A}_i e_i^0\right| \le \epsilon_G\left(A_0 + \sum_i w_i\mathscr{A}_i\right).$$

If the recovered angular intervals give a unit vector e with $e\cdot e_b, e\cdot e_i^0 \ge \nu > \epsilon_G$, then

$$-\, e\cdot\nabla\Gamma(Z) \ge (\nu - \epsilon_G)\left(A_0 + \sum_i w_i\mathscr{A}_i\right) > 0. \tag{9.51}$$

This is the aggregate gradient lower bound. The transverse numerator at the next insertion uses, in addition, the prescribed adjacent angle: its parent term is at least $c_\theta\sigma_m^2\theta_{m+1}^{\mathrm{in}}/5$, and the complete transverse error is at most $\varepsilon_s\sigma_m^2\theta_{m+1}^{\mathrm{in}}$ by the next-index gradient comparison in (9.42). Thus it is positive, since $8\varepsilon_s \le c_\theta$. The full-collar extension is exactly the datum test (9.31) at that index. The aggregate cone inequality alone is not used to infer a sign in a different direction. The normalized-to-unit-ring conversion is $\nabla_y\Gamma = \nabla_x\Gamma'$, followed by $\nabla_{z,r}\Gamma = \mathrm{diag}(1,r)\nabla_x\Gamma'$. Thus the physical gradient gains the factor $\min(1, r_-)$, and the meridional vorticity components are $(-r^{-1}\partial_{x_1}\Gamma', \partial_{x_2}\Gamma')$. The prescribed-ring scaling is the one in Lemma 2.2.

Throughout, the means of both rows of (9.20) remain in the residuals. Their global phase integrals and compact label-vector primitives use the same positive operations; the reciprocal-symbol mean, the slow circulation-square mean, and both terminal elliptic expressions are included. The fixed compact force recovery of Section 6 therefore applies to this unchanged complete finite family. The argument has proved the low-order implication under its scalar conditions. The higher-order force allocation remains

$$r_* = k + 4J_m + 9, \quad r_X = r_* + 2, \quad r_F = r_* + 4, \qquad B = L_m = r_F + 3, \quad L_i = L_{i+1} + 2J_i + 5 + B.$$

No higher-order growth estimate or choice of an infinite parameter sequence is asserted here.

10. Higher derivatives of the corrected finite family

Integration by parts prevents a loss of time derivatives in the correction equations. We combine that identity with the complete coefficient and level count to produce the mixed estimates used in the physical force bound. We use the finite family and scalar hypotheses of Proposition 9.3. Older-history estimates are uniform over every allowed current trial parameter. Estimates for the youngest completed history use its selected parameter; no return selector is differentiated. These quantifiers apply to every statement below. All constructions and estimates refer to the magnified unit-ring coordinates of Lemma 2.3. In particular

$$\mathsf{A} = \operatorname{diag}(c_g^{-1}, 1), \quad c_g = 1 + 2\varepsilon_{\mathrm{geom}} x_2, \quad W = \varepsilon_{\mathrm{geom}} c_g^{-2}, \quad \Gamma_0 = (2\varepsilon_{\mathrm{geom}})^{-1}.$$

The product $2W\Gamma_{<i}$, not two independent bounds for its factors, is used in the linear coefficient. The physical principal strain retains $|\zeta_i|^2/\kappa_i$. There are two material domains. At a young index m, the coefficient, fibre and complete-level estimates hold on the support ball B_{ℓ_m} at its own finite orders; its larger smooth collar $B_{2\ell_m}$ is used only for insertion data. For an older index $i < m$, the larger-range arrays are on the constant-envelope core

$$\mathcal{B}_i^\flat = B_{\rho_i^{\mathrm{core}}}, \qquad g_i \equiv 1 \quad \text{on } \mathcal{B}_i^\flat, \tag{10.1}$$

with the prescribed radius of the preceding section. For the usual envelope this is $\rho_i^{\mathrm{core}} = \ell_i/2$; keep the actual first-layer core when it is smaller. We use the time intervals of (9.8): W_l is one insertion window and $I_m = [\tau_m, t_{\mathrm{end}}]$ is the elapsed lifetime to the current finite horizon. The older estimates are on $\mathcal{B}_i^\flat \times \mathbb{T} \times I_m$. In the separate infinite-history assertion below, $I_m = [\tau_m, T_*)$. Means and primitives are formed on the entire phase circle before restriction to a physical phase cell. Pointwise vector and tensor bounds use the Euclidean norm and its induced multilinear norms. Component estimates are summed with the explicit dimension factors below. A profile bound is uniform on its indicated domain at the specified time.

The finite admitted orders are nonnegative integers satisfying

$$k_1 \le k_2 \le \cdots \le k_m, \qquad k_j \le Q_j/c_Q, \qquad J_j = 2k_j + 8. \tag{10.2}$$

The same inclusion is imposed on any later finite segment being used. The separate fixed-layer all-order assertion additionally assumes $k_l \longrightarrow \infty$ on the infinite selected history. These conditions concern admission of orders, not a uniform bound above the admitted orders. A maximal schedule realizing these inclusions is given after the fixed order function is defined.

For a young force order $0 \le k \le k_m$ at layer m, retain the allocation

$$\begin{gathered} r_* = k + 4J_m + 9, \quad r_X = r_* + 2, \quad r_F = r_* + 4, \\ B = r_F + 3, \quad L_m = r_F + 3, \quad L_i = L_{i+1} + 2J_i + 5 + B \quad (i < m), \qquad \iota_i = 3J_i + 4. \end{gathered} \tag{10.3}$$

An older level n, on (10.1), is estimated for $b \le B$, $p \le \iota_i - 2n$, $|\alpha| \le L_i - 2n - 1$. Phase and label allowances are separate. The young global force calculation uses the fixed profiles through $r_* + 1$ on the whole phase circle, not the older polynomial-cell allowance. Derivative orders on older circulation and streamfunction are respectively $r_F + 1$ and $r_F + 3$. The scale letters are those of the common scalar construction: $\bar\beta = 1 - \beta$, $\delta_i = \Pi_i^{\mathsf{d}} N_i^{-\mathsf{h}}$, $\mathsf{d} \ge 2$, $0 < \mathsf{h} \le \bar\beta$, and, for $i \ge 2$, $\ell_i = N_{i-1}^{-\mathsf{a}}$, $\Pi_i = N_{i-1}^{\mathsf{p}} (\log N_i)^{\mathsf{r}}$. Keep the separately defined first radius and first clock. We use their scalar comparisons $\mathsf{p} \ge \mathsf{a} > 1 + \beta/2$, $\Pi_i \le N_i^{\mathsf{v}}$, and the positive small-power conditions stated below; v is this prescribed upper exponent, not a velocity. In expressions involving an initial growth interval, replace an index below zero by the fixed base clock.

10.1. **Refunded equations and complementary orders.** Temporarily suppress the layer index, and set

$$Q_n = (V_n, \widetilde{W}_n), \quad U_n = (N\sigma)^{-1} \chi \mathcal{I} \widetilde{W}_n, \quad \chi = \kappa^{-1}, \quad \sigma = \sigma_{i-1}.$$

Here $\mathcal{I}$ is the literal mean-zero phase primitive and $\mathsf{P} = 1 - \langle\cdot\rangle_s$. Their sup norm constants are $\pi/2$ and 2. Both are applied on the entire phase circle and commute with label and time differentiation. Write

$$C = H\mathsf{A}(X)H^T, \quad \mathfrak{l} = \operatorname{div}_a C, \quad q = \Gamma_{<i} \circ X, \quad h = He_1, \quad \varpi = W \circ X.$$

The divergence $\mathfrak{l}$ is a vector. The exact reduced vorticity coefficient splits as

$$b = \xi_{<i} \circ X = \alpha b^{\mathrm{rot}} + b^{\mathrm{nr}}, \qquad b^{\mathrm{rot}} = (1 + c_g^{-1}) \circ X,$$

on the base region; b^{nr} contains every older correction. The scalar α is the time derivative of the base angle.

Define $Z_n = (N/\sigma)(\partial_s \widetilde{W}_n + \mathcal{Z}_n)$. Then

$$\begin{aligned}\mathcal{Z}_n = N^{-1}\big[&2(Cp)\cdot D_a(\chi\widetilde{W}_{n-1}) + (p\cdot\mathfrak{l})\chi\widetilde{W}_{n-1}\big] \\ &+ N^{-2}\big[C : D_a^2(\chi\mathcal{I}\widetilde{W}_{n-2}) + \mathfrak{l}\cdot D_a(\chi\mathcal{I}\widetilde{W}_{n-2})\big].\end{aligned} \tag{10.4}$$

Negative indices are zero. The pair levels above J vanish, but $\mathcal{Z}_{J+1}$ and $\mathcal{Z}_{J+2}$ do not.

Put $h_n^\flat = (0, -\mathcal{I}\mathsf{P}\mathcal{Z}_n)$. Let f_n^{nr} be the pair obtained from the complete source rows (9.20) by removing the outer derivative of $\mathcal{Z}_n$, replacing b by b^{nr}, and applying $-\mathsf{P}$ and $-\mathcal{I}\mathsf{P}$, respectively, after the vorticity row has been multiplied by σ/N. The removed scalar row is exactly αK_n, where

$$K_n = \begin{pmatrix} 0 \\ -\mathcal{I}\mathsf{P}\left[N^{-1}\chi\widetilde{W}_{n-1}\partial_\perp b^{\mathrm{rot}} + N^{-2}\{\chi\mathcal{I}\widetilde{W}_{n-2}, b^{\mathrm{rot}}\}_a\right]\end{pmatrix}. \tag{10.5}$$

Thus no derivative of the actual base-vorticity factor is omitted. With $\mathcal{B} = \begin{pmatrix} 0 & -d\chi/\sigma \\ \sigma c\zeta_1 & 0\end{pmatrix}$, the exact equation is

$$\begin{aligned}Q_n(t) = h_n^\flat(t) &+ \mathcal{P}(t,t_0)[Q_n(t_0) - h_n^\flat(t_0)] \\ &+ \int_{t_0}^t \mathcal{P}(t,s)[\mathcal{B}h_n^\flat + f_n^{\mathrm{nr}} + \alpha K_n](s)\,ds.\end{aligned} \tag{10.6}$$

Indeed $\partial_s\mathcal{P}(t,s) = -\mathcal{P}(t,s)\mathcal{B}(s)$; integration by parts gives the positive $\mathcal{B}h_n^\flat$ term. At insertion the bracketed datum is zero. At a later window it is inherited, and cannot be discarded. The same formula retains both quadratic circulation terms, the older-vorticity terms and every mean.

Here is the derivative calculation used for each integral. Its central weight is $a_i(t) = Y_{*,i}\rho_i(t)$; its propagator bound is $2C_R\rho_i(t)/\rho_i(s)$. For a source of size A, with the weight ρ_i removed, write its scalar derivative bounds as $F(b,r)$. Let $c_{p,0}$ be the normalized integrated positive-label coefficient bounds, and $c_{p,d}^\sharp$ the pointwise coefficient bounds with their rates removed. The relative output bounds at size $\Pi_i A$ may be computed by

$$\begin{aligned}Y(0,r) &= 2C_R\left[F(0,r) + c_{*,i}\sum_{p=1}^r \binom{r}{p} c_{p,0} Y(0,r-p)\right], \\ Y(b+1,r) &= F(b,r) + \mu_B \sum_{d=0}^b\sum_{p=0}^r \binom{b}{d}\binom{r}{p} c_{p,d}^\sharp Y(b-d,r-p).\end{aligned} \tag{10.7}$$

The harmless component factor may be included in F; μ_B includes the masses of the rate lists below. At time zero isolate the undifferentiated matrix and apply weighted Duhamel to each label derivative. The weights at the integration variable cancel. At positive order use the differentiated equation. This proves (10.7), first in label order and then in time order. The coefficient and partner orders are complementary: $(d+p) + (b-d+r-p) = b+r$. To obtain time order b of the integral requires the source only through $b-1$, whereas the direct $h_n^\flat$ requires order b. All its pair levels are lower than n. In particular the scalar row requires α only through $b-1$.

10.2. **Rate factors and scalar bounds.** Use the scalar floor of (9.9), with a distinct name:

$$r_i = C_\psi(R_0+2) + 2(2\overline{C}_W + c_{\mathcal{B}})\sigma_{i-1}, \qquad \mathscr{R}_i = \max(1, r_i),$$
$$c_{\mathrm{fl}}\sigma_{i-1} \le \mathscr{R}_i \le C_{\mathrm{fl}}(1+\sigma_{i-1}), \qquad \mathscr{R}_i \le \Pi_i. \tag{10.8}$$

Here the letters and containing radius R_0 are those already used in the explicit low-order floor; the amplitude pair remains R_i. One may take $c_{\mathrm{fl}} = 2(2\overline{C}_W + c_{\mathcal{B}})$ and $C_{\mathrm{fl}} = 1 + C_\psi(R_0+2) + c_{\mathrm{fl}}$. The last inequality uses the accepted scalar test $r_i \le \Pi_i$ and $\Pi_i \ge 1$. In particular the floor is a calculated number, not a freely chosen large rate or the norm of an unknown matrix.

A rate list is a finite positive sum $\mathbf{a} = (a_\nu @ X_\nu)$, where $X_\nu \ge \mathscr{R}_i \ge 1$. Its moment is $M_{\mathbf{a}}(b) = \sum a_\nu X_\nu^b$ for $b \ge 1$, and $M_{\mathbf{a}}(0) = 1$. Equal rates retain their family labels. If $b = \sum b_j > 0$, weighted arithmetic–geometric mean gives

$$\prod_j M_{\mathbf{a}_j}(b_j) \le \sum_{j:b_j>0} \left(\prod_{h\ne j: b_h>0} \Sigma(\mathbf{a}_h) \right) M_{\mathbf{a}_j}(b), \qquad \Sigma(\mathbf{a}) = \sum_\nu a_\nu. \tag{10.9}$$

Indeed the product of the selected rates raised to their orders is at most their sum raised individually to order b; summing the independent amplitudes proves the formula. At zero order use the product of actual sizes, not a list mass. If S_f bounds $|f|$ and $\mathbf{L}_f$ is its list of absolute derivative amplitudes, define its effective size by

$$\widehat{S}_f = \max\{S_f, \Sigma(\mathbf{L}_f)\}. \tag{10.10}$$

For a normalized list $\mathbf{a}_f$ at size S_f, the absolute list is $S_f \mathbf{a}_f$. Thus this definition keeps the independently specified value bound and the derivative-list mass, including when the value bound is smaller.

An integrated entry $a@X$ of a source of size A becomes $a/(\Pi_i X)@X$ at output size $\Pi_i A$. The equality $AX^b = (\Pi_i A)(\Pi_i X)^{-1} X^{b+1}$ proves this rule. The first-order endpoint is supplied by the floor. For the scalar row, if $|\alpha| \le S_0$ and $|\alpha^{(d)}| \le S_1 P_\alpha(d+1) M_{\mathbf{a}_\alpha}(d+1)$ for $d \ge 1$, normalize its size by $\Pi_i(S_0 + \mathscr{R}_i S_1)A_K$. The differentiated scalar factor costs

$$\frac{S_1}{\Pi_i(S_0 + \mathscr{R}_i S_1)} \le (\Pi_i \mathscr{R}_i)^{-1},$$

and the undifferentiated scalar contributes $S_0/[\Pi_i(S_0 + \mathscr{R}_i S_1)X]$. Their sum is at most $(\Pi_i \mathscr{R}_i)^{-1}$. The derivative orders in the scalar product are $(d+1) + (b-d) = b+1$, precisely the output order. None of these identities replaces a later rate by an intrinsic old-layer rate.

10.3. **Production of the coefficient arrays.** We give a scalar prescription, including its normalizations. A derivative position is either a time derivative, a phase derivative, or one of the two label derivatives. For a product distribute the labelled derivative positions among its factors. For a composition use

$$D^{\mathcal{S}} f(Z,t) = \sum_{T \subset \mathcal{S}_t} \sum_{\pi \in \mathrm{Part}(\mathcal{S}\setminus T)} (\partial_t^{|T|} D^{|\pi|} f)(Z,t)[D^{B_1} Z, \ldots, D^{B_{|\pi|}} Z]. \tag{10.11}$$

Include the empty partition. Contract componentwise bounds by summing the two spatial components, or the three components of (s,a). For a positive scalar use

$$D^{\mathcal{S}} u^{-1} = \sum_\pi (-1)^{|\pi|} |\pi|! u^{-1-|\pi|} \prod_{B\in\pi} D^B u. \tag{10.12}$$

Replace signs by plus signs and each input by its already calculated bound. These formulas, the exact identities below and (10.7) define nonnegative scalar arrays; they do not mean the supremum of an unestimated field. The zero-order inputs are the scalar windows of Proposition 9.3. Fixed profile derivatives and the explicit derivatives of the metric are the remaining leaves. Rates are reduced by (10.9), keeping their family tags.

Write $X_j = Y_j + M_j(a + \Xi_j)$, and let A_j^{inv} denote the inverse label map. To avoid confusing it with an amplitude, the superscript is retained throughout this subsection. Write $\psi = \vartheta$ for the base rotation angle of (2.7); thus $\psi(0) = 0$ and $\dot\psi = \alpha$. In the magnified coordinates centered at the ring, put

$$O = e^{\psi J}, \qquad \bar M_j = O^T M_j, \qquad \bar Y_j = O^T Y_j.$$

The base velocity is $\dot\psi J x$, so differentiation gives

$$\bar M_j' = \bar D_j \bar M_j, \qquad \bar Y_j' = O^T \sum_{i<j} v_i(Y_j),$$

where $\bar D_j = O^T \sum_{i<j} D_y v_i(Y_j) O$. In particular

$$\bar M_j^{(b)} = \sum_{d<b} \binom{b-1}{d} \bar D_j^{(d)} \bar M_j^{(b-1-d)} \quad (b \geq 1). \tag{10.13}$$

The inverse satisfies the corresponding equation with reversed factors and a minus sign. Restore O by (10.11). Its input is ψ through b, not $\dot\psi$ through b.

We first specify where old quantities in these calculations are evaluated. For $i < j$, let $a_i^c[j] = A_i^{\mathrm{inv}}(Y_j, t)$. For $|a'| \leq 2\ell_j$ and $0 \leq u \leq 1$, the accepted containing-tube tests (9.35) give

$$\begin{aligned} |A_i^{\mathrm{inv}}(Y_j + uM_j a', t)| &\leq \mathfrak{o}_i^j + 2H_{i,0}^{\mathrm{bd}} K_j \ell_j < \rho_i^{\mathrm{core}}, \\ |N_i p_i \cdot A_i^{\mathrm{inv}}(Y_j + uM_j a', t)| &\leq N_i \mathfrak{p}_i^j + 2N_i \mathfrak{z}_{ij} \ell_j < s_c. \end{aligned} \tag{10.14}$$

Indeed integrate the inverse Jacobian, respectively its phase contraction, along the rigid segment. Their unscaled bounds are the calculated $H_{i,0}^{\mathrm{bd}}$ and $\mathfrak{z}_{ij}$ from the preceding section, with the directional parent estimate for the latter. The tests are made with the relaxed offsets before producing the younger layer; a first exit from the indicated core would contradict these strict bounds. The containing region $\mathcal{P}$ and its base-distance condition are retained. Since the low-order map bound has $|a + \Xi_j(a,t)| < 2\ell_j$ for $|a| \leq \ell_j$, the same argument covers the actual younger material image. It also covers the young insertion collar at insertion, where $\Xi_j = 0$.

For differences of two center evaluations we retain the separate accepted connecting-segment clause: the whole segment joining those centers lies in $\mathcal{P}$ and pulls back into each indicated older core and phase cell. Endpoint containment alone is not used. For example, if $\mathsf{s}_q^{\mathrm{sep}}$ bounds $|C_q - C_{q-1}|$, one explicit sufficient verification of its label part is

$$\min(\mathfrak{o}_i^q, \mathfrak{o}_i^{q-1}) + H_{i,0}^{\mathrm{bd}} \mathsf{s}_q^{\mathrm{sep}} < \rho_i^{\mathrm{core}}.$$

The phase part follows by the same integral with its actual contracted inverse-Jacobian bound. This sufficient comparison is not substituted for the accepted segment clause when that clause has already been verified more precisely. A physical core image need not be convex. The cases $C_q = Y_q$ and the selected full-flow point $C_{m+1} = Z$ use their respective inherited offset data. Thus rigid remainders, old scalar compositions, and all two-point differentials below use only the older domains in (10.1).

The rigid non-affinity on $B_{2\ell_j}$ is calculated from the complete old streams:

$$\widetilde v_j(a', t) = M_j^{-1} \int_0^1 (1-u) D_y^2 \sum_{i<j} \sum_{n=0}^{J_i} J\nabla U_{i,n}^{\natural}(Y_j + uM_j a', t)[M_j a', M_j a']\, du. \tag{10.15}$$

For one label derivative use the first-order Taylor remainder; for two or more use the differentiated velocity directly. The affine rotation has disappeared before estimating. Thus all differentiated terms retain a non-affine old factor. The common-history bounds are

$$|D_{a'}^q \widetilde v_j| \leq f_{0,q} \eta_j \mathfrak{d}_j(t) \ell_j^{1-q}, \quad \int_{I_j} \mathfrak{d}_j \leq 1, \quad \mathfrak{d}_j(t) \leq C\sigma_{j-2}, \quad \eta_j f_{0,1} \leq \tfrac{1}{8}.$$

The numbers $f_{0,q}$ are obtained by expanding (10.15). The marked low-order normalization is the one in the common lower-order proposition, including its contracted parent transverse estimate; it is not replaced by independent norms of the two frames. For a positive time order the density is replaced by the floor $\mathscr{R}_j$.

The time conversion is important. At $y = Y_j + M_j a'$, put $a_i = A_i^{\mathrm{inv}}(y,t)$, $s_i = N_i p_i \cdot a_i$. Exactly,

$$\dot a_i = H_i(y)\left[\sum_{i\le h<j} v_h(y) - M_j \widetilde v_j(a')\right], \qquad \dot s_i = N_i p_i \cdot \dot a_i. \tag{10.16}$$

Use $\partial_t + \dot s_i \partial_{s_i} + \dot a_i \cdot D_{a_i}$ on an old profile. The own fast velocity is parallel to $J\zeta_i$, hence contributes zero to $\dot s_i$. This cancellation is made before further differentiation. The evaluated region has bounds

$$|a_i| \le |a_i^c[j]| + 2K_i K_j \ell_j, \qquad |s_i| \le |s_i^c[j]| + 2\zeta_+ N_i K_j \ell_j.$$

Both summands are retained; the sharper unscaled inequalities (10.14), not these coarsenings with fixed letters, certify containment. The scalar conversion estimates obtained from the complete low-order sums are

$$\begin{aligned} \ell_i^{-1}|\dot a_i| &\le C\{N_i^{-\min(\bar\beta+\mathsf{h},\,2-3\beta/2)+o(1)} + N_{j-1}^{-(\mathsf{a}-\beta)+o(1)}\},\\ |\dot s_i| &\le C\{N_i^{-\bar\beta+o(1)} + N_{i+1}^{-\min(\bar\beta+\mathsf{h},\,2-3\beta/2)+o(1)}\\ &\qquad + N_{j-1}^{-(\mathsf{a}-\beta)+o(1)} + N_{j-1}^{2+3\beta/2-\mathsf{h}-2\mathsf{a}+o(1)}\} \le \mathscr{R}_j. \end{aligned} \tag{10.17}$$

Here and below an $o(1)$ in a scale comparison denotes the explicit grandparent and logarithmic powers in that comparison, bounded by a fixed constant divided by Q_{j-1}; the corresponding finite test can always be made before this coarsening. The last term uses the pointwise bound on $\mathfrak{d}_j$. Relative to $\mathscr{R}_j \ge c N_{j-1}^{\beta/2}$ its exponent is $2+\beta-\mathsf{h}-2\mathsf{a} < 0$. It need not be at most one. Differentiating (10.16) at order $b-1$ uses old fields and the rigid remainder only through $b-1$.

For clarity, the flow arrays themselves can be generated without any high-order smallness condition. At time zero isolate the linear one-block term in the differentiated equation $\Xi_t = \widetilde v_j(a+\Xi,t)$ and use the factor $E = e^{\eta_j f_{0,1}}$. For $r \ge 2$ an upper array is

$$x_{0,r} = E \sum_{\substack{\pi\in \mathrm{Part}([r])\\ |\pi|\ge 2}} 2^{|\pi|} f_{0,|\pi|} \prod_{B\in\pi} (\mathbf{1}_{|B|=1} + \eta_j x_{0,|B|}),$$

with $x_{0,0} = f_{0,0}$, $x_{0,1} = E f_{0,1}$. At positive time order use (10.11) on a set of $b-1$ time derivative positions and r label derivative positions. This gives $|\partial_t^b D_a^r \Xi_j| \le \eta_j x_{b,r} \ell_j^{1-r} M_{\mathbf{a}_j}(b)$. The outer zero-time factor uses $\mathfrak{d}_j \le \mathscr{R}_j$, and the remaining floor supplies the last time order. Every time index on the right is smaller than b.

Area preservation gives the exact, two-dimensional inverse formula

$$H_j = (I + J(D_a\Xi_j)^T J^T) M_j^{-1}.$$

Produce $\zeta = H^T p$, $C = H\mathsf{A}(X)H^T$, $\kappa = p^T C p$, $\chi = \kappa^{-1}$, q, b^{nr}, $c = 2\varpi q$ and $d = (Jp)\cdot D_a q$ by the preceding rules. The low-order margin gives $\kappa \ge \zeta_-^2/6$, which supplies the denominator in (10.12). Piola gives

$$\mathfrak{l}_i = \sum_{r,k=1}^{2} \mathsf{A}_{rr}(X) H_{kr} \partial_{a_k} H_{ir}. \tag{10.18}$$

Thus the vector $\mathfrak{l}$ retains a curvature factor; no bare derivative of the metric remains. Its full time derivatives are nevertheless taken. Also

$$D_a b^{\mathrm{rot}} = -2\varepsilon_{\mathrm{geom}} c_g(X)^{-2} (D_a X)^T e_2.$$

The direct rotation dependence of C retains $O^T(\mathsf{A}(X)-I)O$, whereas that of h or ζ_1 has no such small multiplier. These distinctions are preserved by the tagged product rule.

The mixed domain needed for this calculation is $b \le B$, $b+r \le B+L_j+2$ for Ξ_j. An old stream factor has total order at most $B+L_j+3$. For its level $n \le J_i$, $L_i - 2n - 1 \ge L_{i+1} + B + 4$, so (10.3) pays this derivative and every intermediate conversion. A phase derivative vanishes only when the actual core polynomial degree says so. Its coefficients still include the full-circle means and primitives.

10.4. **Intrinsic normalization and the complete level depth.** For each output in the preceding calculation divide by its actual prescribed size. The resulting constants are denoted $P_{j,n}(b,p,r)$; a pair is divided by $Y_{*,j}\rho_j(t)\delta_j^n$, a stream additionally by $(N_j\sigma_{j-1})^{-1}$. No division by a realized field is involved.

We first specify the intrinsic center units. For a positive scalar expression E in the complete low-order arrays, write $\mathrm{sz}_i(E)$ for the following numerical expression. Expand its products using (9.5) and the complete profile sums in (8.4), with the calculated arrays of (9.36)–(9.38). Extract the normalized coefficients as in (9.24) and (9.25), then replace each such coefficient by one. Retain the fixed comparison letters, every carrier, clock, radius and amplitude factor, the phase-radius factors, and each finite level sum. In particular the stream units are $Y_{*,i}\delta_i^n/(N_i\sigma_{i-1})$, the circulation units are $Y_{*,i}\delta_i^n$, and each label derivative retains ℓ_i^{-1}. For inverse curvature use the expanded inverse recursion following (9.14); its non-affinity factor is retained. The scalar interval and phase-radius bounds in this prescription are the full-history bounds fixed for layer i. Thus $\mathrm{sz}_i(E)$ is independent of the later observing window.

The four intrinsic units are

$$\begin{aligned} &\mathfrak{x}_i = C_X \Pi_i^{2\mathsf{d}} N_i^{\beta-2\mathsf{h}}/\sigma_{i-1}, \qquad \mathfrak{g}_i = C_G C_\Theta N_i^{\beta}\delta_i^2, \\ &\mathfrak{g}'_i = C'_G K_i \ell_i^{-1} C_\Theta N_i^{-\bar\beta}\delta_i, \\ &\mathfrak{m}_i = \mathrm{sz}_i\Big(2N_i z_{+,i} H_{i,0}^{\mathrm{bd}} U_{sa,i} + (H_{i,0}^{\mathrm{bd}})^2 U_{aa,i} + A_{i,2}^{\mathrm{bd}}(N_i U_{s,i} + U_{a,i})\Big). \end{aligned} \tag{10.19}$$

The three terms defining $\mathfrak{m}_i$ are respectively the mixed phase-label Hessian, the label Hessian, and the inverse-curvature term of the full non-phase strain. Principal slow derivatives and all correction levels are included. This is a scalar size for the full matrix; $d_i^{0,\mathrm{bd}}$ is instead the contracted unmatched angular value, and $\mathfrak{d}_j(t)$ above is a time density. The fixed letters C_X, C_G, C'_G include the fixed geometric, profile and tail factors in the aligned and gradient rows.

For the four numerical center rows $X_i^{\mathrm{bd}}, G_i^{\mathrm{al,bd}}, D_i^{\mathrm{mis,bd}}, G_i^{\mathrm{mis,bd}}$ and their first-rate arrays in (9.38), include the principal displacement in (9.39) and its circulation counterpart. Use the local frames of that calculation, the positive derivative rules above and the first-rate families in (9.9). Divide the value bounds by their respective units $(\mathfrak{x}_i, \mathfrak{g}_i, \mathfrak{m}_i, \mathfrak{g}'_i)$. For a first-rate entry also remove its displayed rate and its ordinary or rotation-family amplitude, including that family's displayed geometric and non-affinity factors. Let $c_{i,\nu}$ be this finite collection of normalized numerical bounds, evaluated at the certifying range $i+1$, and define

$$a_{\Delta,i} = \max\big(1, \max_{\nu} c_{i,\nu}\big). \tag{10.20}$$

An absent rate entry is omitted. All inputs are the calculated arrays and fixed scalar history bounds, so $a_{\Delta,i}$ depends on neither a later observing window nor a later requested order. Its comparison with $G(i+1,k_{i+1})$ is proved in (10.29) below.

The following four scalar comparisons specify the normalization of the old inputs:

$$
\begin{array}{c|l}
\text{old input} & \text{quantity tested in its output unit} \\
\hline
\text{non-affinity} & \eta_j f_{0,r}, \quad K_j \eta_j f_{0,2}/\zeta_- \\
\text{old scalar variation} & \ell_j \max(K_j \mu_{<j}, \lambda_{\Gamma,j}, \lambda_{\mathrm{met},j}) \\
\text{center corrections} & a_{\Delta,i}(\mathfrak{x}_i, \mathfrak{g}_i, \mathfrak{m}_i, \mathfrak{g}'_i)/S_{\mathrm{out}} \\
\text{complete old levels} & \displaystyle\sum_{i,n,\nu} a^{\mathcal{R}}_{i,n,\nu} \delta_i^n S^{\mathcal{R}}_{i,n,\nu}/S_{\mathrm{out}}.
\end{array}
\tag{10.21}
$$

Here $a^{\mathcal{R}}$ is calculated by the derivative position rules including all subsequent contractions; $\mathcal{R}$ identifies the actual term in (10.15), (10.16), the coefficient formulas, or the center formulas (9.36). The positive units $S^{\mathcal{R}}$ and S_{out} are the displayed powers of the carriers, radii and clocks in those formulas. Thus (10.21) is an explicit finite-sum prescription, not an extra maximum over unrelated old field norms. The inequalities are the corresponding scalar comparisons (9.25), with their fixed output letters. More explicitly the low flow tests are $\eta_j \max(1, f_{0,0}, f_{0,1}) \le 1/8$ and $K_j \eta_j f_{0,2} \le \zeta_-/2$; the own first-label tests set the second row at most one. For the young coefficient calculation through its admitted order, sufficient additional tests are, with the first-generation convention $G(1, k_1) = 1$,

$$G(m, k_m) \max\{K_m \eta_m/\zeta_-, \eta_m, K_m \mu_{<m} \ell_m, \lambda_{\Gamma,m} \ell_m, \lambda_{\mathrm{met},m} \ell_m, \lambda_{R,m} \ell_m\} \le 1.$$

The intrinsic low-order partition test remains the one in the common proposition. These are own admitted-order tests, not refinement of every old layer at every future reading range. The third row is required to satisfy the particular center comparisons printed below. The fourth row is bounded by the fixed output letter of its indicated term; its full relative tail inequalities are printed next. Every entry is calculated by the finite positive formulas before making its comparison.

The intrinsic value of an old level and each of its low-order coefficients is calculated at its certifying index $i+1$. In particular the full level-one curvature is divided by $N_i^2 Y_{*,i} \delta_i/\sigma_{i-1}$; its tail uses δ_i^{n-1}. The aligned even correction uses δ_i^{n-2}, not δ_i^n. Keep the finite tests (9.24) for every used first-rate entry. Their even tail is at most $\frac{4}{3} A^{\mathcal{R}}_{i,2} \delta_i^2$, their odd tail above level one at most $\frac{2}{3} A^{\mathcal{R}}_{i,2} \delta_i^2$, and the leading odd term is retained separately. Its center phase factor satisfies $S_i/\delta_i \le C \Pi_i^{2-\mathsf{d}} N_i^{-(\bar\beta - \mathsf{h})}$. These bounds apply to the complete levels, not just levels one and two.

The principal scalar $|\zeta|^2/\kappa$ remains a multiplicative coefficient of the principal shear, with its positive interval from (9.41). In the four rows of (10.21) the relevant small powers are, respectively,

$$\mathsf{a} - 2\beta, \quad \mathsf{a} - \tfrac{3}{2}\beta \text{ and } \mathsf{a} - \tfrac{1}{2}\beta, \quad 2\mathsf{h}_* - \tfrac{1}{2}\beta, \quad \bar\beta + \mathsf{h}_*,$$

with $\mathsf{h}_* = \mathsf{h} - \mathsf{dv}$. The packet and reciprocal variations also use $\mathsf{c} = \mathsf{a} + \mathsf{h} - 1 - 3\beta/2 > 0$. For the complete-level tests the additional power is $(n-1)\mathsf{h}_*$ or $(n-2)\mathsf{h}_*$. These follow directly by inserting $\ell_j = N_{j-1}^{-\mathsf{a}}$, $\delta_i = \Pi_i^{\mathsf{d}} N_i^{-\mathsf{h}}$, $Y_{*,i} \le C_\Theta N_i^{-\bar\beta}$, and $\sigma_i \asymp N_i^{\beta/2}$. The numerical Eulerian curvature rate is the one defined in (9.30); its factor K_j is retained when converting to a rigid label derivative. The unit $\mathfrak{m}_i$ retains precisely the scale terms of (9.38). Removing their normalized constants therefore leaves the same small-power comparisons; the relative tails are still tested with δ_i^{n-1} or δ_i^{n-2}, as appropriate. The contracted transverse estimate is used before this substitution. The common scalar hypotheses ensure positivity of these powers.

For a larger reading range it is not legitimate to repeat this absorption at an arbitrarily old carrier. Instead use the intrinsic zero-order bound and retain the ratio of the new normalized bound to the intrinsic one. Denote its logarithm by $s \ge 0$. At positive order r, $e^s \le e^{rs}$. Products then charge s on the sum of their actual orders, once. Center readings are undilated products; old level gains enter only through the first two rows of (10.21). This is the only surplus used below. In particular no zero-order amplitude of an old frame or old field is inserted into a new seed amplitude.

In the next lemma the layer j is fixed and suppressed. The single order r denotes total time, phase and label order: $P_n(r)$ is the maximum of $P_{j,n}(b, p, |\alpha|)$ over admissible triples $b + p + |\alpha| \le r$.

Their normalization by prescribed sizes is the one given above; the fixed order-only profile factors remain. The maximum uses only the rectangles in (10.3).

Lemma 10.1 (Complete depth, with normalized shifts)**.** *Let $\phi(0) = 0$ be increasing and superadditive, and suppose the coefficient, already produced propagator, and seed arrays, normalized by their prescribed sizes, have majorant $\mathcal{A}e^{\phi(r)}$, where $\mathcal{A} \geq 1$ is a numerical majorant amplitude. This hypothesis includes the fixed order-only profile factors at their actual derivative orders, with the preceding intrinsic zero-order normalization. A complete level of index n, after all lower elliptic substitutions and the refund, is bounded by*

$$P_n(r) \leq \mathcal{A}^{8n+1}\,[C(n+1)]^{C(n+1)}[C(n+1)]^{Cr}\exp\{\phi(r+4n)\}, \tag{10.22}$$

for the arrays just defined. The constant C depends on the fixed scalar and propagator letters only. The same rule with its grade applies to the terminal composites and sources. The shifted majorant factors as $\exp\phi(4n)\exp\{\phi(r+4n)-\phi(4n)\}$, with the first factor in the amplitude. All fixed profile factors are included at the shifted orders in this majorant.

Proof. Fully expand a term until only seed and coefficient factors remain. A first-shift linear node has one child and at most two extra derivatives after the direct refund. A second-shift node has one child and at most three. A fast quadratic node with principal vorticity has at most two extra derivatives; after substituting a lower elliptic factor it has at most four, but its grade has increased by at least one. The slow quadratic node has the same four-derivative bound and shift at least two. The scalar row has the extra primitive order exactly accounted for in (10.5). Thus each internal node of grade loss d adds at most $4d$ to the sum of the leaf orders. The losses telescope to n. There are at most n internal nodes, $n+1$ seed leaves, and at most $4n$ coefficient or propagator leaves, since an expanded source has at most three coefficients and two pair factors. The exponent $8n+1$ also allows the fixed component sums in the direct refunded term. The positive label-Duhamel recursion is order-conservative and does not alter the grade.

Choose a plane tree, choose one of the finitely many source types at each node, and distribute the derivative positions. The first two choices are bounded by $[C(n+1)]^{C(n+1)}$; the distribution by $[C(n+1)]^{Cr}$. The order-only profile factors belong to the input majorants. If the leaf orders, including the derivatives introduced by the rows, are r_s, then $\sum_s r_s \leq r+4n$ and

$$\prod_s e^{\phi(r_s)} \leq e^{\phi(\sum_s r_s)} \leq e^{\phi(r+4n)}.$$

Complementary orders in (10.7) are essential here. This proves (10.22), retaining the full shift inside the profile majorant. The safe value four includes both terminal elliptic pieces. With a sharper grade count three can replace four in several rows, but even then the amplitude is $e^{\phi(3n)}$, not $e^{n\phi(3)}$. $\square$

10.5. **An upper bound for the generation constants.** We use the general flow estimate at every generation in the following upper bound. This does not require any additional refinement condition. Refined intrinsic estimates remain available for the low-order sizes. The less precise general estimate is sufficient for the scalar carrier comparisons.

For $g(x) = x\log(1+x)$, set

$$\phi_{\Lambda,\beta_0,M}(r) = \Lambda[g(M+r)-g(M)] + \beta_0 r.$$

These functions are normalized, increasing and superadditive. Their exact shift and a useful bound are

$$\phi(r+e)-\phi(e) = \phi_{\Lambda,\beta_0,M+e}(r), \qquad \phi(r) \leq r\{\Lambda[1+\log(1+M+r)]+\beta_0\}. \tag{10.23}$$

Moreover a dilation by an integer d is bounded by replacing (Λ,β_0) with $(d\Lambda, d\beta_0 + d(2+\log d)\Lambda)$. This follows by integrating g', using $1+M+dx \leq d(1+M+x)$.

Here is a conservative count for the scalar prescriptions above. It is stated explicitly so that the generation multiplier can be checked independently of the choice of the derivative range. In a normalized jet of positive total order r, the expansion of a rigid old profile, including conversion, has earlier total order at most $3r+2$. Indeed, a conversion either assigns one time derivative position to that profile, turns it into one spatial derivative position, or differentiates one of the two explicit drift factors in (10.16); the last alternative spends that time derivative position and adds at most the stream-to-velocity derivative and one frame derivative. The undifferentiated drift uses its scalar size. Induction on the number of time derivative positions gives this bound.

In a flow jet of order r, the expanded rigid factors have number at most $2r-1$ and sum of orders at most $3r-2$: isolate the linear term at zero time order and induct on the blocks of (10.11); for positive time order there are only $r-1$ derivative positions. Substitution therefore has earlier total order at most $13r-8$.

Adjugation, products and reciprocals conserve total order. One additional spatial derivative for a scalar gradient or $\mathfrak{l}$ is bounded by doubling this positive-order allowance. In the label-Duhamel expansion, each additional coefficient has at least one label derivative position, so its extra derivative costs at most the number of those derivative positions. Consequently the bound dr, with $d=64$, covers the full coefficient and propagator calculation. The zero-order terms in this count are always the intrinsic sizes, not the expansion of a large reading-range amplitude. The subset, partition and component counts are at most $e^{C_{\mathrm{cnt}} r\log(1+r)+C_{\mathrm{cnt}} r}$, with the fixed profile factors kept separately at their actual derivative orders: an ordered partition is bounded by $r!2^r$, and each of the finitely many operations above adds a fixed number of such counts.

Fix a young range m, put $K=L_1$, $R=256(K+1)$, and $J^{\Sigma}=\sum_{i<m}(J_i+1)$. This scalar evaluation argument covers the sum of all earlier orders in the final young coefficient production, including its dilation by d. It does not require extra field derivatives beyond (10.3). Every individual derivative remains in that allocation; only the majorant is evaluated at the sum. In fact $K\le m(13k_m+69)$. Let $\mathcal{A}_j^{\mathrm{gen}}$ and $\phi_j=\phi_{\Lambda_j,\beta_j,\mathfrak{M}_j}$ bound the normalized arrays through generation $j<m$. These $\mathcal{A}_j^{\mathrm{gen}}$ are scalar majorant amplitudes, distinct from the physical amplitudes A_q in the center equations below. They may be made nondecreasing. The following explicit upper recursion implements the count:

$$\begin{aligned}
\Lambda_j^* &= d\Lambda_{j-1}+C_{\mathrm{cnt}},\\
\beta_j^* &= d\beta_{j-1}+d(2+\log d)\Lambda_{j-1}+d\log\mathcal{A}_{j-1}^{\mathrm{gen}}+C_{\mathrm{cnt}}(1+\log\mathcal{A}_{0,j}^{\mathrm{seed}}),\\
\phi_j^* &= \phi_{\Lambda_j^*,\beta_j^*,\mathfrak{M}_{j-1}},\\
\mathfrak{M}_j &= \mathfrak{M}_{j-1}+4J_j,\\
\Lambda_j &= \Lambda_j^*+C_{\mathrm{cnt}}(J_j+1),\\
\beta_j &= \beta_j^*+C_{\mathrm{cnt}}(J_j+1)\log(J_j+2),\\
\log\mathcal{A}_j^{\mathrm{gen}} &= (8J_j+1)\log\mathcal{A}_{0,j}^{\mathrm{seed}}+\phi_j^*(4J_j)+C_{\mathrm{cnt}}(J_j+1)\log(J_j+2).
\end{aligned}\tag{10.24}$$

Increase any right side to the preceding value when necessary. At $j=1$ the same formulas start from $\mathcal{A}_0^{\mathrm{gen}}=1$, $\Lambda_0=\beta_0=\mathfrak{M}_0=0$; the rigid base is affine. The positive ratio in the preceding subsection is at most $\mathcal{A}_{j-1}^{\mathrm{gen}}$, and is charged only in the slope $d\log\mathcal{A}_{j-1}^{\mathrm{gen}}$. The coefficient seed $\mathcal{A}_{0,j}^{\mathrm{seed}}$ contains only fixed letters and fixed profile data; the four true-size comparisons removed old amplitudes from it. For this application, first bound the fixed profile factors at their actual shifted orders by the finite seed norms specified in (10.25) below. The resulting input bound is $\mathcal{A}_{0,j}^{\mathrm{seed}}e^{\phi_j^*(r)}$. Lemma 10.1 and (10.23) then give (10.24), including the full $\phi_j^*(4J_j)$ term. This is an upper recursion, so its larger counting constant C_{cnt} may be chosen once by taking the sum of the finitely many component and partition constants just described.

The fixed time and profile weights will be specified below. Their finite evaluation, together with the seed norms, is required to obey the carrier-free order-function test

$$\mathcal{K}_{\mathrm{ho}}(k_m)\le\log N_m.\tag{10.25}$$

Normalize it so that $\log \mathcal{A}^{\mathrm{seed}}_{0,j} \le C_{\mathrm{cnt}} + 2\log\log N_m$ for all $j < m$. This is legitimate on the old cores: the spatial profile leaves are derivatives of the core polynomials of degree at most $3J_j + 4$, with their full-phase coefficient norms retained. Their maximal profile order and the time order $B + 3$ are fixed functions of k_m, by (10.2), not of L_1. The row derivatives in the tree are included in these allocated orders before the fixed profile norms are evaluated; the scalar argument $r + 4n$ does not increase any profile's derivative cap. The label envelope is constant on (10.1). The larger label allowance only differentiates the geometric maps and the polynomial coefficients. Thus $\mathcal{K}_{\mathrm{ho}}$ has no layer or carrier as an argument. No derivative of an older cutoff on its outer annulus occurs in this comparison. The young support-ball calculation retains its global profile norms at its own finite orders. Full-support derivatives of a fixed older layer are treated separately at the end, with constants allowed to depend on that layer and order.

Choose

$$L = \max\{2 + \log d,\ 1 + \log(1 + R + 4J^{\Sigma})\}, \qquad Z^{\mathrm{gen}}_j = \log \mathcal{A}^{\mathrm{gen}}_j + L\Lambda_j + \beta_j.$$

By (10.23), $\phi^*_j(4J_j) \le 4J_j(L\Lambda^*_j + \beta^*_j)$, and $L\Lambda^*_j + \beta^*_j \le 2dZ^{\mathrm{gen}}_{j-1} + C_{\mathrm{cnt}}(L + 1 + \log \mathcal{A}^{\mathrm{seed}}_{0,j})$. The harmless running maximum is bounded by one additional copy of Z^{gen}_{j-1}. Hence, with $d = 64$,

$$Z^{\mathrm{gen}}_j \le \mu_j Z^{\mathrm{gen}}_{j-1} + C_1(J_j + 1)^2[L + 1 + \log\log N_m], \qquad \mu_j = 1024(J_j + 1). \tag{10.26}$$

This recomputes the multiplier using the whole shift $\phi^*_j(4J_j)$. No estimate of it by $J_j\phi^*_j(4)$ has been used.

Take the fixed schedule integers, if necessary, with $q_0 \ge 2^{20}$, $c_Q \ge 2^{18}$, $Q_j = q_0 + j$, $k_j \le Q_j/c_Q$ and $J_j = 2k_j + 8$. Then, literally,

$$\mu_j = 2048k_j + 9216 \le Q_j/128 + 9216 \le Q_j/4.$$

Only lower bounds on these integers have been strengthened; none of the scale exponents or the construction is changed. Solving (10.26) and using $\phi_j(R) \le RZ^{\mathrm{gen}}_j$ gives an explicit bound for

$$G(1, k_1) = 1, \qquad G(m, k_m) = \exp \max_{1 \le j < m} \{(R + 1)\log \mathcal{A}^{\mathrm{gen}}_j + \phi_j(R)\} \quad (m \ge 2).$$

The factor $R + 1$ on the logarithmic amplitude also covers the number of positive-order earlier factors in a final young coefficient product; superadditivity covers their total order. We obtain

$$\log G(m, k_m) \le C_2(R + 1)m(J_m + 1)^2(L + 1 + \log\log N_m) \prod_{j=2}^{m-1} \mu_j. \tag{10.27}$$

Here and in the following ray comparison $m \ge 2$; the product is empty for $m = 2$. With $y = \log N_1$ and $N_j = N^{Q_j}_{j-1}$, it follows that

$$\frac{\log G(m, k_m)}{\log N_{m-1}} \le C_3(m + q_0)^{12} 4^{-(m-2)} \frac{1 + \log y}{y}. \tag{10.28}$$

All constants depend only on the fixed choices. For example a single constant for the right side is $C_3 \sum_{n \ge 0}(n + q_0 + 2)^{12} 4^{-n}$, a finite explicitly defined number. The series equals $C_3[(z\partial_z + q_0 + 2)^{12}(1 - z)^{-1}]_{z=1/4}$. The logarithmic factors discarded in obtaining the twelfth power are bounded by their positive arguments. Thus this is a uniform comparison in m, not merely a limit with m fixed.

In particular, for every fixed positive exponent c, one bare condition on y, of the form $C(c)(1 + \log y)/y \le c$, implies both

$$G(m, k_m)(Q_m \log N_{m-1})^2 N^{-\mathsf{c}/2}_{m-1} \le 1, \qquad G(i + 1, k_{i+1})a_{\Delta,i} \le N^c_i. \tag{10.29}$$

For the second inequality apply (10.28) at $i + 1$: the intrinsic construction gives $a_{\Delta,i} \le G(i + 1, k_{i+1})$, so two copies suffice. The base is N_i, not N_{i+1}. This proves that every full-depth tail test above and every positive-power scalar comparison can include its certifying-range constant. There is no assertion that $\log G$ is polynomial in the layer number.

10.6. **The joint time-order induction.** We specify the scalar conditions for the center rows as well as their production. For $v \geq i+1$ define the ordinary ratio $\upsilon_{i,v} = \sigma_i/\sigma_{v-1}$; the augmented family is

$$\upsilon^+_{i,i} = 1, \qquad \upsilon^+_{i,v} = \upsilon_{i,v} \quad (v \geq i+1).$$

In particular both have value one at $v = i+1$; only the augmented family has the extra $v = i$ entry. These are respectively the ordinary and augmented ratios in (9.9). For $v < i$ the augmented entry is absent. A family on W_l is truncated at $v = l$; the infinite notation is used only when summing its moments. The rotation family is $(\lambda^*_v @ \max(1, X_v))_v$, with total amplitude at most one. Each of the four center corrections has its actual size in (10.19), multiplied by $a_{\Delta,i}$, and list that size times $(\upsilon^+_{i,v} @ X_v)_v$. The additional rotation amplitude is $(\varepsilon^{\mathrm{def}}_{\mathrm{geom}} + \varepsilon_{\xi,i})$ times that size for the aligned quantities, with the tube-flatness letter added for the misaligned quantities; here $\varepsilon_{\xi,i} = (\Pi_i \mathscr{R}_i)^{-1}$. This notation for the geometric defect denotes the explicit metric variation on the fixed tube, not a change of the magnification parameter.

In this subsection $P_q = -T_q(0,t) > 0$ is the positive central circulation amplitude, $A_q = N_q|\zeta_q(0,t)|P_q$ its gradient amplitude, and $\varrho_q = \log|\zeta_q(0,t)|$ the logarithmic stretch. The central weight is still ρ_q, and $S_q = C_\Theta N_q^{-\bar\beta}$ is a fixed upper bound for P_q after its own pulse. The central weight and the logarithmic stretch are different quantities. The scalar aligned correction is denoted $\mathcal{X}_i$, distinct from the flow map X_i. The scalar rate X_q, when used after the sign @ or in a moment, is the X^{rate}_q of the preceding section. For $q \geq 2$, write

$$s_q = \theta^{\mathrm{in}}_q, \qquad \Gamma^{\mathrm{gr}}_q = \sigma_{q-1}\sin s_q, \qquad \mathsf{D}_q = \Lambda_q, \qquad X_q = \mathsf{D}_q\Gamma^{\mathrm{gr}}_q, \qquad \lambda^*_q = \mathsf{D}_q s_q.$$

The duration factor D_q is distinct from every label-derivative allowance L_i. At $q = 1$ retain exactly $\mathsf{D}_1 = \Lambda_{\mathrm{ret},1}$, $X_1 = \Lambda_{\mathrm{ret},1}\sigma_0 \sin\theta^{\mathrm{in}}_1$ and $\lambda^*_1 = c_p\Lambda_{\mathrm{ret},1}\sin\theta^{\mathrm{in}}_1$. These first-pulse numbers are not identified with the first logarithmic growth length. The scalar X_q remains distinct from the map $X_q(a,t)$.

Write $e(\varphi) = (\sin\varphi, \cos\varphi)$, and use $\varphi_i = \phi_i$ for the own-center angle introduced in Section 9. Thus $e_i = e(\varphi_i) = \zeta_i(0,t)/|\zeta_i(0,t)|$. For the base put $e_0 = e_b = e(\varphi_0)$. The local direction at the reading center Y_q is

$$e'_i[q] = \frac{\zeta_i(A^{\mathrm{inv}}_i(Y_q,t),t)}{|\zeta_i(A^{\mathrm{inv}}_i(Y_q,t),t)|} = e(\varphi'_i[q]), \qquad E'_i[q] = (e'_i[q], Je'_i[q]).$$

Use continuous angle lifts consistent with the insertion angles, and put

$$\begin{aligned} \theta_{iq} = \varphi_q - \varphi_i, \qquad \theta'_i[q] = \varphi'_i[q] - \varphi_i,\\ \theta^\flat_{iq} = \varphi_q - \varphi'_i[q] = \theta_{iq} - \theta'_i[q], \qquad \vartheta_q = \theta_{q-1,q} \quad (q \geq 2). \end{aligned} \tag{10.30}$$

Thus ϑ_q is the earlier adjacent angle θ_q, not the base angle ϑ. At a selected insertion, $M_q(\tau_q) = I$ and the held parent has angle zero, so $\vartheta_q(\tau_q) = \theta^{\mathrm{in}}_q$. The local angle $\varphi'_i[q]$ is the value of ϕ^{loc}_i at the specified displaced label. The first relative angle $\theta_{01} = \theta^{\mathrm{in}}_1$ is constant because the first covector is transported by the rigid base. For the base take $\varphi'_0[q] = \varphi_0$, so $\theta'_0[q] = 0$. Since $e'(\varphi) = -Je(\varphi)$ and $\dot e_b = \dot\psi J e_b$,

$$\varphi_0(t) = \varphi_0(0) - \psi(t), \qquad \psi = \theta_{0c} - \varphi_c + \varphi_0(0).$$

For the prescribed first-layer choice $e_b(0) = (0,1)$, $\varphi_0(0) = 0$. This is the normalization in which the last identity is $\psi = \theta_{0c} - \varphi_c$.

Read an old correction at $A_i^{\mathrm{inv}}(Y_q,t)$, and write $[q]$ on that value. Components of a misaligned tensor are measured in $E_i'[q]$. The exact strain decomposition is

$$D_{<q}(Y_q) = \dot{\psi} J + \sum_{1\le i<q} \big(\mathfrak{q}_i(0)\Omega_i + \mathcal{X}_i[q]\big) J e_i'[q] \otimes e_i'[q] + \sum_{1\le i<q} E_i'[q] \underline{D}_i^{\mathrm{mis}}[q] E_i'[q]^T. \tag{10.31}$$

The displacement of $\mathfrak{q}_i\Omega_i$ is included in $\mathcal{X}_i[q]$ by (9.39); $\mathfrak{q}_i(0)\Omega_i$ remains the principal term. Similarly put $A_i^+[q] = A_i - G_i^{\mathrm{al}}[q]$. Then

$$\mathcal{G}_q^{\mathrm{tot}} = \sum_{0\le i<q} A_i^+[q] \sin\theta_{iq}^\flat - \sum_{1\le i<q} (Je_q)^T E_i'[q] \underline{G}_i^{\mathrm{mis}}[q]. \tag{10.32}$$

The base is the exact $i=0$ term, with $A_0^+[q] = A_0$. Indeed $E_i'[q]^T e_q = (\cos\theta_{iq}^\flat, -\sin\theta_{iq}^\flat)$; this also fixes the sign of the transverse gradient in (10.32).

For later use put $D_i^+ = G(i+1, k_{i+1}) a_{\Delta,i}$. The finite scalar center tests, with a sufficiently small fixed $\varepsilon_{\mathrm{sch}}$, are

$$\begin{gathered} D_i^+ \mathfrak{r}_i \le \tfrac14 \min(1, c_W)\varepsilon_{\mathrm{sch}}\sigma_i, \qquad D_i^+ \mathfrak{g}_i \le \varepsilon_{\mathrm{sch}}\zeta_+\sigma_{i-1}\sigma_i, \\ \sum_i D_i^+(\mathfrak{m}_i + \sqrt{2}\mathfrak{g}_i') \le \varepsilon_{\mathrm{sch}}, \\ \sum_i D_i^+(\mathfrak{r}_i + \mathfrak{g}_i/\sigma_i) s_{i+1} \le \varepsilon_{\mathrm{sch}}. \end{gathered} \tag{10.33}$$

The unmatched angular parent uses its contracted number $d_i^{0,\mathrm{bd}}$ from (9.38), with the next-index transverse estimate of the common proposition: $D_i^+ d_i^{0,\mathrm{bd}} \le \varepsilon_{\mathrm{sch}}\lambda_{i+1}^*$. It is not bounded by an uncontracted larger Hessian. Two-point differences use the same inequality with the explicitly calculated sum of their segment differentials. In the units of (10.19) its multiplier is

$$\varrho_{i,q} = (K_i\ell_i^{-1} + C_{\mathrm{ph}} N_i K_i \zeta_+ \delta_i^{-1})\mathsf{s}_q^{\mathrm{sep}}, \qquad i \le q-2.$$

Thus require $\sum_{i\le q-2} D_i^+(\mathfrak{r}_i + \mathfrak{m}_i)\varrho_{i,q} \le \varepsilon_{\mathrm{sch}}\lambda_q^*$. The local-direction differences have sizes $\mathfrak{r}_{i,q}, \mathfrak{r}_{i,q}'$ obtained from (9.27) and its segment version. Their test, in the angular equation, is

$$2C_W \sum_{i\le q-2} \sigma_i D_i^+ \mathfrak{r}_{i,q}' + 2|\Omega_{q-1}|\vartheta_q D_{q-1}^+ \mathfrak{r}_{q-1,q} \le \varepsilon_{\mathrm{sch}} s_q \Gamma_q^{\mathrm{gr}}.$$

These are all actual center contractions; the common low-order statement supplies their intrinsic forms. Insertion of D_i^+ is permitted by (10.29). Their least exponent is $\bar\beta + \mathsf{h}_* - \beta/2 > 0$; the full-level relative tail factors remain those of (9.24).

The remaining center factors are $c_q = 2W(Y_q)\Gamma_{<q}(Y_q)$ and $\widehat{\kappa}_q = \kappa_q(0)/|\zeta_q(0)|^2$. Their derivatives are calculated from these formulas, with the reciprocal rule for $\widehat{\kappa}_q^{-1} = \mathfrak{q}_q$. There are three contributions to their nonconstant center size: the complete old circulation values, drift through label and metric factors, and drift through the phase. Explicit upper sums are

$$\begin{aligned} \varepsilon_{\mathrm{ctr},q} = C &\sum_{i<q} \{C_\Theta N_i^{-2\bar\beta}\Pi_i^2 + D_i^+ \mathfrak{c}_i\} \\ &+ \frac{C}{\sigma_{q-2}} \sum_{i<q} (C_L + K_i\ell_i^{-1})\mathfrak{q}_{i,q}^{\mathrm{vel}} \\ &+ \frac{C}{\sigma_{q-2}} \sum_{i<q} C_\Theta N_i^\beta \left[\mathfrak{q}_i^{\mathrm{slow}} + \sum_{i<h<q} \mathfrak{q}_{h,q}^{\mathrm{vel}}\right], \\ \mathfrak{q}_{i,q}^{\mathrm{vel}} &= C\sigma_i |s_i^c[q]|/N_i + D_i^+ \mathfrak{q}_i^{\mathrm{cor}}, \\ \mathfrak{q}_i^{\mathrm{slow}} &= C|\Omega_i|\lambda_{R,i} K_i/N_i^2 + D_i^+ \mathfrak{q}_i^{\mathrm{cor}}/(N_i\ell_i). \end{aligned} \tag{10.34}$$

Here $\mathfrak{c}_i$ and $\mathfrak{q}_i^{\rm cor}$ are respectively the complete odd circulation value and correction velocity numbers calculated from (9.36); their scale is $N_i^{-\bar\beta-\mathsf{h}_*+o(1)}$. The phase coefficient N_i^{β} in the third line is not small. It multiplies the slow own velocity, since the fast own velocity cancels in the phase drift. The three lines have least exponent $\bar\beta+\mathsf{h}_*$, $\min(2-3\beta/2,\bar\beta+\mathsf{h}_*)$, and $\min(2-3\beta/2,2\bar\beta+\mathsf{h}_*)$, respectively; the cross terms in the third line have the stronger exponent $Q_{i+1}(\bar\beta+\mathsf{h}_*)-\beta$. Require $\varepsilon_{{\rm ctr},q}\le K_{\rm ctr}\varepsilon_{\rm sch}$. The absolute metric defect is kept separately inside the positive interval (9.41). None of these quantities is assigned a negative power of the newest carrier of the reading layer.

Differentiation of a moving reading uses $\frac{d}{dt}A_i^{\rm inv}(Y_q,t)=H_i\sum_{i\le h<q}v_h$. Its b-th derivative requires the velocities through $b-1$. For a difference of two readings use the mean-value formula on the whole segment, differentiating both endpoint paths and the segment. Thus its rate list is the union of the old layer list and the separation list $v^+_{q-1,v}$, with the coefficient $\varrho_{i,q}$ just specified. Local direction differences have the same property. Raising rates below X_q costs total amplitude at most three, because $2+\sigma_i\sum_{v\ge i+2}\sigma_{v-1}^{-1}\le 3$.

Here are the primary differential equations, written to exhibit all their inputs. Put $\Omega_i^{\rm eff}[q]=\mathfrak{q}_i(0)\Omega_i+\mathcal{X}_i[q]$, and let $\mathcal{D}_q^{\rm mis}=\sum_{1\le i<q}E_i'[q]\underline{D}_i^{\rm mis}[q]E_i'[q]^T$. The two contractions are

$$\pi_q^{\varrho}=-e_q^T(\mathcal{D}_q^{\rm mis})^Te_q,\qquad \pi_q^{\varphi}=(Je_q)^T(\mathcal{D}_q^{\rm mis})^Te_q.$$

The covector equation $\dot\zeta_q=-D_{<q}(Y_q)^T\zeta_q$ then gives

$$\begin{aligned}\dot\varrho_q&=\sum_{1\le i<q}\Omega_i^{\rm eff}[q]\sin\theta^\flat_{iq}\cos\theta^\flat_{iq}+\pi_q^{\varrho},\\ \dot\varphi_q&=-\dot\psi-\sum_{1\le i<q}\Omega_i^{\rm eff}[q]\sin^2\theta^\flat_{iq}+\pi_q^{\varphi},\\ \dot\Omega_q&=-c_qA_q\sin\varphi_q,\\ (\log P_q)'&=-\mathfrak{q}_q\mathcal{G}_q^{\rm tot}\Omega_q/A_q.\end{aligned}\tag{10.35}$$

The relative-angle equation is the difference of the second line at the two reading centers. The common rotation cancels. Use $\sin^2u-\sin^2v=\sin(u+v)\sin(u-v)$ and (9.40). Its terms are the factored old sum, the unmatched parent, the two-point differences, and the local direction differences. No term is omitted.

For reference the resulting principal lists, in the notation above, are

(10.36)

quantity	size	amplitude at X_v
$\vartheta_q,\ 2\le q\le l$	Cs_q	$s_qv^+_{q-1,v},\ v\ge q$
$\varrho_q,\ e^{\pm\varrho_q},\ 2\le q\le l$	C	$(\Gamma_q^{\rm gr}/X_v)v^+_{q-1,v},\ v\ge q$
$P_q,\ 1\le q<l$	S_q	$S_q(\sigma_{q-1}/\sigma_q)v^+_{q,v},\ v\ge q$
$\Omega_q,\ 1\le q<l$	$C\sigma_q$	$\sigma_qv^+_{q,v},\ v\ge q+1.$

All rate indices are truncated at $v=l$. The first-layer angle and stretch use $\theta_{01}=\theta_1^{\rm in}$, $\varrho_1=0$; its controlled angle has the separate first-pulse list. The active pair $q=l$ is estimated by its variable-coefficient matrix below, separately from the last two rows. The displayed rows have a separate corrector-borne family of these shapes. The controlled tilt is zero on a hold, has list $s_l@X_l$ on a blend, and $\lambda_l^*@X_l$ on a pulse. Its actual blend size may be smaller than the list mass; use the latter in products. The principal metric coefficient multiplies these lists by its bounded size and adds its own small rotation/center family. It is never included in an additive-corrector size.

We verify the placement of the new terms. In the angle equation, the factored old sum is bounded by $Cs_q\sum_iD_i^+\mathfrak{m}_i$; the unmatched and two-point terms use $D_i^+d_i^{0,{\rm bd}}$, $\sum D_i^+(\mathfrak{x}_i+\mathfrak{m}_i)\varrho_{i,q}\le\varepsilon_{\rm sch}\mathsf{D}_qs_q$, and $\mathsf{D}_q\le X_v$ pays the integration. The direction term uses its displayed $s_q\Gamma_q^{\rm gr}$ test. In the radius equation the raised amplitude is at most $3\sum_iD_i^+\mathfrak{m}_i$, which after integration is at most $3\varepsilon_{\rm sch}\Gamma_q^{\rm gr}/X_q$. At later rates use $v^+_{i,v}\le v^+_{q-1,v}$. The two primary amplitude equations have the extra bounded factors c_q and $\mathfrak{q}_q$. For the first the ratio after integration is $C\varepsilon_{\mathrm{ctr},q}\sigma_q\widehat{S}_{\sin\varphi_q}/X_{q+1}$, with effective size as in (10.10). The blend, hold and pulse estimates of Lemma 9.2, and its subsequent old-angle bounds, give

$$\frac{\sigma_q\widehat{S}_{\sin\varphi_q}}{X_{q+1}} \le C \qquad (1 \le q < l).$$

On a blend the value of the controlled tilt may be smaller than its list mass; on a pulse this ratio may be of order one. The immediate transition uses $\sigma_q s_{q+1} \le CX_{q+1}$; the pulse uses $\sigma_q\lambda^*_{q+1} \le CX_{q+1}$. On later windows the old-angle and rotation-list comparisons give the same bound. For the second its integrand size is at most $C\sigma_{q-1}\Gamma^{\mathrm{gr}}_q/\sigma_q$, so its extra entry divided by X_v is inside the third list in (10.36). These verifications use actual sizes and list masses, not a high-order constant in a low-order test.

In the adjacent-angle equation, let π^ϑ_l denote the remainder after the central aligned shear sum is removed: it consists of $\pi^\varphi_l - \pi^\varphi_{l-1}$ and the two-point and local-direction differences just listed. The signed aligned corrections $\mathcal{X}_i$ remain in the aligned sum. The blend equation includes all three additional terms

$$-\eta_b\pi^\vartheta_l - \eta_b\pi^\rho_l\tan\varphi_l - (1-\eta_b)\pi^\rho_{l-1}\tan\varphi_{l-1}.$$

The tangent factors have size Cs_l on the blend. Its derivatives follow by the composition rule; all their rates are at most X_l. On a pulse use the exact prescribed relation $\sin\varphi_l = z_\mu e^{-\varrho_l}$, since the pulse branch in (7.2) prescribes $\zeta_{l,1} = z_\mu$. The function z_μ has the fixed pulse-profile derivatives. The top pair uses the matrix with entries $\mathfrak{q}_l\mathcal{G}^{\mathrm{tot}}_l/(|\zeta_l|\sigma_{l-1})$ and $c_l\sigma_{l-1}\zeta_{l,1}$. Its derivative of order b follows from matrix derivatives through $b-1$ by (10.13). Positivity for reciprocals and the compact ranges of $\tan, \arcsin, \exp$ are supplied by the strict low-order windows.

Fix a nonnegative time order n. Assume the produced estimates through n on every existing window, with that window's admitted cap. At order $n+1$ proceed as follows:

(1) Produce adjacent angles and radii, then the primary Ω, P from (10.35). They use complete center rows only through n.
(2) Produce the controlled tilt, the top pair, relative directions and $A_q = N_q|\zeta_q|P_q$. Produce the rotation primitive $\psi = \theta_{0c} - \varphi_c + \varphi_0(0)$, where $c = l-1$ or l is the controlled index on that window. The fixed initial angle contributes no positive-order derivative.
(3) In increasing layer index use (10.13)–(10.18), and in increasing level index use (10.6), to produce the field arrays at $n+1$. Produce their complete center values, two-point differences, local directions and the factors $c, \widehat{\kappa}$.
(4) Form Ω^{eff}, A^+, and finally (10.32) at order $n+1$.

The last row is a terminal output at this order. The next primary step reads it only at the previous order. The direct refund reads only lower levels at the same order. In a moving-reading drift, even a younger layer is used only at the previous time order. The base is the complete size and first-gauge statement of Proposition 9.3; the step 0 to 1 produces the window form from that base. On a window with a lower cap retain only the orders that it admits; its inherited datum enters the zero-time history, not an unavailable higher derivative. This proves the induction without assuming an independently available all-order control or field family.

All constants are produced at these same ranks. More formally, let Φ^X_{n+1} be the positive derivative position polynomial of the exact equation for row X, divided by its listed size and rate, using the comparisons just proved. In rank order set its new constant equal to $\max(1, C_n, \Phi^X_{n+1})$. Then take the maximum of this finite list. The outer-function constants are the maxima of their derivatives on the fixed low-order intervals; fixed cutoff derivatives through $n+3$ are additional leaves. This is an explicit recursive definition of the carrier-independent order constants C^{pot}_n, before adjoining the generation factor G. The field part uses exactly the scalar polynomials already printed above. An order weight dominating any such finite sequence can, for example, be chosen as

$$w(n) = (n!)^2 \exp\{n \max_{r\le n} \log(2 + C_r^{\mathrm{pot}})\}, \qquad w_t = w. \tag{10.37}$$

and enlarged once by a fixed exponential so that each binomial convolution has bound $3w(n)$. This follows from $\sum_{r=0}^{n} \binom{n}{r}(r!)^2((n-r)!)^2 \le 3(n!)^2$ and superadditivity of the exponent. Separate phase and label weights w_s, w_a are defined in the same way from the fixed profile derivatives, with all three weights equal to one at zero. They are functions of order only.

10.7. **Four forms of the derivative estimates.** The construction just proved has four distinct uses. In its window form, keep the native rates, their amplitudes and the separate corrector family. No gain is asserted between equal rates. For the old two-rate form on I_m, use

$$\Lambda_{j,m}(b) = \Pi_j^{\min(b,1)} \Pi_m^{(b-1)_+}, \qquad j \le m.$$

It is a Leibniz algebra, and $\Pi_j \Lambda(b_1)\Lambda(b_2) \le \Lambda(b_1 + b_2 + 1)$. Thus

$$\begin{aligned} |\partial_t^b M_j^{\pm 1}| &\le K_j C_b^{(m)} \Lambda_{j,m}(b), \\ |\partial_t^b D_{<j}(Y_j)| &\le C_b^{(m)} \Pi_j \Lambda_{j,m}(b), \\ |\partial_t^b \nabla\Gamma_{<j}(Y_j)| &\le C_b^{(m)} \sigma_{j-1}^2 \Lambda_{j,m}(b). \end{aligned} \tag{10.38}$$

For $j < m$, the range-dependent corrector constant in $C_b^{(m)}$ is kept separate; it is not absorbed at Π_j.

For $m \ge 2$, the young form, $j = m$, has a genuine smaller clock. Put

$$\widehat{\Pi}_m = Q_{m+1}^4 (\log N_m)^2 \sqrt{C_\sigma} N_{m-1}^{\beta/2}, \qquad G(m, k_m)\widehat{\Pi}_m / \Pi_m \le \tfrac{1}{2} \quad (m \ge 2). \tag{10.39}$$

The moments proved below allow each differentiated window entry to use $\widehat{\Pi}_m$ before comparison with Π_m. An expanded term with r differentiated corrector factors carrying range constants, at respective orders $b_s \ge 2$, has total order $b \ge \sum b_s \ge 2r$. Its excess relative to the carrier-independent order polynomial is bounded by

$$G(m, k_m)^r (\widehat{\Pi}_m / \Pi_m)^b \le 2^{-r}.$$

Product, partition, integration and linear-matrix recurrences all preserve this count, since orders are complementary; zero-order factors use their intrinsic size and first orders have the intrinsic constant. The finite sum of the terms is included in C_b^{pot}; hence the young versions of (10.38) have constant at most $2C_b^{\mathrm{pot}}$. Purely spatial range constants in the young coefficients retain their marked curvature or first-label variation and are paid by (10.21) with (10.29). A gap is not assigned to an undifferentiated or purely spatial entry. At $m = 1$ there is no inherited range-dependent corrector constant. The native Π_1 moments below give the young bounds with the fixed profile-order constants directly.

For $\Pi_m = N_{m-1}^{\mathsf{p}} (\log N_m)^{\mathsf{r}}$, the logarithm of the left side of (10.39) is at most

$$-\big(\mathsf{p} - \beta/2 - \mathsf{c}/2\big) \log N_{m-1} + 4 \log Q_{m+1} + (2 - \mathsf{r}) \log\log N_m + C.$$

The coefficient is positive: $\mathsf{c}/2 < \mathsf{a}/2$, $\mathsf{p} \ge \mathsf{a} > 1 + \beta/2$. The uniform ray in (10.28) therefore also ensures (10.39).

Finally, on a later window $W_l \subset I_m$, layers $m \le i < l$ are certified by their own range $i + 1$. Their D_i^+-inclusive entries in (10.33) and the difference tests are already small at base N_i. They reach an old row only through relative angles, the controlled tilt and the rotation primitive. The integration rule gives their own amplitudes divided by their own rates. Thus they fit the same lists, even when l is arbitrarily large. This fourth form does not use range m's constant for a younger layer. It also does not replace a future rate by the intrinsic rate of the old layer receiving it.

10.8. **Clock moments and fixed-layer finiteness.** Here are the precise clock conditions used above. The schedule has $\sigma_i \asymp N_i^{\beta/2}$ for $i \geq 1$, and $N_i = N_{i-1}^{Q_i}$ for $i \geq 2$. The generic growth-clock comparison is $\sigma_{l-1} s_l \leq C_{\mathrm{gr}} \mathsf{D}_l \sigma_{l-2}$, where C_{gr} is a fixed scalar letter. Hence on a generic window

$$\lambda_l^* X_l^n = \mathsf{D}_l^{n+1} \sigma_{l-1}^n s_l (\sin s_l)^n \leq C_{\mathrm{gr}}^{n+1} \mathsf{D}_l^{2n+2} \frac{\sigma_{l-2}^{n+1}}{\sigma_{l-1}}.$$

The first window is kept as its own finite term. Require the scalar schedule comparisons $\sum_l \lambda_l^* \leq 1$ and $\upsilon_{i,l} \leq C\lambda_l^*$ for $l \geq i+2$. For $m \geq 2$, require $X_l \leq \widehat{\Pi}_m$ for $l \leq m+1$, $(\log N_m)^{10} \leq \widehat{\Pi}_m$, and $(\log N_m)^{30} \leq \Pi_m$. The last inequality follows on one fixed bare ray from $\mathsf{p} > 0$, since $\log N_m = Q_m \log N_{m-1}$ and $\Pi_m \geq N_{m-1}^{\mathsf{p}}$; it does not involve a field norm. For the two initial clock terms impose

$$\lambda_1^* X_1^r + \lambda_2^* X_2^r \leq \tfrac{1}{2} \Pi_1^r \quad (r = 1, 2), \qquad \sum_{h \geq 0} e^{-\beta(q_0+1)^h y/4} \leq \tfrac{1}{2}.$$

The last scalar series is bounded by a geometric series on the same bare ray; no later profile occurs in it.

Here is the comparison for the raised rates, including the initial indices. Write $\widehat{X}_{i,l} = \max(\mathscr{R}_i, X_l)$. The scalar floor (10.8) satisfies, for $1 \leq i \leq m$, $m \geq 2$,

$$\mathscr{R}_i \leq \widehat{\Pi}_m, \qquad \widehat{X}_{i,l} \leq \widehat{\Pi}_m \quad (l \leq m+1). \tag{10.40}$$

To see the first inequality for $m \geq 2$, use the scalar clock upper bound and increasing carriers to obtain

$$\mathscr{R}_i \leq C_{\mathrm{fl}}(1 + \sigma_0 + \sqrt{C_\sigma} N_{m-1}^{\beta/2}).$$

The ratio of this bound to $\widehat{\Pi}_m$ is at most $C_{\mathrm{fl}}(2+\sigma_0)/[Q_{m+1}^4 (\log N_m)^2]$ after $\sqrt{C_\sigma} N_1^{\beta/2} \geq 1$. It is at most one on a single scalar ray in $y = \log N_1$. At $m = 1$ impose its literal fixed-data comparison

$$\max(1, r_1) \leq Q_2^4 y^2 \sqrt{C_\sigma} N_0^{\beta/2};$$

its left side is fixed, so the same ray can include it. The second inequality in (10.40) now follows from the stated head-rate comparison for $m \geq 2$, including the separate first-pulse rate X_1. Also $\mathscr{R}_i \leq \Pi_i$ and the accepted own-window tests $X_l \leq \Pi_i$ for $l \leq i+1$ give the corresponding intrinsic head bound. Thus neither raising a rate nor adjoining the explicit floor introduces a new field estimate or an arbitrary large clock.

For a generic tail put $x = \log N_{l-2}$, $Q = Q_{l-1}$; require

$$\mathsf{D}_l \leq x^4, \qquad Q \leq x, \qquad \frac{x}{\log x} \geq \frac{128[1 + |\log c_\sigma| + |\log C_\sigma|]}{\beta} Q. \tag{10.41}$$

All these are scalar clock tests. The last one propagates after one fixed base threshold, because $Qx/\log(Qx) \geq (Q/2)x/\log x$ when $Q \leq x$. Retain the derivative cap $B + 3 \leq Q_m$. For the first future term we also use the sharper scalar input (11.22),

$$\mathsf{D}_{m+2} \leq Q_{m+2}^3 \log N_m \quad (m \geq 1),$$

together with $Q_{m+2} \leq (q_0+3)m$ and $m^2 \leq \log N_m$. These comparisons are realized by the single sequence in Section 12. They are additional to the generic tail test (10.41).

For $m \geq 2$, $1 \leq n \leq Q_m$, split each moment at $m+1$ and $m+2$. The first part is bounded by its amplitude mass times $\widehat{\Pi}_m^n$. The first term beyond it has

$$\frac{\sigma_m^{n+1}}{\sigma_{m+1}} \leq C^n N_m^{\frac{\beta}{2}(n+1-Q_{m+1})} \leq C^n,$$

since $n+1 \leq Q_{m+1}$. Put $x_m = \log N_m$. The remaining factor satisfies

$$\mathsf{D}_{m+2}^{2n+2} \leq Q_{m+2}^{6n+6} x_m^{2n+2} \leq C^n x_m^{5n+5} \leq C^n x_m^{10n} \qquad (n \geq 1).$$

Here the middle inequality uses $Q_{m+2} \le (q_0+3)m$ and $m^2 \le x_m$. It is therefore at most $C^n\widehat{\Pi}_m^n$. For $l \ge m+3$ one has $n \le Q-2$. The logarithm of the remaining summand is at most

$$(8n+8)\log x + C(n+2) - \tfrac{\beta}{2}(Q-n-1)x \le -\tfrac{\beta}{4}x$$

after the fixed ray in (10.41), enlarged to include the displayed fixed constants. The tail is summable since $x \ge (q_0+1)^{l-3}y$. The same argument applies to $\upsilon_{i,l}$, starting at $l=i+1$, using its comparison with λ_l^* beyond $i+1$. It proves its moment bound $C^n\widehat{\Pi}_m^n$. The first moment of $\upsilon_{i,l}$ is at most $C\Pi_j$: the head has bounded amplitude mass and rates at most Π_j, and the tail has the preceding exponential bound at order one. At $m=1$ the head uses the native Π_1, including the separate first-pulse, augmented own, base and fixed-floor entries. The first future term is at most $C^n y^{10n}$, and hence at most $C^n\Pi_1^n$, since $\Pi_1 = N_0^{\mathsf{p}}y^{\mathsf{r}}$ with $\mathsf{r} \ge 10$. The same exponential tail applies thereafter. Thus all first-index moments have their native $C^n\Pi_1^n$ bound; no logarithmic-head comparison with $\widehat{\Pi}_1$ is used. The two initial tests and that exponential tail give $\sum \lambda_l^* X_l \le \Pi_1$ and $\sum \lambda_l^* X_l^2 \le C\Pi_1^2$. For $m \ge 2$, $n \ge 2$, the first tail's logarithmic power satisfies $(\log N_m)^{10n} \le \Pi_m^{n-1}$; for $n \ge 3$ it also satisfies $(\log N_m)^{10n} \le \Pi_m^{n-2}$. Both follow from the thirtieth-power test. Thus

$$\begin{aligned} &\sum_{l \ge i+1} \upsilon_{i,l} X_l^n \le C^n \Pi_j \Pi_m^{n-1}, \qquad 0 \le i \le j, \quad 1 \le j \le m, \quad 1 \le n \le Q_m, \\ &\sum_l \lambda_l^* X_l^{b+1} \le C^{b+1}\Pi_j \Lambda_{j,m}(b), \quad 1 \le j \le m, \quad 0 \le b \le Q_m - 1. \end{aligned} \tag{10.42}$$

For the latter at $b=0,1$ use the first two moments directly; for $b \ge 2$ factor $X_l^2 X_l^{b-1}$ in the head and use the same tail estimate. These are the ordinary-list moment inputs of (10.38), with $j \ge 1$. For the augmented list of a layer $i \ge 1$ retain, rather than drop, the extra own term X_i^n. It has both required bounds since $X_i \le \Pi_i \le \Pi_j \le \Pi_m$ and $X_i \le \widehat{\Pi}_m$ for $m \ge 2$; at $m=1$ use $X_1 \le \Pi_1$. Finally for any positive amplitudes a_l,

$$\sum_l a_l \widehat{X}_{i,l}^n \le \sum_l a_l X_l^n + \mathscr{R}_i^n \sum_l a_l. \tag{10.43}$$

The ordinary, augmented and rotation amplitude sums are bounded by the fixed scalar comparison letters. By (10.40), the extra term is at most $C\widehat{\Pi}_m^n$ for $m \ge 2$, and at most $C\Pi_1^n$ for $m=1$. For an old two-rate row it is at most $C\Pi_j\Pi_m^{n-1}$, since $\mathscr{R}_i \le \Pi_i \le \Pi_j$. For the rotation moment at order $b+1$, it is at most $C\Pi_j\Lambda_{j,m}(b)$: use $\mathscr{R}_i \le \Pi_j$ for $b=0,1$, and $\mathscr{R}_i^{b+1} \le \Pi_j^2\Pi_m^{b-1}$ for $b \ge 2$. The separate floor entry $1@\mathscr{R}_i$ obeys the same bounds. This proves the augmented and floor-raised forms actually used by the coefficient and scalar-source recursions, with all initial terms retained.

For the following separate assertion assume an infinite selected history with these scalar conditions at every index and $k_l \to \infty$. For a fixed arbitrary n, eventually $Q_{l-1} \ge n+2$, so the same summable tail applies. The finitely many earlier terms are finite; they need not be small. By the additional admission hypothesis, sufficiently late windows admit the required order in the joint induction, which gives $\psi^{(n)}$ with this moment. On the finitely many earlier windows the smooth finite control equations give bounded derivatives of each order on their compact intervals. Thus every fixed derivative of ψ is bounded on the entire preterminal interval. The exponential carrier identity is essential: a mere cubic lower bound for successive σ_l would not justify these moments.

For a fixed layer, now induct on the layer number without any uniformity claim. On its whole support use the prescribed smooth cutoff's derivatives at this fixed order, with constants allowed to depend on that layer and order. They are not estimated by $G(m,k_m)$ on an old outer annulus. Its older velocity has every derivative bounded by the preceding fixed-layer conclusion and the just proved rotation bound. Differentiating its flow equation in space and time, integrating the highest linear spatial term and using lower orders for all other terms gives finite bounds on its finite lifetime. The determinant is one and the low-order inverse bound remains in force. The scalar coefficient formulas and (10.12) then give bounded derivatives of every order. Finally (10.6), at its fixed finite depth, gives the same for every correction and terminal composite. This proves fixed-layer all-order finiteness. It is distinct from the uniform admitted-order estimates and does not enlarge their order caps.

10.9. **Substitution in the physical force bound.** We finish by making the actual inputs to Proposition 6.2 explicit. At the young index use the geometric arrays through r_F, the coefficient and reciprocal arrays through r_X, the homogeneous amplitude arrays through r_*, and the complete level arrays on (10.3), all on the young support ball and the full phase circle. The rigid inputs are on $B_{2\ell_m}$, whose older pullbacks were checked in (10.14); insertion data use the collar only at insertion. Their bounds have just been produced. The weighted estimate is pointwise in the central amplitude:

$$|\partial_t^b \partial_s^p D_a^\alpha Q_{m,n}(s,a,t)| \le P_{m,n}(b,p,|\alpha|) Y_{*,m} \rho_m(t) \delta_m^n \ell_m^{-|\alpha|} M_{\mathbf{a}_m}(b).$$

Taking a supremum may replace ρ_m by one, but none of the time estimates differentiated that upper bound. For the activation power below we impose the seed and short-ramp convention (11.17). The scalar growing-line comparison over that ramp, proved in Proposition 11.1, gives $Y_{\mathrm{act},m} \le C_{\mathrm{rp}} N_m^{-k_m - \mathsf{e}}$. For a different prescribed negative seed the unspecialized activation term remains $E_{r_*} \Pi_m Y_{\mathrm{act},m}$, as in (11.24). Use $\mathcal{T} = \max(|I_m|, \Pi_m^{-1})$, not just $|I_m|$, throughout the force recursion.

Take the sizes $\chi, C_0, L_0, G_0, B_0, c_0, h_0, \varpi_*, z_0$ to be the positive scalar outputs of the coefficient calculation for $\chi, C, \mathfrak{l}, D_a q, D_a b, c, h, \varpi, \zeta_1$. In B_0, add the actual $S_0 |D_a b^{\mathrm{rot}}|$ to the nonrotational bound. At positive orders use the scalar shifted row, not an assumed unshifted bound for α. Let C_* be the maximum of the resulting normalized young arrays after their proven spatial and time absorptions. It is bounded by a fixed function of r_X and the profile derivatives, independent of m. Choose $\lambda = \ell_m^{-1}$, increasing it to one if needed; the first-label tests ensure it dominates the actual label rates. Keep the actual numerical values in (6.3). In particular its fast circulation-square condition is $\mathcal{T} \sigma \varpi_* z_0 Y_* / \delta \le 1$, not an inverse-frequency estimate.

The pair and source constants a_n, b_n, s_n, and $H_{r_*}, D_{r_*}, E_{r_*}$, are now obtained by the literal recursions (6.4) and (6.5); their inputs are the just produced scalar arrays. The coarser unrefunded estimates used by this finite consumer have enough reserved time orders, so the direct outer time derivative in (6.3) is estimated without altering the analytic refund above. For $n > J_m$ put $a_n = 0$, but calculate b_n through $J_m + 2$ and s_n through $2J_m + 4$. These instructions retain every terminal term.

To display both means put $\mathfrak{A}_n = a_n Y_* \delta^n$, zero outside $[0, J_m]$, and $\mathfrak{B}_n = H_{r_*}(z_1^* \mathfrak{A}_{n-1} + z_2^* \mathfrak{A}_{n-2})$, where

$$z_1^* = \chi(2C_0\lambda + L_0)/N, \qquad z_2^* = \chi(C_0\lambda^2 + L_0\lambda)/N^2.$$

The actual nonnegative bounds for the mean primitives are

$$\begin{aligned} M_n^\Gamma &= H_{r_*}\left[\frac{\chi}{\sigma}\sum_{u+v=n-1}\mathfrak{A}_u\mathfrak{A}_v + \frac{\lambda\chi}{N\sigma}\sum_{u+v=n-2}\mathfrak{A}_u\mathfrak{A}_v\right], \\ M_n^\xi &= H_{r_*}\left[\frac{N\chi}{\sigma^2}\sum_{u+v=n-1}\mathfrak{A}_u\mathfrak{B}_v + \frac{\lambda\chi}{\sigma^2}\sum_{u+v=n-2}\mathfrak{A}_u\mathfrak{B}_v \right. \\ &\qquad \left. + \left(\frac{\chi\Lambda_K}{2\sigma^2} + \varpi_* h_0\right)\sum_{u+v=n-2}\mathfrak{A}_u\mathfrak{A}_v\right]. \end{aligned} \tag{10.44}$$

Here Λ_K is the calculated first-label rate of the reciprocal. The normalized lower elliptic terms occur in $\mathfrak{B}_v$. The last line retains both the reciprocal derivative term and the circulation-square term. The means use the entire phase circle even when the subsequent reading is on a polynomial cell.

Consequently the scalar force bound delivered to the next step is

$$\begin{aligned}\mathfrak{F}_{m,k} = C_{k,\mathcal{D}} C_k^{\mathrm{ev}} \Big[\Lambda_{\mathrm{ph}}^k (1 + N/\sigma) \left(E_{r_*} \Pi C_{\mathrm{rp}} N^{-k_m - \mathsf{e}} + \frac{C_\Theta N^{-\bar{\beta}}}{\mathcal{T}} \sum_{g=J_m+1}^{2J_m+4} s_g \delta^g \right) \\ + 2\lambda \Lambda_{\mathrm{slow}}^k \sum_{n=1}^{J_m} (M_n^\Gamma + M_n^\xi) \Big], \\ \Lambda_{\mathrm{ph}} = 1 + \Pi + (N+\lambda)(K + V_X), \\ \Lambda_{\mathrm{slow}} = 1 + \Pi + \lambda (K + V_X).\end{aligned} \tag{10.45}$$

The physical force satisfies $\|f_m\|_{C^k_{x,t}} \le \mathfrak{F}_{m,k}$. Indeed the two scalar residuals are exactly the activation terms, terminal graded sources and the two means in the unchanged finite cancellation identity. The reduced-vorticity normalization is undone by N/σ. Apply the existing compact inverse to activation plus remainder together and to each mean, as required by its compatibility statement. Its fixed constant is $C_{k,\mathcal{D}}$; no new inverse is needed.

For completeness, (10.45) exhibits the young powers without hiding the slow terms. Since $\delta = \Pi^{\mathsf{d}} N^{-\mathsf{h}}$, its oscillatory bracket is bounded by a fixed order function times

$$\Lambda_{\mathrm{ph}}^k (1 + N/\sigma) \left[\Pi N^{-k_m - \mathsf{e}} + \mathcal{T}^{-1} \Pi^{\mathsf{d}(J_m+1)} N^{-\bar{\beta} - \mathsf{h}(J_m+1)} \right].$$

Its slow bracket is bounded by a fixed order function times

$$\lambda \Lambda_{\mathrm{slow}}^k N^{-2\bar{\beta}} \left[\frac{\chi}{\sigma}(1 + \lambda/N) + \frac{N\chi}{\sigma^2}(z_1^* + z_2^*)(1 + \lambda/N) + \frac{\chi \Lambda_K}{\sigma^2} + \varpi_* h_0 \right]. \tag{10.46}$$

These bounds use the finite sums of a_n, s_n, not an assertion that termination controls their constants. There is no phase-frequency factor in Λ_{slow}.

Here is a precise choice of the fixed order function used in (10.25). At an integer k put $J = 2k+8$, use the allocation (10.3) for the young coefficient and profile orders, and form the finite positive formulas for the primary order constants, the young coefficient constants, H, D, E, C^{ev}, and a_n, b_n, s_n through their terminal indices. In these formulas replace each tested dimensionless quotient by one, each young absorbed corrector constant by $2C_b^{\mathrm{pot}}$, and keep all fixed profile norms through the stated orders. Let $\mathcal{K}_{\mathrm{ho}}(k)$ be the exponential of one plus the maximum of the logarithms of these positive numbers and of the corresponding constants in (10.44)–(10.46), made nondecreasing in k. This is a finite scalar recursion using only fixed data. It is fixed before any carrier is chosen. The range factor G is excluded only at the specific places where (10.29) or (10.39) has already paid it; it is never included in this order function.

One may now use the maximal admitted-order rule

$$k_m = \max\{k \in \mathbb{N}_0 : k \le Q_m / c_Q,\ \mathcal{K}_{\mathrm{ho}}(k) \le \log N_m\}. \tag{10.47}$$

Take the initial carrier so that the set is nonempty. Since $\mathcal{K}_{\mathrm{ho}}$ is one fixed nondecreasing function and $Q_m, \log N_m$ increase, these sets are nested; hence the finite inclusions in (10.2) hold. If both upper limits tend to infinity, any fixed integer eventually belongs to the set, so $k_m \to \infty$. This verifies the admission assertions for this rule, without asserting that all the other scalar tests have already been realized by one infinite sequence.

Proposition 10.2 (Higher-order finite implication)**.** *Suppose the finite corrected family satisfies the hypotheses and strict scalar tests of Proposition 9.3, with the all-trial older and selected-youngest quantifiers stated at the start of this section. Retain its containing-tube and connecting-segment conditions on the actual cores* (10.1)*, and the finite admission inclusions* (10.2)*. Impose the complete relative-tail and four-route tests stated above at their certifying indices, the displayed center tests, the clock and profile tests, and the finite scale quotients* (6.3)*. Then the joint induction produces the* *actual coefficient, complete-level and center arrays on* (10.3)*: older larger-range arrays on* $\mathcal{B}_i^\flat \times \mathbb{T} \times I_m$*, and young own-order arrays on* $B_{\ell_m} \times \mathbb{T} \times I_m$*, with insertion data on their prescribed collar. They have the four derivative forms just proved. Their generation constants obey* (10.24)*–*(10.29)*; at the young index the constants of the consumed time rows are at most* $2C_b^{\mathrm{pot}}$*. The physical force obeys* (10.45)*. If the scalar conditions hold on an infinite selected history and every fixed order is eventually admitted, as for* (10.47)*, each fixed layer on its full support and the rotation primitive have bounded derivatives of every fixed order on their whole preterminal lifetimes.*

Proof. The coefficient calculation and the refunded recursion prove the field step. The four intrinsic comparisons and Lemma 10.1 prove its normalized bounds; solving the explicit generation recurrence proves their uniform carrier comparison. The rank-ordered argument proves the simultaneous center and control step. The moment split proves the four forms and fixed-layer assertion. Finally (10.44) and the complete residual identity give (10.45). Each estimate was proved for precisely the earlier ranks used by the next one, so these are simultaneous conclusions, not mutually assumed hypotheses. □

The remaining parameter task is scalar: choose one sequence realizing the common low-order tests, the positive denominator and support windows, the relative full-depth tests, the $i+1$-inclusive center comparisons, the clock and order-function tests, and $\mathfrak{F}_{m,k} \le \epsilon_{m,k}$ with the desired summable bounds for each fixed k. The estimates above prove a uniform bare ray for their generation and moment parts, not the choice of that whole infinite sequence. All-trial assertions remain at the all-trial scope of the common lower-order proposition; its selected-history bounds, including the instantaneous central weight and strict returns, are used only on that selected history. No limiting Euler theorem or terminal uniqueness statement is asserted here.

11. Continuation and return of the finite quantitative inputs

The preceding estimates concern the same complete finite solution. We now specify their scalar hypotheses together and prove that they return the inputs of the material, correction and control propositions. In particular a high derivative bound is not used to start the low-order continuation argument. Older-layer estimates are uniform over every current trial $0 \le \mu \le c_p$ with its common selected past. Estimates for the youngest completed layer and its subsequent full-flow path use the selected return only. These quantifiers apply throughout; the return selector is never differentiated.

11.1. **Data, domains and scalar comparisons.** Fix a finite number of insertions, the smooth profiles and compact recovery operator of Section 5, and the magnified unit-ring chart of Lemma 2.3. Fix a compact containing region inside $\mathcal{P}$, with positive distance from its boundary and from $c_g = 0$. If L_F contains the prescribed compact physical profiles and recovery supports in this chart, take $0 < \varepsilon_{\mathrm{geom}} \le (16 L_F)^{-1}$, as in Lemma 2.3. The base datum, the first angle, the first return compression, and all comparison constants are fixed. The base is the one in (9.1); in particular its circulation level is $(2\varepsilon_{\mathrm{geom}})^{-1}$. At each insertion the profiles are smooth on $B_{2\ell_i}$, supported in B_{ℓ_i}, and have the prescribed flat time activation. The constant-envelope core is exactly $B_{\rho_i^{\mathrm{core}}}$ in (10.1); the usual choice $\rho_i^{\mathrm{core}} = \ell_i/2$ does not replace the smaller first core. On this core the phase profile has its prescribed linear part for $|s| < s_c$. Means and phase primitives are always taken on $\mathbb{T}$ before this restriction. The time intervals I_i, W_i, G_i are those of (9.8). In particular $I_i = [\tau_i, t_{\mathrm{end}}]$ includes all elapsed later evolution of layer i up to the current finite horizon. Every use of a growth or return window is restricted to its existing portion.

The paths Y_q and Z solve respectively the older and full velocity equations (7.6). Write $A_i^{\mathrm{inv}} = X_i^{-1}$ for the inverse map, to distinguish it from the positive amplitude A_i below. Every occurrence of an old profile at either path includes its own inverse map. An estimate on the rigid set $Y_q + M_q B_{2\ell_q}$, an estimate on the transported image of B_{ℓ_q}, and an estimate on a straight segment between two observation points have their different meanings. The connecting-segment clause below is not inferred from its endpoints.

For a requested force order $k \le k_m$ use

$$\begin{gathered} 0 \le k_1 \le \cdots \le k_m, \qquad k_i \le Q_i/c_Q, \qquad J_i = 2k_i + 8, \\ r_* = k + 4J_m + 9, \qquad r_X = r_* + 2, \qquad r_F = r_* + 4, \\ B = L_m = r_F + 3, \qquad L_i = L_{i+1} + 2J_i + 5 + B, \qquad \iota_i = 3J_i + 4. \end{gathered} \tag{11.1}$$

The older rectangles are $b \le B$, $p \le \iota_i - 2n$, $|\alpha| \le L_i - 2n - 1$. They apply on the old cores, not on old cutoff annuli. Young estimates have total mixed order as in (6.5)–(6.6) on $\mathbb{T} \times B_{\ell_m}$. We use Euclidean tensor norms and the normalized profile norm (3.8); its weight is $\rho_i(t) = |R_i(0,t)|/Y_{*,i}$, never differentiated. Physical norms are $C^k_{x,t}$ as in Section 6.

Here is a precise convention for evaluating the conditions. Start with the prescribed profile derivatives, scalar intervals, phase and label radii, and inherited data. Use the positive product, partition and reciprocal rules (9.6), (9.7), (10.11), (10.12). Apply them to the signed formulas before estimating their terms. Use (9.12), (9.18), (9.22) at low order, and the recursions (10.7), (10.24) at the higher orders. Restore the displayed powers of ℓ_i before multiplying a derivative by a radius. Integrals use the prescribed scalar history, including its inherited portion. Denote the resulting upper number for an expression E by $\mathcal{U}(E)$, and its computed positive lower number by $\mathcal{L}(E)$. A zero expression has upper number zero. These are finite formulas in scalar data, not suprema of the unknown solution. For example $\mathcal{U}(|D_a\zeta_i|)$ is the unscaled entry of the inverse-map array. For a two-point difference one first writes its segment integral, so $\mathcal{U}$ retains the separation and the exact contraction.

All tests for stage q are made before its current trial is run. The already selected σ_i, $i < q$, may be used as recorded scalar data. Their defining convention is the principal sum

$$A_i = -N_i T_i(0,t)|\zeta_i(0,t)|, \quad e_i = \zeta_i(0,t)/|\zeta_i(0,t)|, \quad \sigma_m^2 = \Big|-A_0 e_b - \sum_{i \le m} A_i e_i\Big|(t_m^b). \tag{11.2}$$

The sum at the selected endpoint has all activations equal to one. It is not redefined to be the complete circulation gradient at Z; the difference remains in (9.49). In a bound involving a not-yet-selected later scale use the endpoint of

$$c_\sigma N_i^\beta \le \sigma_i^2 \le C_\sigma N_i^\beta \tag{11.3}$$

which enlarges that bound. The current instance of (11.3) is a conclusion, not a hypothesis on its trial. Truncation of a prescribed nonnegative history integral only decreases its upper number. A finite version of each list below ends at the last window in use. Its infinite version consists of the same tests for every finite initial sequence.

Scales and control margins. Take $N_1 > 1$, put $N_i = N_{i-1}^{Q_i}$ for $i \ge 2$, and $Q_i = q_0 + i$, with $q_0 \ge 2^{20}$, $c_Q \ge 2^{18}$. The scale conventions of (10.3) are retained, with $0 < \beta < 1$: $\bar\beta = 1 - \beta$, $\delta_i = \Pi_i^{\mathsf{d}} N_i^{-\mathsf{h}} \le 1/2$, $\mathsf{d} \ge 2$, $0 < \mathsf{h} \le \bar\beta$, $\ell_i = N_{i-1}^{-\mathsf{a}}$ for $i \ge 2$, and $\Pi_i = N_{i-1}^{\mathsf{p}}(\log N_i)^{\mathsf{r}} \le N_i^{\mathsf{v}}$. Require $\mathsf{p} \ge \mathsf{a} > 1 + \beta/2$ and positivity of

$$\mathsf{h}_* := \mathsf{h} - \mathsf{d}\mathsf{v}, \quad \mathsf{a} - 2\beta, \quad 2\mathsf{h}_* - \beta/2, \quad \bar\beta + \mathsf{h}_* - \beta/2, \quad \mathsf{c} := \mathsf{a} + \mathsf{h} - 1 - 3\beta/2. \tag{11.4}$$

Keep the first radius and clock as defined in Section 9, with its fixed base linear-phase half-width $0 < s_0 \le s_c$ and first-collar constant $C_\ell > 0$. In particular

$$\ell_1 = \frac{s_0}{C_\ell N_0 \Lambda_\ell}, \qquad \Lambda_\ell \ge (\log N_1)^2, \qquad \ell_1^{-1} \le \Pi_1. \tag{11.5}$$

The first variation integral, rather than a later-index power estimate, is tested below.

Write $s_i = \theta_i^{\mathrm{in}}$. The growth length Λ_i is not the derivative allowance L_i in (11.1). For $i \ge 2$, $s_i = \Lambda_i \sigma_{i-2}/\sigma_{i-1}$, $\Gamma_i^{\mathrm{gr}} = \sigma_{i-1} \sin s_i$, and the return compression is Λ_i. The first compression $\Lambda_{\mathrm{ret},1}$

remains fixed. The finite scalar schedule used in the low-order comparison is the following, with empty sums zero:

$$
\begin{gathered}
\Lambda_j^{-1} \le \varepsilon_s, \quad 100 \le \Lambda_j/\Lambda_{j-1} \le Q_j^2 \quad (j \ge 2), \\
s_j,\ \Lambda_j s_j,\ \Lambda_j^4 \sigma_{j-2}^2/\sigma_{j-1},\ \sigma_{j-2}/\sigma_{j-1} \le \varepsilon_s \quad (j \ge 2), \\
\Lambda_j s_j \le \varepsilon_s s_{j-1}\ (j \ge 3), \qquad \sum_{j>i} s_j \le \varepsilon_s s_i, \\
h_j := s_{j-1}\sigma_{j-2}\ (j \ge 3), \quad h_2 = 0, \qquad h_j \Lambda_j/\Gamma_j^{\mathrm{gr}} \le 2\varepsilon_s, \\
\sum_{i=0}^{j} \sigma_i^\nu \le (1+2\varepsilon_s)\sigma_j^\nu \quad (\nu = 1,2), \qquad \sum_{i=1}^{j-2} \sigma_i s_{i+1} \le 2h_j, \\
\sum_{i=1}^{j-1} \sigma_i s_{i+1}^2 \le 2\varepsilon_s \Lambda_2^{-2}, \qquad \sum_{i=1}^{j-1} \sigma_i^2 s_{i+1} \le 2\sigma_{j-1}^2 s_j, \\
A_0 \le \varepsilon_s N_1^\beta, \qquad A_0/(\sigma_{j-1}^2 s_j) \le \varepsilon_s/\Lambda_j, \quad 2\sigma_{j-1}^2 \le \varepsilon_s N_j^\beta \quad (j \ge 2), \\
\sigma_i \ge \max(\varepsilon_s^{-1}\sigma_{i-1}, \sigma_{i-1}^3), \quad 1, \ell_j^{-1}, C_\Pi \sigma_{j-1}^2, (\Gamma_j^{\mathrm{gr}})^{\pm 1} \le \Pi_j, \\
\Gamma_j^{\mathrm{gr}} \le X_j, \qquad X_j \le \varepsilon_s X_{j+1}, \qquad X_{j+1} \le \varepsilon_s \Pi_j, \\
(\Lambda_1 + 2 + \Lambda_{\mathrm{ret},1}^{-1})/\Gamma_1^{\mathrm{gr}} + C_T/\sigma_0 \le \Pi_1.
\end{gathered}
\tag{11.6}
$$

Only available scale indices occur in this display; an occurrence of σ_j at the next insertion is tested by (11.3) beforehand. The first pulse obeys $\Lambda_{\mathrm{ret},1} \ge 2c_p\|\eta_2\|_\infty$, $0 < s_1 \le 1/8$, and $c_p\Lambda_{\mathrm{ret},1}\|\eta_2\|_\infty \sin s_1 \le 1/4$.

Use the fixed strict intervals stated before (9.4), including $0 < c_W < 1/3$, $c_p > \bar{\upsilon} > 3/2$, $C_W > 1 + \bar{\upsilon}$, $C_W^{\mathrm{all}} > 1 + c_p$, and $0 < c_\theta < 2/3 < 1 < C_\vartheta$, $0 < \zeta_- < 1 < 3/2 < \zeta_+$, where $\zeta_\pm = z_\pm$ of the low-order section. More explicitly, take

$$
\begin{gathered}
0 < z_* < \min(\zeta_-, \sqrt{15}/4), \quad 0 < c_A < \cos(2s_1), \quad C_\varphi > c_p\|\eta_2\|_\infty, \\
I_0 = \int_0^1 \frac{dt}{(1+t^2/2)^2(1+t)}, \quad 0 < G_- < \min(1 + I_0, 1 + \log 2), \\
G_+ > \max(1 + \bar{\upsilon}, 2 + \Lambda_{\mathrm{ret},1}^{-1}), \quad c_\sigma = e^{G_-}, \quad C_\sigma = \zeta_+ e^{G_+}, \\
C_S > 1, \quad C_T = 2C_S, \quad C_K > 2\zeta_+\zeta_-^{-2}.
\end{gathered}
\tag{11.7}
$$

The ratio and central matrix-size constants are the finite upper endpoints calculated on these intervals. Enlarge C_Θ once to contain their product with e^{G_+} and the fixed factor C_Y in the low-order normalization. The tolerance is chosen by the following finite rule. In the scalar proof of Lemma 9.2 and the proof of Proposition 7.7, replace every coarse interval by its stated relaxed endpoints and collect the finitely many coefficients multiplying the error tolerance, each divided by its corresponding positive model margin. Let $C_{\mathrm{ctl}} \ge 1$ be one plus their sum. Choose

$$
\begin{gathered}
0 < \varepsilon_s \le \min\{1/4, (32C_\vartheta)^{-1}, c_\theta/8, (4C_{\mathrm{ctl}})^{-1}\}, \qquad 0 < \varepsilon_{\mathrm{sch}} \le \varepsilon_s/(4C_{\mathrm{ctl}}), \\
0 < \delta_{\mathrm{ret}} \le \delta_0(c_p, \bar{\upsilon}), \qquad C_{\mathrm{ctl}}\delta_{\mathrm{ret}} < 1/4, \qquad \delta_{\mathrm{ret}} \le (c_p - \bar{\upsilon})/(2c_p), \\
0 < \varepsilon_{\mathrm{fac}} \le 1/100, \qquad 8\varepsilon_{\mathrm{fac}} \le \delta_{\mathrm{ret}}, \qquad \Lambda_m^{-1} \le \delta_{\mathrm{ret}} \quad (m \ge 2).
\end{gathered}
\tag{11.8}
$$

Here δ_0 is the finite threshold in the displayed generic comparison proof, with its fixed pulse profiles. Thus this choice is independent of the correction constants and of unknown fields.

Complete low-order arrays and their domains. Require (9.24), (9.25), (9.31), (9.35), and (9.42), evaluated at their printed relaxed factor two and at every consumed low-order entry. In addition impose the following unscaled comparisons:

$$
\begin{gathered}
\eta_i \max(1, F_{i,0}, F_{i,1}) \le 1/8, \qquad \eta_i \sum_{s=2}^{9} \frac{18^s}{s!} \max_{q\le s} F_{i,q} \le (16e^2)^{-1}, \\
|z_{ij}^{\mathrm{in}}|^{\mathrm{bd}} \le z_+, \quad \int B_{ij}^{\mathrm{bd}} \le 1/2, \quad \mathfrak{D}_{ij} \le z_+/2, \quad 2\ell_j K_j^2 A_{i,2}^{\mathrm{bd}} \le z_+/2, \\
S_i/\delta_i \le C_{\mathrm{ph}} \Pi_i^{2-\mathsf{d}} N_i^{-(\bar\beta-\mathsf{h})}, \qquad \sum_{q>i} |W_q|^{\mathrm{bd}} \le 2C_T/\sigma_{i-1}, \\
\mathcal{U}(|D_{a'}[\Gamma_{<i}(Y_i + M_i a', t)]|)\ell_i \le \Gamma_{-,i}, \\
0 < \Gamma_{-,i} \le \mathcal{L}(\Gamma_{<i}), \qquad \mathcal{U}(\Gamma_{<i}) \le \Gamma_{+,i}.
\end{gathered} \tag{11.9}
$$

The letters in the second line are the initial, integral and curvature numbers of (9.27). The last line is evaluated from the base and complete profile values on the containing region; the signed product $2W\Gamma_{<i}$ is still evaluated by (9.2). If positivity of the total circulation after insertion is used, also require $\mathcal{U}(|\gamma_i|) \le \Gamma_{-,i}/2$. These value tests concern positive circulation, not its large gradient.

The base-distance test is that the computed center and rigid-tube enclosures are compactly inside the fixed containing region. The old-core tests are exactly (9.35); for every two-point consumer retain separately

$$
A_i^{\mathrm{inv}}([C_q(t), C_{q-1}(t)], t) \Subset B_{\rho_i^{\mathrm{core}}}, \qquad |N_i p_i \cdot A_i^{\mathrm{inv}}| < s_c \quad \text{on that segment.} \tag{11.10}
$$

This notation denotes the segment enclosure certified by the scalar inverse-map and separation arrays, not an additional unknown-field assumption. One may certify the already required clause by subdividing the segment and using those arrays. For example its endpoint label bound plus its computed inverse-Jacobian bound times the segment length is a sufficient expression when it is smaller than ρ_i^{core}; this particular sufficient expression is not required if the original segment enclosure is already certified. The same convention applies to its phase bound. No convexity of the nonlinear core image is asserted.

The older-gradient rate in (9.42) is specified without a realized derivative. Compute the positive first derivative of $\sum_{i<q} A_i \sin(\phi_q - \phi_i)$ on growth, using the factored relative-angle equations and the old scalar rate bounds in the proof of Lemma 9.2, then divide by $\sigma_{q-1}^2 s_q$. Include the base summand separately and take its sum with the scalar logarithmic rates of A_{q-1}, $|\zeta_q|$ and the adjacent angle. For $q \ge 2$, call its sum with $C_{\mathrm{in}}\Gamma_1^{\mathrm{gr}}\mathbf{1}_{q=2}$ the number $\mathfrak{s}_q$, where $C_{\mathrm{in}} \ge 1$ is the fixed first-parent rate constant from the first-history calculation. At $q=1$ use $\mathfrak{s}_1 = 0$: there is no older correction and the central growth coefficients are constant. In particular the first-parent contribution at $q=2$ is not deleted with the empty older shear sum. Require

$$
\mathfrak{s}_q \le C_s \Gamma_q^{\mathrm{gr}}/\Lambda_q. \tag{11.11}
$$

This is the older-clock sum meant in (9.42). In its calculation each differentiated old summand is $A_i' \sin d_{iq} + A_i \cos d_{iq}(\phi_q' - \phi_i')$; the factored difference is taken before absolute values. Thus the definition does not charge an unfactored shear sum.

Weighted history, return and the next datum. Use the complete coefficient histories (9.46) and both pulse costs (9.47). If $a_i^\pm, b_i^\pm$ are the prescribed positive growth windows and e_i^{bd} their logarithmic-ratio rate, require

$$
\begin{gathered}
K_i^{\mathrm{eig}} \ge \max\{\sqrt{a_i^+/b_i^-}, \sqrt{b_i^+/a_i^-}, 1\}, \quad 0 < \rho_i^{\mathrm{eig}} < 1, \\
2\sqrt{a_i^- b_i^-}\, \rho_i^{\mathrm{eig}} > e_i^{\mathrm{bd}}(1 + (\rho_i^{\mathrm{eig}})^2), \qquad C_R \text{ satisfies (9.48)}.
\end{gathered} \tag{11.12}
$$

The datum is the growing eigenline. The first hold has its actual nonzero upper-row integral in this formula, and all later windows belong to the tail. Fix $c_{*,i} = \int m_i^{\mathrm{bd}}$ using (9.28); require its full double-history estimate (9.29) and the first-layer integral from (11.5), together with the pointwise $\mathfrak{n}_i \le 1$ and datum tests of (9.31).

For the generic return retain the definitions preceding Lemma 8.3. Use the upper arrays of the complete jets for $\mathfrak{E}_i^D, \mathfrak{E}_i^G, \mathfrak{S}_i, \mathfrak{A}_i$ in (8.20)–(8.21); the angle bounds are the corresponding interval bounds from the relative-angle and segment arrays. Require the two sums in (8.21) to be at most $\delta_{\rm ret}$. For each of the thirteen factors printed after that equation, calculate the upper bound for its difference from one by the same signed formulas and require it to be at most $\varepsilon_{\rm fac}$. Require the computed entry deviations

$$\mathcal{U}(|\mathfrak{z}_e - 1|), \quad \mathcal{U}(|\mathcal{W}(t^1)|), \quad \mathcal{U}(|\upsilon(t^1) - 1|) \le \delta_{\rm ret}. \tag{11.13}$$

These are the outputs of growth and transition calculated with the relaxed errors, not hypotheses on an unconstructed return solution. The first return instead uses its exact scalar formulas. To specify its short-length test, let $c_\pm, c_r^+, k_\pm, v_\pm$ be the computed positive coefficient and entry-ratio endpoints in Lemma 7.5, and put

$$\begin{gathered} V_+ = \max\{v_+, \sqrt{c_r^+/k_-}\}, \qquad V_- = v_-/(1 + k_+ v_-), \qquad B_{\rm ret} = 2(V_+ + c_p c_+ + 1), \\ c_p c_- > V_+, \qquad k_+ B_{\rm ret}^2 / \Lambda_{{\rm ret},1} \le \min(1, V_-/2). \end{gathered} \tag{11.14}$$

These are exactly the positive coefficient and short-length inequalities on that fixed first box.

For completeness, make the scale and anchor comparisons explicit too. Let $a_m^{{\rm end},-}$, $a_m^{{\rm end},+}$ be the computed endpoint bounds for A_m: begin with $-T_m = N_m^{-\bar\beta}$ at the hit, integrate the scalar transition and selected-return logarithmic gain bounds, and multiply by N_m and the endpoint covector bounds. Let $S_m^{\rm old} = A_0 + \sum_{i<m} \mathcal{U}(A_i(t_m^b))$ using the inherited amplitude intervals. Require

$$\begin{gathered} c_\sigma N_m^\beta < a_m^{{\rm end},-}, \qquad a_m^{{\rm end},+} + S_m^{\rm old} < C_\sigma N_m^\beta, \\ S_m^{\rm old} \le \tfrac{1}{2} \varepsilon_s a_m^{{\rm end},-}. \end{gathered} \tag{11.15}$$

These are the scalar endpoint inclusions associated with the strict gain margins (11.7). Their lower gain is $1 + I_0$ in a generic stage and $1 + \log 2$ in the first stage, with the calculated perturbations retained. No derivative or norm of a future field is used in these three tests.

For holding and reinsertion require the explicitly computed $\varepsilon_i^G \le \varepsilon_G$ of (9.49), (9.50), and the cone margin $e \cdot e_i, e \cdot e_b \ge \nu > \varepsilon_G$ certified by the scalar angular intervals. Keep separately the next-index contracted comparison

$$\mathcal{U}(\text{transverse error at } Z) \le \varepsilon_s \sigma_m^2 s_{m+1}, \qquad \text{parent lower contribution} \ge c_\theta \sigma_m^2 s_{m+1}/5. \tag{11.16}$$

The expressions are the contraction of (9.36) with the new transverse direction, using (9.38) and the selected full-flow offsets. Also test the two positive insertion coefficients on $B_{2\ell_{m+1}}$ by (9.31). In particular this is not a full-matrix smallness condition.

The seed convention and activation test are

$$T_i^{\rm seed} = -N_i^{-k_i - \mathsf{e}}, \qquad Y_{{\rm act},i}^{\rm bd} := N_i^{-k_i - \mathsf{e}} \sqrt{1 + (u_i^+)^2} \exp(g_i^+ / \Gamma_i^{\rm gr}) \le C_{\rm rp} N_i^{-k_i - \mathsf{e}}. \tag{11.17}$$

Here u_i^+ and g_i^+ are the upper ratio $\widehat{\Omega}_i / T_i$ and logarithmic growth rate supplied by the scalar growing-line comparison on the short activation window. Thus $C_{\rm rp}$ is a fixed ramp constant. The growing-line bounds $u_i^+ \le 1 + O(\varepsilon_s)$, $g_i^+ \le (1 + O(\varepsilon_s))\Gamma_i^{\rm gr}$ give this bound directly, without a bound over the whole growth interval. Fix $\mathsf{e} > \bar\beta$. The desired logarithmic gain and the scalar test placing activation before its target are $\Lambda_i = (k_i + \mathsf{e} - \bar\beta) \log N_i$ and $\Lambda_i > g_i^+ / \Gamma_i^{\rm gr}$, respectively.

Higher constants, true sizes and clock moments. Use the full generation recursion (10.24) and the majorant $G(m, k_m)$ of (10.27), with the fixed-order profile function $\mathcal{K}_{\rm ho}$ in (10.25). Require (10.29), (10.39), and $\mathcal{K}_{\rm ho}(k_m) \le \log N_m$, as well as all four finite true-size rows (10.21). The complete-level row uses (9.25) at each actual term and retains the relative tails (9.24), including their first rates. Set $G(1, k_1) = 1$; there is no older generation at the first layer. In the second inequality of (10.29) fix c to be one sixteenth of the minimum of the five positive numbers in (11.4) and $\bar\beta$. Thus no exponent in that test remains unspecified. For the own young calculation also require

$$\begin{gathered}\eta_j \max(1, f_{0,0}, f_{0,1}) \le 1/8, \qquad K_j\eta_j f_{0,2} \le \zeta_-/2,\\ \ell_j \max(K_j\mu_{<j}, \lambda_{\Gamma,j}, \lambda_{\mathrm{met},j}) \le 1,\\ G(m,k_m)\max\{K_m\eta_m/\zeta_-, \eta_m, K_m\mu_{<m}\ell_m, \lambda_{\Gamma,m}\ell_m, \lambda_{\mathrm{met},m}\ell_m, \lambda_{R,m}\ell_m\} \le 1.\end{gathered} \tag{11.18}$$

The density in the non-affinity representation is the produced nonnegative function, with $\int \mathfrak{d}_j \le 1$ and $\mathfrak{d}_j \le \mathscr{R}_j$. Test the exact drift formula (10.16) in its units:

$$\ell_i^{-1}\mathcal{U}(|\dot a_i|) \le C_{\mathrm{cv}}\mathscr{R}_j, \qquad \mathcal{U}(|N_ip_i\cdot \dot a_i|) \le \mathscr{R}_j, \qquad 4\zeta_+N_{j-1}^{1-\mathsf{a}} \le 1. \tag{11.19}$$

The displayed exact positive expressions, not the coarsened $o(1)$ notation in (10.17), are used for this test. In particular the pointwise nonlinear remainder need only be below its scalar rate, not below one.

Require (10.33). Define $D_i^+ = G(i+1,k_{i+1})a_{\Delta,i}$, and use the separation and direction numbers specified immediately after that equation. The remaining comparisons there are

$$\begin{gathered}D_i^+d_i^{0,\mathrm{bd}} \le \varepsilon_{\mathrm{sch}}\lambda_{i+1}^*, \qquad \sum_{i\le q-2} D_i^+(\mathfrak{x}_i+\mathfrak{m}_i)\varrho_{i,q} \le \varepsilon_{\mathrm{sch}}\lambda_q^*,\\ 2C_W\sum_{i\le q-2}\sigma_iD_i^+\mathfrak{r}'_{i,q} + 2\Omega_{q-1}^{\mathrm{box}}\vartheta_q^{\mathrm{box}}D_{q-1}^+\mathfrak{r}_{q-1,q} \le \varepsilon_{\mathrm{sch}}s_q\Gamma_q^{\mathrm{gr}},\\ \varepsilon_{\mathrm{ctr},q} \le K_{\mathrm{ctr}}\varepsilon_{\mathrm{sch}}.\end{gathered} \tag{11.20}$$

In the second line Ω^{box} is the absolute endpoint of its appropriate relaxed all-trial interval and $\vartheta_q^{\mathrm{box}} = 2C_\vartheta s_q$. In (10.34), likewise replace $|\Omega_i|$ and $|s_i^c[q]|$ by their scalar interval and phase-radius bounds. This defines the last line entirely before the trial. Its geometric and reciprocal defects are evaluated from c_g, $2W\Gamma$, $\kappa/|\zeta|^2$ and their derivatives using (9.15), not by assigning corrector decay to $|\zeta|^2/\kappa$.

For clarity the clock conditions, including the unlabelled first terms, are collected here. Set $\mathscr{R}_i = \max(1, r_i)$, with the literal r_i before (9.9); use $X_1 = \Lambda_{\mathrm{ret},1}\Gamma_1^{\mathrm{gr}}$, $\lambda_1^* = c_p\Lambda_{\mathrm{ret},1}\sin s_1$, and $X_l = \Lambda_l\Gamma_l^{\mathrm{gr}}$, $\lambda_l^* = \Lambda_l s_l$ for $l \ge 2$. The ordinary list begins at $l = i+1$, whereas the augmented list has the additional term $1@X_i$. Both retain the separate floor $1@\mathscr{R}_i$, for $i \ge 1$. The ordinary base list is $\upsilon_{0,l} = \sigma_0/\sigma_{l-1}$, $l \ge 1$; its first term is $1@X_1$, and the rest is dominated by the index-one list. Require

$$\begin{gathered}N_1 \ge N_0, \quad \Pi_i \le \Pi_{i+1}, \quad \mathscr{R}_i \le \Pi_i, \quad X_l \le \Pi_i \ (l \le i+1),\\ \sum_l \lambda_l^* \le 1, \quad 2 + \sigma_i\sum_{l\ge i+2}\sigma_{l-1}^{-1} \le 3,\\ \sum_{l\ge i}\upsilon_{i,l}^+\max(\mathscr{R}_i, X_l) \le C_1\Pi_i, \qquad \sum_l \lambda_l^*\max(\mathscr{R}_i, X_l) \le C_1\Pi_i,\\ \upsilon_{i,l} \le C_\lambda\lambda_l^* \ (l \ge i+2), \qquad \mathscr{R}_i, X_l \le \widehat\Pi_m \quad (i \le m, l \le m+1,\ m \ge 2),\\ (\log N_m)^{10} \le \widehat\Pi_m, \qquad (\log N_m)^{30} \le \Pi_m \quad (m \ge 2),\\ \lambda_1^*X_1^r + \lambda_2^*X_2^r \le \tfrac12\Pi_1^r \quad (r = 1,2), \qquad \sum_{h\ge 0}\exp[-\beta(q_0+1)^h\log N_1/4] \le \tfrac12,\\ \max(1, r_1) \le Q_2^4(\log N_1)^2\sqrt{C_\sigma}N_0^{\beta/2}.\end{gathered} \tag{11.21}$$

Here $\widehat\Pi_m$ is exactly that of (10.39); both logarithmic-head comparisons are for $m \ge 2$. At $m = 1$, use the separate first terms and native Π_1 moments proved in Section 10, including the base, augmented own, rotation and fixed-floor entries. Retain (10.41) for the generic tail, its $B+3 \le Q_m$ order condition, and the finite first-future comparison

$$\mathsf{D}_{m+2} \le Q_{m+2}^3 \log N_m, \qquad \mathsf{D}_1 = \Lambda_{\mathrm{ret},1}, \qquad \mathsf{D}_l = \Lambda_l \quad (l \ge 2). \tag{11.22}$$

The last line distinguishes pulse duration from derivative-list amplitudes. For finite histories the infinite positive series in (11.21) is just a prescribed scalar upper bound for their tails; it contains no future fields.

Finally form all young coefficient sizes, fixed-profile constants and level constants in Section 6 from these arrays. Require every quotient in (6.3) to be at most one, with

$$\mathcal{T}_m = \max(|I_m|^{\mathrm{bd}}, \Pi_m^{-1}), \qquad \lambda_m = \max(1, \ell_m^{-1}), \qquad Y_{*,m} = C_\Theta N_m^{-\bar\beta}. \tag{11.23}$$

The finite time bound includes the history to the last window under consideration. The scalar C_* is the maximum of the produced normalized young entries through r_X and the relative data through r_*, with the fixed profile derivatives included as in (6.4). These definitions, the explicit comparisons above, and the referenced finite tables are the complete sufficient scalar list. No assumed mixed field norm is an entry in it.

11.2. **The joint finite statement.**

Proposition 11.1 (Finite continuation and quantitative input return)**.** *Suppose the preceding data and scalar comparisons hold for a finite initial sequence. Begin each insertion from its common corrected selected past. Then the actual finite Euler construction exists through growth, transition and both return pulses, for the full compact interval $0 \le \mu \le c_p$ of the current trial parameter. Its older layers satisfy the low-order and admitted higher-order estimates on every such trial. The zero set of the final principal vorticity is a nonempty compact subset of $(0, c_p)$; its least member selects a causal next history. No derivative of that choice is asserted.*

On the selected history the youngest complete layer, its hold and its full-flow insertion path satisfy these estimates too. The complete circulation gradient has the positive cone lower bound (9.51), and the distinct next-index transverse bound supplies the growing datum on its insertion collar. The scalar growth-scale intervals (11.3) are returned.

For every $k \le k_m$, the same fields supply the amplitude inputs through r_*, the coefficient inputs through r_X, the rigid gradients and nonlinear remainder through r_F, and the primitive older orders r_F+1, r_F+3. The actual corrections, terminal composites and both means satisfy (6.5), (6.6) and (6.7). Their physical force obeys

$$\begin{aligned} \|f_m\|_{C^k_{x,t}(I_m)} \le C_{k,\mathcal{D}} C_k^{\mathrm{ev}} \Bigg[\Lambda^k_{\mathrm{ph}} (1 + N_m/\sigma_{m-1}) \left(E_{r_*} \Pi_m Y^{\mathrm{bd}}_{\mathrm{act},m} + \frac{C_\Theta N_m^{-\bar\beta}}{\mathcal{T}_m} \sum_{g=J_m+1}^{2J_m+4} s_g \delta_m^g \right) \\ + 2\lambda_m \Lambda^k_{\mathrm{slow}} \sum_{n=1}^{J_m} (M_n^\Gamma + M_n^\xi) \Bigg], \end{aligned} \tag{11.24}$$

with the literal constants and rates of Section 6. By (11.17) this is at most the specialized expression (10.45). These assertions retain the exact physical scaling to any prescribed positive radius and any prescribed height.

Separately, on an infinite selected history satisfying the same scalar conditions at every finite initial sequence and $k_m \to \infty$, each fixed layer has bounded derivatives of every fixed order over its whole preterminal lifetime. The bound may depend on that layer and order. This last conclusion does not assert uniform older estimates on cutoff annuli or beyond the admitted rectangles, an infinite parameter choice, or summability of the complete force.

Proof. We give the dependency argument, keeping the long coefficient and constant calculations in their preceding lemmas.

Local family and low-order production. Proposition 7.2 starts the actual finite family from its common past. Its spatial allocation can be enlarged to every finite order needed here. It provides smooth finite solutions and continuous dependence on the current pulse parameter, not the desired uniform constants. Fix the comparison phase and label radii before starting the family. For each complete-center or displacement error let $\beta_\nu(t)$ be its positive expression from (9.38) or (9.34), including its inherited datum. Temporarily impose $2\beta_\nu$, the relaxed scalar boxes, and the prescribed strict domain inclusions on an initial segment.

On this segment Lemma 9.2 applies with exactly (11.6), (11.11), (9.42) and the fixed margins (11.8). Its angular calculation uses the factored old matrices, the unmatched parent contraction, and the two-point and direction errors separately. For $1 \le i \le m-2$, write $\mathcal{X}_i$ for the aligned scalar correction in (9.43), distinct from the flow X_i. Then

$$\mathfrak{q}_i\Omega_i + \mathcal{X}_i \ge c_W\left[\frac{1}{2(1+\varepsilon_s)} - \frac{1}{4}\right]\sigma_i > 0, \qquad \mathfrak{q}_i = |\zeta_i|^2/\kappa_i.$$

The principal factor is positive and bounded, not a decaying additive error. Its full matrix is retained in the matrix and propagator consumers. The scalar comparison returns the stricter amplitude, angle, component and ratio intervals. On growth the quotient $\widehat{\Omega}/T$, divided by its positive equilibrium, has an inward vector field at both endpoints. Hence $T < 0$ and its logarithmic derivative lies between the specified positive growth rates. A target logarithmic gain is reached once, transversally, between the corresponding two scalar times. The transition comparison returns (11.13) and the coefficient factors. Nothing in this step presupposes existence of the end of return.

In increasing layer order, (9.35) places the young rigid tube in each required old core. The Taylor integral (10.15) therefore uses actual old-core derivatives. The flow recursion gives $|\Xi| \le \ell/8$, $\|D_a\Xi\| < 1/4$, and the full inverse map. In particular the nonlinear image of B_ℓ stays inside the rigid $B_{2\ell}$ region; these are not identified. The coefficient recursion retains the vector $\operatorname{div}_a C$, inverse curvature, the rotation-dependent reduced vorticity, and the product $2W\Gamma$. The tube and variation tests give the required unscaled first-label numbers.

Here is precisely the weighted subargument being used. The growing eigenline, the cone inequality in (11.12), and (9.48) give $\|P_0(t,s)\| \le C_R\rho(t)/\rho(s)$. The middle interval includes the transition and both pulses. A selected hold contributes $1 + H_i^{\mathrm{hist}}$, with H_1^{hist} retained, and the entire later tail contributes its exponential. These are the four parts of the proof underlying (9.48). Before selecting the current return only its existing parts are used. For a fixed label, mean value and (9.28) give $|\mathcal{B}(a,t) - \mathcal{B}(0,t)| \le m(t)$. Weighted variation of constants gives

$$\sup_{s\le u\le t}\frac{\rho(s)}{\rho(u)}\|P_a(u,s)\| \le C_R + C_R c_* \sup_{s\le u\le t}\frac{\rho(s)}{\rho(u)}\|P_a(u,s)\| \le 2C_R.$$

The same identity gives a propagator difference at most $2C_R^2 c_*\rho(t)/\rho(s)$. Including the relative datum error gives $|R(a,t) - R(0,t)| \le 3C_R^2(c_* + c_{\mathrm{dat}})|R(0,t)| < |R(0,t)|/2$. This uses exactly the central history, integrated first-label variation and datum hypotheses of the initial proof of Lemma 3.1. No mixed or peak conclusion of that larger lemma is used to establish this step. Higher label derivatives follow by the triangular pair recursion and the first time derivative by its equation.

For $i < m$, now form every level through J_i, in increasing level order, using (9.21). Its direct term reads a lower level and its integral reads the already produced pair coefficient. The complete relative tails in (9.24) and all true-size routes give the full center formulas, not a truncation at level two. In the phase offset the own fast velocity cancels before estimating; integrate this offset first, then the full label offset. The results are precisely β_ν, not $2\beta_\nu$. At an insertion the earlier full-flow offset is inherited, not reset. Estimates on two-point differences use (11.10) throughout. If an integral begins at zero, its undivided integral inequality is used at that time.

These strict recoveries exclude a first exit. They also give bounded α, positive symbols and circulation, a nonzero current pair and all active $\zeta_{j,2}$ denominators separated from zero. The physical support lies strictly in the fixed domain. On the compact finite interval, smooth dependence of the flow and the strict inclusion margins provide an intermediate collar $\ell_i < \ell_i' < 2\ell_i$ with compact transported image. The choice is uniform on the compact trial set. This supplies (5.1); it does not assert transport bounds on all of $B_{2\ell_i}$. The prescribed smooth profiles extend past the endpoint. Proposition 7.3 therefore excludes a finite terminal endpoint before the asserted horizon. The growth hit, transition and both pulses consequently belong to one common continuous trial family.

Return, selected holding and the next insertion. The exact return equations are (8.19). Their error sums use complete local jets and retain every aligned angle factor. The thirteen-factor tests give $|c_i-1|, |d_i| \le \delta_{\mathrm{ret}}$, with $c_4 = c_m(0,t)z_e/\sin s_m$ and $c_6 = c_m(0,t)$ still time- and trial-dependent. Proposition 7.7 recovers the strict geometric intervals and $T_m < 0$ on all trials. On the negative pulse, in its rescaled variable,

$$V(y) = v_1 - \mu \int_0^y c_6\eta_2 - \Lambda_m^{-1} \int_0^y \mathfrak{k} V^2.$$

At $\mu = 0$ reciprocal comparison gives a positive endpoint. At $\mu = c_p$, its endpoint is at most $\bar{v} - c_p(1-\delta_{\mathrm{ret}}) < 0$. The first return has the same two signs from its exact positive coefficient comparison. Continuity gives a nonempty zero set; it is closed in a compact interval and avoids both endpoints. Its least element defines the causal return. This selection is made once; it is not differentiated with respect to inherited data.

On that member $\zeta_{m,1} = \widehat{\Omega}_m = 0$ throughout the prescribed hold, so the principal pair is held. The complete youngest correction can now be formed and estimated on this selected history. Continue Z with that complete field. Its offset equation is (9.32); the same phase-first, then full-label, calculation supplies the next insertion point and its inherited old offsets. No displacement is reset at this step.

The complete gradient is not frozen: (9.49) and the cone and ramp tests give

$$-e \cdot \nabla\Gamma(Z,t) \ge (\nu - \varepsilon_G)\left(A_0 + \sum_i w_i A_i\right) > 0.$$

This is the complete-field holding estimate. The different contraction (11.16), with $8\varepsilon_s \le c_\theta$, gives positive insertion coupling in the new transverse direction. Its collar test supplies the full growing datum. Thus the complete next state is the state used by the next application of Proposition 7.2. The recovered angular intervals, with $s_1 \le 1/8$, also give $e_i \cdot e_m > 0$ at this endpoint. Consequently the norm in (11.2) lies between A_m and $A_m + S_m^{\mathrm{old}}$. Equation (11.15) returns (11.3) and $1 - \varepsilon_s/2 \le A_m(t_m^b)/\sigma_m^2 \le 1$. The complete-gradient lower bound was proved separately above and is not substituted into this clock definition.

Higher derivatives on that same family. The low-order result is the base, not an assumed higher-order window theorem. At time order $n+1$, (10.35) reads complete centers only through n; it produces adjacent angles, radii and principal amplitudes. The control and top pair follow, then the frame and map in increasing layer order, then the refunded equations in increasing correction level. Their direct terms read only lower levels at the same order; the integrated scalar rotation row reads α through the preceding order. Form the complete effective center values and total transverse gradient last. They enter the next primary step only at its previous order. A younger velocity in a moving reading likewise occurs only at that previous time order. Thus this is the actual time-order, internal-rank, layer and level induction of Section 10.

The enlarged rectangles (11.1) pay the old primitive orders used on the containing rigid tube. The exact nested-tube and segment conditions justify each domain of evaluation. The contracted unmatched angular term is differentiated as an identity; mixed tensors containing ζ_i vanish in that contraction. Its small bound does not replace the full matrix in the other consumers. The multiplicative principal metric and the full direction term in (11.20) remain in the differentiated lists.

The positive recurrence (10.24), its full-depth tree count and (10.29) bound all resulting constants by the prescribed majorant at their true sizes. An intrinsic old coefficient uses its own certifying carrier N_i, whereas the comparison for a new generation uses N_{m-1}. The proof does not absorb an arbitrarily old factor at the newest carrier. Purely spatial young factors are paid by (11.18). At positive time order the gap (10.39) applies to the total differentiation order of the marked factors in each monomial. The finite polynomial count in the generation proof is retained before this absorption; a geometric sum alone would not count those monomials.

The moment split in (10.42) uses the head, the first future term, and the exponential tail. Conditions (11.21)–(11.22) supply each part, with its literal first compression and its floor. The raised-moment identity (10.43) retains the extra amplitude mass; it does not silently remove the own term of the augmented list. These are precisely the ordinary, augmented and rotation moments consumed by (10.38). Consequently the four higher-order estimate forms and their fixed-order constants apply to the same complete fields just continued above.

Return of the material and force inputs. On the young support the own spatial tests and the time gap give the relative amplitude derivatives through r_*, including mixed pointwise coefficient variation separately from $\int m$. The central weighted bound is the one already proved, not a newly assumed propagator. The map estimates supply $M^{\pm 1}$, D, $\dot{Y}$, the central covector and the rigid nonlinear remainder through r_F, as required by (4.5). The first zero-time remainder has its integrable density; positive time derivatives have their produced clock bounds. The full inverse, its curvature, and the metric and older-scalar products then supply (4.11) through r_X, with the sharper Piola identity (4.12). In particular $\nabla_a b$ retains $\alpha D_a b^{\mathrm{rot}}$. The primitive supplies are $r_F + 1$ for circulation and $r_F + 3$ for streamfunction. The initial $B_{2\ell}$ datum calculation is separate from the subsequent young support estimates. Fixed smooth-profile derivatives and the fixed-order factor are included before defining C_*.

The pointwise weight remains $Y_* \rho(t) = |R(0,t)|$ in all these differentiated equations. On activation, the scalar logarithmic growth and ratio bounds integrate over its length $(\Gamma_m^{\mathrm{gr}})^{-1}$, giving exactly (11.17); the fibre comparison supplies the fixed relative factor used by the activation profiles. This is the short-ramp growth calculation, not replacement of the instantaneous weight by its lifetime maximum inside derivatives.

Every quotient in (6.3) now concerns produced quantities. In particular its fast circulation-square row is $\mathcal{T}_m \sigma_{m-1} \varpi_* z_0 Y_* / \delta_m \le 1$, with no inverse-frequency gain. Proposition 6.1 therefore applies at all retained levels. Pairs stop at J_m, elliptic composites continue through $J_m + 2$, and raw terminal sources through $2J_m + 4$. The largest source order at force order k is $k + 2(2J_m + 4) = r_* - 1$, leaving the stated extra derivative for activation and the mean primitive. The two means are exactly (6.7); the reciprocal-symbol derivative and circulation-square terms of the vorticity mean remain. The mean primitive is estimated through $k + 1$, its divergence through k, and complete gradient and Hessian evaluation through $k + 2$.

Finally physical differentiation uses the complete material drift HX_t. Its oscillatory and mean rates are respectively Λ_{ph} and Λ_{slow}; no phase frequency is charged to a mean. The compact inverse is applied to activation plus terminal remainder together and to each compatible mean separately. This proves (11.24) by the actual proof of Proposition 6.2. The fixed chart and the unit-ring scaling give the asserted physical interpretation.

For the final, separate infinite-history assertion fix a derivative order. Eventual admission and the exponential clock tail give the required rotation derivative on every sufficiently late window. The finitely many earlier windows are a smooth finite beginning. The resulting fixed-order rotation bound, followed by induction on the fixed layer, gives its flow derivatives, inverse derivatives, coefficient derivatives and the finite correction hierarchy over its whole finite preterminal lifetime. On its outer annulus use its fixed smooth cutoff with a constant depending on that layer and order, as in the last part of Section 10. This proves fixed-layer finiteness and nothing outside the stated quantifiers. $\square$

12. One sequence and the smooth physical force

We choose the parameters in Proposition 11.1 once. The choice will work at every finite stage, and its admitted derivative orders will tend to infinity. Throughout this section all estimates preceding the final physical change of variables are in the magnified unit-ring coordinates of Lemma 2.3.

12.1. **Fixed data and the initial frequency.** Fix the compact recovery operator, the smooth cutoff and pulse profiles, and a compact containing region on which the base has the form

$$\Gamma^b = \Gamma_0 - A_0 e_b(t)\cdot \mathbf{y}, \qquad \Psi^b = \alpha(t)|\mathbf{y}|^2/2, \qquad \Gamma_0 = (2\varepsilon_{\mathrm{geom}})^{-1}.$$

Here $\mathbf{y}$ denotes the two-dimensional magnified coordinate. The actual base fields are their prescribed compact extensions. Fix $A_0 > 0$, $\sigma_0 = \sqrt{A_0}$, and the first pulse compression $\Lambda_{\mathrm{ret},1}$ so that (11.14) holds with strict margins. First choose the pulse amplitude interval and its positive coefficient endpoints; increasing the fixed compression makes the second inequality in that test strict. Then choose $s_1 > 0$ sufficiently small to meet the two first-angle conditions following (11.6). Set $\Gamma_1^{\mathrm{gr}} = \sigma_0 \sin s_1 > 0$. This compression is not the first logarithmic gain.

Choose the remaining fixed windows, $G_\pm, c_\sigma, C_\sigma$, the cone margin, and the tolerances by (11.7)–(11.8). Choose the containing and recovery radii first, and then one positive $\varepsilon_{\mathrm{geom}}$ sufficiently small for Lemma 2.3 and for the fixed metric margins. In particular the product $2W\Gamma^b$, not separate large bounds for its two factors, is used in these choices. The finitely many strict margins remain positive after these choices. The central four-part history calculation (9.46)–(9.48) now fixes C_R; the first hold contributes its fixed, possibly nonzero, upper-row integral. It is not required to tend to zero. Also fix the base carrier $N_0 > 0$ from (2.7), and the base linear-phase half-width $0 < s_0 \le s_c$ and first-collar constant $C_\ell > 0$ of Section 9. These first-layer data are fixed before $\mathcal{K}_{\mathrm{ho}}$.

Here is a convenient non-extremal choice of exponents:

$$\beta = \frac{1}{8}, \quad \bar\beta = \mathsf{h} = \frac{7}{8}, \quad \mathsf{a} = 4, \quad \mathsf{p} = 8, \quad \mathsf{q} = 10, \quad \mathsf{r} = 40, \quad \mathsf{d} = 16, \quad \mathsf{e} = 8. \tag{12.1}$$

The auxiliary q is only the upper exponent for the gauge. Choose fixed integers $c_Q \ge 2^{18}$, $q_0 \ge 2^{20}$, enlarging them in the finite inequalities below, and put

$$Q_i = q_0 + i, \qquad \mathsf{u} = \mathsf{q}/c_Q, \qquad \mathsf{v} = \mathsf{q}/(q_0+1), \qquad \mathsf{h}_* = \mathsf{h} - \mathsf{d}\mathsf{v}.$$

We require

$$\mathsf{d}\mathsf{v} < \frac{1}{16}, \qquad 8\mathsf{u} + 100\mathsf{v} < \frac{1}{16}, \qquad \frac{9}{c_Q} + \frac{51}{q_0+1} < 1. \tag{12.2}$$

Any finite constants in the power comparisons below are fixed before the last enlargement of q_0. In particular, if C_o is the largest fixed exponent of an earlier carrier in those comparisons, also require $C_o/(q_0+1) < 1/64$. These are finite choices: none involves a derivative order or a realized field. The five numbers in (11.4) are then positive; in particular

$$\mathsf{h}_* > \frac{13}{16}, \qquad \mathsf{a} - 2\beta = \frac{15}{4}, \qquad \mathsf{c} = \mathsf{a} + \mathsf{h} - 1 - \frac{3}{2}\beta = \frac{59}{16}.$$

Let $c > 0$ have exactly the value specified after (11.17), namely one sixteenth of the minimum of those five numbers and $\bar\beta$. Fix C_s by the older-gradient calculation below, and then the profile-order function $\mathcal{K}_{\mathrm{ho}}$ of (10.25) with these data, including the fixed clock-moment constants. It is fixed before y and every depth.

For $y = \log N_1$, define

$$\begin{gathered} N_1 = e^y, \qquad N_i = N_{i-1}^{Q_i} \quad (i \ge 2), \\ \Pi_1 = N_0^{\mathsf{p}} y^{\mathsf{r}}, \qquad \Pi_i = N_{i-1}^{\mathsf{p}} (\log N_i)^{\mathsf{r}} \quad (i \ge 2), \\ \ell_1 = \frac{s_0}{C_\ell N_0 y^2}, \qquad \ell_i = N_{i-1}^{-\mathsf{a}} \quad (i \ge 2), \qquad \delta_i = \Pi_i^{\mathsf{d}} N_i^{-\mathsf{h}}. \end{gathered} \tag{12.3}$$

The calculation below enlarges the threshold in y to meet the first-collar tests (8.23); N_0, s_0, C_ℓ remain fixed. There is no recurrence defining N_1 from N_0. Set, before constructing any trajectory,

$$k_i = \max\{k \in \mathbb{N}_0 : k \le Q_i/c_Q,\ \mathcal{K}_{\mathrm{ho}}(k) \le \log N_i\}, \qquad J_i = 2k_i + 8,$$
$$T_i^{\mathrm{seed}} = -N_i^{-k_i-\mathsf{e}}, \qquad \Lambda_i = (k_i + \mathsf{e} - \bar{\beta}) \log N_i. \tag{12.4}$$

For $i \ge 2$, the insertion angle and growth rate are chosen causally from the already selected clocks:

$$s_i = \Lambda_i \sigma_{i-2}/\sigma_{i-1}, \qquad \Gamma_i^{\mathrm{gr}} = \sigma_{i-1} \sin s_i.$$

Every estimate involving a future clock uses its two prescribed endpoints $c_\sigma N_i^\beta$ and $C_\sigma N_i^\beta$.

Lemma 12.1 (Uniform elementary comparisons)**.** *There is one finite y_0, depending only on the fixed choices, such that, for $y \ge y_0$, the sets in (12.4) are nonempty and nested, $k_i \to \infty$, and*

$$x_i := \log N_i = y \prod_{j=2}^{i} Q_j, \qquad x_i \ge (q_0+1)^{i-1} y, \qquad Q_i \le x_{i-1} \quad (i \ge 2),$$
$$\Pi_i \le N_i^{\mathsf{q}/Q_i} \le N_i^{\mathsf{v}}, \qquad \delta_i \le N_i^{-\mathsf{h}_*} \le \frac{1}{2}. \tag{12.5}$$

Every fixed finite family $Cx^A e^{-bx}$, $b > 0$, is uniformly small at $x = x_i$, and its sum over i is small, on one such ray. The same assertion holds with a fixed power of Q_i or i. It also holds for the generation factors in the two comparisons (10.29), with exactly their stated bases.

Proof. Nonemptiness follows from $y \ge \mathcal{K}_{\mathrm{ho}}(0)$. Both upper limits in (12.4) increase to infinity; thus every fixed integer eventually belongs to the set. The frequency identity gives the first line of (12.5). Choose y so large that

$$40 \log(Q_2 y) \le 2y, \qquad \frac{\mathsf{q}}{Q_1} y - 40 \log y \ge \mathsf{p} \log N_0.$$

The first inequality propagates, because $x/\log x$ increases for $x > e$ and

$$\frac{Qx}{\log(Qx)} \ge \frac{Q}{2} \frac{x}{\log x} \quad (Q \le x).$$

It proves the gauge estimate for $i \ge 2$; the second is exactly its exceptional first instance. The assertions for δ_i follow. For any $A \ge 0, b > 0$, the function $x^A e^{-bx/2}$ is bounded and eventually decreasing. Hence

$$\sum_{i \ge 1} C x_i^A e^{-b x_i} \le C_{A,b} \sum_{i \ge 1} e^{-b(q_0+1)^{i-1} y/2} \le C_{A,b} \frac{e^{-by/2}}{1 - e^{-by/2}}.$$

Fixed powers of Q_i and i are absorbed by a fixed power of x_i, after increasing y.

For the remaining assertion use the full, already proved generation bound (10.28), not a bound polynomial in the generation:

$$\frac{\log G(m, k_m)}{\log N_{m-1}} \le C_3 (m + q_0)^{12} 4^{-(m-2)} \frac{1 + \log y}{y} \qquad (m \ge 2). \tag{12.6}$$

The supremum of the coefficient in m is finite, since it is at most its convergent sum. Thus this ratio is smaller than any fixed positive number on one ray. The extra logarithm of $(Q_m x_{m-1})^2$ divided by x_{m-1} tends to zero uniformly, by the same elementary comparison. This proves the first test (10.29). Apply (12.6) with $m = i + 1$: its denominator is then $\log N_i$. Together with $a_{\Delta,i} \le G(i+1, k_{i+1})$, this proves the second test there, including $i = 1$, whose intrinsic low constant is fixed. No estimate of an old constant has been transferred to a newer base. $\square$

12.2. **Verification of the scalar hypotheses.** All constants below are computed upper bounds for the positive scalar expressions in Section 11. A symbol C is permitted to increase only as a function of the fixed profiles, windows, geometry and the fixed integers. It is not a constant chosen anew at a layer.
Scalar times, angles and histories. Write $b_i^{\mathrm{gain}} = k_i + \mathsf{e} - \bar\beta$. Then

$$7 < b_i^{\mathrm{gain}} \le Q_i/c_Q + 8, \qquad Q_i \le \Lambda_i/\Lambda_{i-1} \le Q_i^2 \quad (i \ge 2).$$

For the upper bound use monotonicity of k_i, $b_{i-1}^{\mathrm{gain}} > 7$, and $b_i^{\mathrm{gain}}/b_{i-1}^{\mathrm{gain}} \le Q_i$. The lower bound exceeds 100. All Λ_i^{-1} are small once Λ_1 is large. The bounds $\sigma_i \asymp N_i^{\beta/2}$ used here mean their prescribed scalar intervals, not an assumption about the current solution. For $i \ge 3$,

$$\frac{\sigma_{i-2}}{\sigma_{i-1}} \le C \exp\{-\tfrac{\beta}{2}(1 - Q_{i-1}^{-1})x_{i-1}\},$$

and the instance $i = 2$ has the fixed numerator σ_0. Consequently every product of a fixed power of Λ_i, Q_i with this ratio tends uniformly to zero. For example,

$$\Lambda_i^4 \frac{\sigma_{i-2}^2}{\sigma_{i-1}} \le C(Q_i x_i)^4 \exp\{-\beta(1/2 - Q_{i-1}^{-1})x_{i-1}\}.$$

The first instance is the same expression with σ_0^2 fixed. These bounds prove the small-angle rows, the cubic clock separation and the first clock comparisons in (11.6).

They also give, uniformly,

$$\Lambda_i s_i \le \varepsilon_s s_{i-1} \quad (i \ge 3), \qquad \sum_{j>i} s_j \le \varepsilon_s s_i.$$

Indeed the ratio of consecutive terms is a fixed polynomial in Q_i, x_i times a negative exponential in x_{i-1}; make this ratio at most $\varepsilon_s/4$ and sum the geometric majorant. The exceptional ratio s_2/s_1 is one separate ray. The remaining sums follow by writing their summands in these same ratios:

$$\sigma_i s_{i+1} = \Lambda_{i+1}\sigma_{i-1}, \qquad \sigma_i^2 s_{i+1} = \Lambda_{i+1}\sigma_i\sigma_{i-1}.$$

Their increasing sums are bounded by twice the last term. The sum with s_{i+1}^2 is decreasing and is bounded by twice its first term, which is smaller than $\varepsilon_s \Lambda_2^{-2}$ on the first ray. Furthermore

$$\frac{h_j \Lambda_j}{\Gamma_j^{\mathrm{gr}}} \le 2 s_{j-1} \quad (j \ge 3), \qquad h_2 = 0.$$

The sums of σ_i and σ_i^2 follow from their successive ratios; the base term is included in the first comparison. The two comparisons involving A_0, and the domination of the old principal sum by the new principal amplitude, are the same negative-power comparisons.

At the first index, $\ell_1^{-1} = C_\ell N_0 y^2/s_0$ and $\Lambda_1 \le (Q_1/c_Q + 8)y$. Hence $\Pi_1 = N_0^8 y^{40}$ dominates ℓ_1^{-1}, the first duration, the fixed rates and their reciprocals, and $X_2 \le C Q_2^4 y^2 \sigma_0$. For every index, $(\log N_i)^{30} \le \Pi_i$, with no exceptional failure at $i = 1$. For $i \ge 2$, $\mathsf{p} > \mathsf{a}, \beta$ gives all the gauge-size rows. The first rate is $X_1 = \Lambda_{\mathrm{ret},1}\Gamma_1^{\mathrm{gr}}$; it is never replaced by $\Lambda_1\Gamma_1^{\mathrm{gr}}$.

The growing-line comparison has uniformly bounded eigenvalue ratios. Its logarithmic drift integral is $O(\varepsilon_s)$; the transition and the two pulse integrals have fixed bounds. The selected hold has zero lower row, its first upper-row integral has a fixed bound, and the later-window integral is a convergent clock sum. Substitution in (9.48) therefore gives the fixed C_R chosen above. The strict cone inequality in (11.12) holds by decreasing the fixed tolerance. On the first growth interval the explicit calculation in Section 8 gives

$$c_{*,1} + c_{\mathrm{dat},1} \le C\ell_1(1 + \Lambda_1) \le C/y.$$

Its later contributions are bounded by $C\ell_1$ times the fixed hold and tail integrals. Thus the full first variation and datum test follow on one ray. This is the integral calculation; it does not use a nonexistent parent carrier for layer one.

Spatial variations and complete corrections. We record the small powers in the actual scalar arrays. This also identifies the carrier used in each comparison. In the next display $x = \log N_{j-1}$ for an insertion at $j \geq 2$; for a correction of layer i, $x = \log N_i$. Here "fixed factors" means a fixed product of powers of x, Q_j and of earlier carriers, with the latter rewritten in the displayed carrier. Its logarithm is at most $C_o x/(q_0+1) + C_o \log(2+x)$. It never includes a generation constant.

The positive product rules of the preceding sections give the following bounds before the final comparison with the prescribed tolerance:

(12.7)

quantity, in its stated output unit	carrier	decay exponent
$\eta_j,\ \eta_j \max_{r\leq 9} F_{j,r}$	N_{j-1}	$\mathsf{a}-2\beta$
$K_j\eta_j,\ c_{*,j}+c_{\mathrm{dat},j},\ \mathfrak{n}_j$	N_{j-1}	c
$\ell_j K_j \mu_{<j},\ \ell_j \lambda_{\Gamma,j}$	N_{j-1}	$\mathsf{a}-3\beta/2$
$\ell_j \lambda_{\mathrm{met},j}$	N_{j-1}	$\mathsf{a}-\beta/2$
$\mathfrak{x}_i/\sigma_i,\ \mathfrak{g}_i/(\zeta_+\sigma_{i-1}\sigma_i)$	N_i	$2\mathsf{h}_*-\beta/2$
$\mathfrak{m}_i$	N_i	$\bar\beta+\mathsf{h}_*-\beta/2$
$\mathfrak{g}'_i$	N_i	$\bar\beta+\mathsf{h}_*$
$d_i^{0,\mathrm{bd}}/\lambda^*_{i+1}$	N_i	$\min(2-3\beta/2, \bar\beta+\mathsf{h}_*-\beta/2)$
first-rate transverse error	N_i	$2\mathsf{h}_*-\mathsf{v}$
individual circulation and center-drift increment	N_i	$\bar\beta$
phase and label displacement	N_i	$\min(\bar\beta, \bar\beta+\mathsf{h}_*, 2-3\beta/2)$.

The first four rows may carry the appropriate factors $G(j,k_j)$, and the second row may carry its square, from the coefficient and propagator calculation. The remaining differentiated correction rows carry $D_i^+ = G(i+1,k_{i+1})a_{\Delta,i}$, not the constant of the arbitrarily late observing window.

Here are the calculations behind the rows and their use. In (9.26), contract with the parent's covector before taking norms. The fast principal velocity has zero contraction. The remaining pure-phase second derivative is the level-one correction; with $\ell_j = N_{j-1}^{-\mathsf{a}}$, its integrated size is

$$C\,\ell_j N_{j-1}^2 N_{j-1}^{-\bar\beta}\delta_{j-1}\sigma_{j-2}^{-1}|I_j| \leq C\,N_{j-1}^{-(\mathsf{a}-2\beta)}\ \text{(fixed factors)}.$$

This is the first row. A coefficient variation or relative growing datum adds at most the factor σ_{j-1} and a logarithmic growth length, giving the exponent $\mathsf{a}-5\beta/2=\mathsf{c}$ in the second. The exact Piola identity (10.18) retains a map-curvature factor. The older scalar gradient and the metric derivative therefore give respectively the third and fourth rows after multiplication by ℓ_j. The derivative of the growing eigenvector is obtained by the reciprocal and square-root rules from these same two positive off-diagonal entries. No independent norm of a second frame is used. The fixed low partition sum is a finite positive combination of the first row; its maximal order is nine.

For the center rows, substitute $Y_{*,i} = C_\Theta N_i^{-\bar\beta}$ and $\delta_i = \Pi_i^{\mathsf{d}} N_i^{-\mathsf{h}}$ in (9.38) and (10.19). The aligned correction has size

$$C\Pi_i^{2\mathsf{d}} N_i^{\beta-2\mathsf{h}}/\sigma_{i-1};$$

division by the positive scale σ_i gives the bound for $\mathfrak{x}_i/\sigma_i$. For the aligned-gradient test,

$$\frac{\mathfrak{g}_i}{\zeta_+\sigma_{i-1}\sigma_i} \leq \frac{C}{\zeta_+\sigma_{i-1}}\Pi_i^{2\mathsf{d}} N_i^{\beta/2-2\mathsf{h}} \leq \frac{C}{\zeta_+\sigma_{i-1}} N_i^{-(2\mathsf{h}_*-\beta/2)}.$$

The first σ_0 is fixed and positive. The slow mixed Hessian has the displayed label and clock factors. The misaligned gradient obeys

$$\mathfrak{g}'_i = C'_G C_\Theta K_i \ell_i^{-1}\Pi_i^{\mathsf{d}} N_i^{-(\bar\beta+\mathsf{h})} \leq C'_G C_\Theta K_i \ell_i^{-1} N_i^{-(\bar\beta+\mathsf{h}_*)}.$$

These formulas apply directly to the two consumers $D_i^+\mathfrak{g}_i$ and $D_i^+\sqrt{2}\,\mathfrak{g}_i'$ in (10.33). Their exponents remain positive after the earlier-carrier and range factors are bounded. In the unmatched angular contraction (9.40), the two mixed tensors containing ζ_i vanish. The retained slow Hessian and inverse-curvature terms, divided by $\lambda_{i+1}^* = \Lambda_{i+1}s_{i+1}$, give the angular row. For differences of two centers, write the segment integral first. It retains the separation factor multiplying that same slow contraction. The factored aligned sum and the local-direction differences retain their sine factors. Their relative exponents are at least $\bar\beta + \mathsf{h}_* - \beta/2$. This proves the three center tests in (11.20), rather than substituting a full-matrix smallness estimate for them.

Differentiate the signed next-index transverse expression once. Its leading correction is the aligned gradient error multiplied by the adjacent sine; division by $\sigma_i^2 s_{i+1}$, followed by its own first-rate factor Π_i, leaves $\Pi_i\delta_i^2$, giving the first-rate row. The remaining terms are the just estimated slow and segment terms. This proves (11.16) and its rate consumer. The central factors are calculated from

$$2W\Gamma_{<j} = c_g^{-2}[1 + 2\varepsilon_{\mathrm{geom}}(\Gamma_{<j} - \Gamma_0)], \qquad \frac{\kappa_i}{|\zeta_i|^2} = 1 + (c_g^{-1} - 1)\frac{\zeta_{i,1}^2}{|\zeta_i|^2}.$$

The variable increments in these expressions give the circulation and center-drift row. Their accumulated defect is their summable sum plus the fixed geometric defect, which is made small when the chart is fixed; it is not required to decay at the newest frequency. In particular the reciprocal $|\zeta_i|^2/\kappa_i$ stays in the principal shear; it is not part of a decaying additive error.

Finally (9.32) is integrated in phase first and in the full label second. The own fast velocity contributes zero to the first equation. The remaining velocities have the last row's powers, with their original inherited endpoints. On an interval starting at insertion the un-divided integral inequality is used. This preserves flat activation and makes no division by a zero seed. Summing the tails uses $\sum_{q>i}|W_q|^{\mathrm{bd}} \le 2C_T/\sigma_{i-1}$, already proved from the duration ratios. For a fixed layer the integrals always start at its original insertion, including on later windows.

All exponents in (12.7) exceed a fixed positive number. Lemma 12.1, with a smaller exponent if necessary, absorbs their fixed factors and their generation factors. For example, the second row with two copies of G is bounded by choosing the uniform ratio in (12.6) smaller than $\mathsf{c}/8$, leaving a positive exponent after both factors. The own young tests (11.18) therefore hold simultaneously at their actual admitted orders. This bounds purely spatial coefficients using their marked curvature or first-label factor.

For completeness, the complete-level comparisons are not truncated at levels one and two. Their intrinsic upper coefficients are bounded by the fixed finite constants included in $\mathcal{K}_{\mathrm{ho}}(k_i) \le x_i$. At a certifying reading range include $D_i^+ \le N_i^c$; the full-depth formula of Lemma 10.1, with its $4n$ shift and its amplitude, is exactly the bound included in that range. Each additional level has the additional factor $\delta_i \le N_i^{-\mathsf{h}_*}$. Thus, after retaining the first one or two levels as required by (9.24), the remaining positive sum is bounded by the corresponding geometric series with ratio at most

$$x_i N_i^{c-\mathsf{h}_*} \le \tfrac{1}{2}.$$

The same argument applies separately to each first-rate and output-normalized route in (9.25); the output ratios of the leading terms are the correction rows of (12.7). This proves both full relative tails and all four comparisons (10.21). No old higher-order smallness is imposed at every future reading order.

Containing regions, rates and return margins. There are two different small spatial comparisons. The rigid tube of layer j, read in an older label i, has label radius bounded by the central matrix ratio times ℓ_j; its phase radius has the additional factor N_i. At the largest old index their powers, apart from the fixed factors defined above, are

$$N_{j-1}^{-\mathsf{a}+\beta/2}/\rho_{j-1}^{\mathrm{core}}, \qquad N_{j-1}^{1-\mathsf{a}+\beta/2}.$$

The first denominator is $\ell_{j-1}/2$, except at the fixed first core. It costs an earlier-carrier power; that first core costs only a fixed factor times y^2. Both comparisons therefore tend uniformly to zero. The other contribution is the inherited center offset. Integrating its phase equation gives

$$S_i \le C_{\mathrm{ph}}\Pi_i^2 N_i^{-\bar{\beta}}, \qquad S_i/\delta_i \le C_{\mathrm{ph}}\Pi_i^{2-\mathsf{d}}.$$

The label displacement satisfies the last row of (12.7). These estimates, including the original insertion-to-present integrals, put every endpoint strictly in half of its old label and phase core. For a segment consumer, apply the same inverse-map derivative array to the segment integral: its extra enclosure radius is the segment length times that array. Its powers are the same two displayed powers and the offset powers. They fit the remaining half margins. This proves (11.10) without assuming the image of a core convex. Earlier labels form a geometric sum, so the bounds are uniform in the number of intervening windows.

The physical center drift and all tube radii have a summable bound $Cy^{C_\circ}N_1^{-\bar{\beta}} + C\ell_1$. They lie in the fixed containing region after one increase of y. The complete circulation values differ from the base there by at most $C\sum_i x_i N_i^{-\bar{\beta}}$. Their first-label variations were estimated above. Consequently the two circulation intervals and their radius tests in (11.9) hold with a fixed positive margin. The fixed compact inverse then has its original support margin. This proves support for the actual compact residuals, not just for a principal oscillatory profile.

For the drift test, the exact decomposition in (10.16) has a central linear part, a nonlinear remainder, and the difference of the two observation velocities. The linear part divided by ℓ_i is the corresponding old clock. The remainder is bounded by $\eta_j\mathfrak{d}_j$ times the just estimated normalized spatial factors, with $\mathfrak{d}_j \le \mathscr{R}_j$; it is thus bounded by its rate, whether that rate is large or small. The last part is the full-flow offset estimate above. The phase bound retains $4\zeta_+N_{j-1}^{1-\mathsf{a}} \le 1$. This gives (11.19) at both positive and zero elapsed time. No pointwise smallness of the rate is asserted.

The older-gradient rate is obtained by differentiating the factored angular sum as prescribed before (11.11). Its adjacent term is bounded by $C\sigma_{q-2}$, and the remaining terms form the same geometric angular and amplitude sums used above. Since $\Gamma_q^{\mathrm{gr}}/\Lambda_q = \sigma_{q-2}(\sin s_q/s_q)$, this proves that test with a fixed C_s. At $q=2$ its separate first-parent term is $C_{\mathrm{in}}\Gamma_1^{\mathrm{gr}} \le C\sigma_0$; at $q=1$ the older coefficient is constant and the rate is zero. Fix C_s to dominate these two constants before the final carrier threshold.

Each of the thirteen generic return factors is a product or quotient of positive central intervals, angular ratios and the two center values already estimated. The identity $\prod a_\nu - 1 = \sum_\nu(a_\nu - 1)\prod_{\mu<\nu} a_\mu$ and the positive reciprocal bound convert these estimates into its required difference from one. The terms are the center rows above, the fixed geometric defect, and $C\Lambda_m^{-1}$. Their sum is smaller than the previously fixed $\varepsilon_{\mathrm{fac}}$. The two return-error sums retain their factored shear and segment terms and have the same bounds. Growth and transition give entry errors bounded by this sum plus $C\varepsilon_s/\Lambda_m$, proving (11.13). The first return instead obeys the fixed first-box calculation, not a large-compression limit. The endpoint logarithmic gain consequently lies strictly between the fixed margins surrounding $1+I_0$ (or $1+\log 2$ at index one) and the corresponding upper endpoint. The old sum is $O(\sigma_{m-1}^2)$, hence is smaller than $\varepsilon_s N_m^{\beta}$. These are precisely the three tests in (11.15).

Finally the short-ramp estimate is used in its actual form

$$Y_{\mathrm{act},i} \le N_i^{-k_i-\mathsf{e}}\sqrt{1+(u_i^+)^2}\exp(g_i^+/\Gamma_i^{\mathrm{gr}}) \le C_{\mathrm{rp}}N_i^{-k_i-\mathsf{e}}. \tag{12.8}$$

The bounded ratio and exponent here concern only the activation window. Since $\Lambda_i > 7x_i$, activation ends before the target is reached. Its complete-gradient contribution is at most a fixed order constant times $N_i^{1-k_i-\mathsf{e}}$ and is smaller than the fixed base gradient. The complete-gradient error and next transverse error were separately estimated above; together with the cone margin they give the holding and reinsertion tests. No estimate over the whole growth interval replaces (12.8).

Clock moments and the young time comparison. Put $\mathsf{D}_1 = \Lambda_{\mathrm{ret},1}$ and $\mathsf{D}_l = \Lambda_l$ for $l \geq 2$. The exact identity behind every generic tail comparison is

$$\lambda_l^* X_l^n = \left(\frac{\sin s_l}{s_l}\right)^n \Lambda_l^{2n+2} \frac{\sigma_{l-2}^{n+1}}{\sigma_{l-1}} \quad (l \geq 2). \tag{12.9}$$

The first term remains $\lambda_1^* X_1^n$. The bounds on k_l, Q_l give $\mathsf{D}_{m+2} \leq Q_{m+2}^3 x_m$, including $m = 1$. For $1 \leq n \leq Q_m$, split the sum at $m+1$. Its head is at most $\Pi_m^n \sum_l \lambda_l^*$. At the next term, $\sigma_m^{n+1}/\sigma_{m+1} \leq C^{n+1}$, since $Q_{m+1} = Q_m + 1$. Its remaining polynomial is bounded by $C^{n+1} x_m^{10n}$, after using $Q_{m+2} \leq (q_0 + 3)m$ and $m^2 \leq x_m$.

For the rest, write $x = x_{l-2}$, $Q = Q_{l-1}$. Here $n \leq Q - 2$. Increase the one initial threshold so that $x/\log x \geq CQ$, where the fixed C absorbs $\Lambda_l \leq x^4$ and the clock endpoints. This comparison propagates, because $\log(Qx) \leq 2\log x$ and $Q/2 \geq (Q+1)/Q$. Taking logarithms in (12.9) then gives

$$\lambda_l^* X_l^n \leq C^{n+1} \exp(-\beta x_{l-2}/4). \tag{12.10}$$

The series of exponentials is at most one half on one ray. For a fixed n outside a present admitted range, start this estimate at the finite index where $Q_{l-1} \geq n+3$. The omitted head is finite. Thus the same single sequence has finite clock moments at every fixed order, without a new initial threshold.

The ordinary and augmented lists reduce to this computation by $\upsilon_{i,l} \leq C_\lambda \lambda_l^*$ for $l \geq i+2$. The terms at i and $i+1$ are kept separately. Their masses and rates are bounded by the displayed $2 + \sigma_i \sum_{l \geq i+2} \sigma_{l-1}^{-1} \leq 3$ and $X_{i+1} \leq \Pi_i$. The base list starts with $1@X_1$, and its remaining masses are bounded by the index-one list. The rotation list has the one-higher-order moment dictated by its differentiated angle equation; apply the same identity with that order. Literal floors are retained by

$$\max(\mathscr{R}_i, X_l)^n \leq \mathscr{R}_i^n + X_l^n, \qquad \mathscr{R}_i \leq \Pi_i.$$

At the maximal requested order the allocation has $B = 9k_m + 48$, so $B + 3 = 9k_m + 51 \leq Q_m$ by (12.2). This also covers the raised moments and the extra primitive orders; their rates have not been identified with unraised ones.

For $m \geq 2$ the head rates obey

$$\mathscr{R}_i, X_l \leq \widehat{\Pi}_m = Q_{m+1}^4 x_m^2 \sqrt{C_\sigma} N_{m-1}^{\beta/2} \quad (i \leq m,\ l \leq m+1).$$

The exponential factor dominates x_m^{10}, and $x_m^{30} \leq \Pi_m$. Hence the first-future polynomial and the finite initial sum give the smaller-clock estimates at these indices. There is no such logarithmic-head assertion at $m = 1$. There the preceding argument uses the native $\Pi_1 = N_0^8 y^{40}$: it contains Cy^{10n}, the separate X_1, the augmented own term and the base term. In particular the legitimate first tests are

$$\lambda_1^* X_1^r + \lambda_2^* X_2^r \leq \tfrac{1}{2}\Pi_1^r \quad (r = 1, 2), \qquad \mathscr{R}_1 \leq Q_2^4 y^2 \sqrt{C_\sigma} N_0^{\beta/2}.$$

The first rates are fixed or at most quadratic in y, so these hold on the same ray. This proves the first ordinary, augmented, base, rotation and floor moments in their own clock.

We also need the two-index, not merely the own-index, conclusion. For $i < m$ and $2 \leq n \leq Q_m$, factor $n-1$ head rates out at Π_m, leaving the already bounded first moment $C\Pi_i$. The first-future and remaining tails are at most $C^{n+1} x_m^{10n} + C^{n+1}$, which is bounded by $C^{n+1}\Pi_m^{n-1}$, since $m \geq 2$ and $10n \leq 30(n-1)$. This gives

$$\sum_{l \geq i} \upsilon_{i,l}^+ \max(\mathscr{R}_i, X_l)^n \leq C^{n+1} \Pi_i \Pi_m^{n-1}.$$

For rotation at order $b+1$, retain its first two moments: the head is at most $C\Pi_1^2 \Pi_m^{b-1}$ for $b \geq 1$. For $b \geq 2$ the tail uses $10(b+1) \leq 30(b-1)$; at $b = 1$ use the native second moment. Thus it is bounded by $C^{b+2} \Pi_i^2 \Pi_m^{b-1}$ for $i \leq m$. The separate floor has exactly the same bound, using $\mathscr{R}_i \leq \Pi_i$.

The order-one ordinary and order-zero rotation cases are their native moments. This is (10.42) together with the raised versions; it retains the actual first-rate factor, not just Π_m^n.

Lastly, at $m \geq 2$,

$$\frac{\widehat{\Pi}_m}{\Pi_m} \leq C Q_{m+1}^4 x_m^{-38} N_{m-1}^{\beta/2-\mathsf{p}}. \tag{12.11}$$

Combining this with the uniform, correct-base generation estimate (12.6) proves (10.39), with room to spare. All purely spatial young factors were already bounded in (11.18); only a term with a time derivative uses this clock ratio. There is no inherited generation at index one.

The coefficient quotients for the force. We verify the complete table (6.3) before coarsening the force. The low arrays and the bounded young constants give the following upper sizes, with fixed constants suppressed:

$$\begin{gathered}\chi, c_0, \varpi_*, z_0, \sigma^{-1} \leq C, \qquad K, h_0, C_0, \sigma, \lambda, \Lambda_K \leq C\Pi, \\ G_0 \leq C\Pi^2, \quad B_0, L_0 \leq C\Pi^3, \quad V_X \leq C\Pi^2, \quad \mathcal{T} = \max(|I|^{\mathrm{bd}}, \Pi^{-1}) \leq C\Pi, \quad \mathcal{T}^{-1} \leq \Pi, \\ z_1^* \leq C\Pi^3/N, \qquad z_2^* \leq C\Pi^4/N^2.\end{gathered} \tag{12.12}$$

For example $C = H\mathsf{A}(X)H^T$ costs two central matrix factors, and $K^2 \leq C\sigma_{m-1}^2 \leq C\Pi_m$; $\mathfrak{l}$ adds one inverse-map derivative; the older scalar gradient costs one clock squared; and its vorticity-gradient bound adds one rate. These are precisely the coefficient entries of (4.11), with the small curvature and radius products already retained. The physical drift V_X is not the phase evaluation rate. The time bound uses the full remaining lifetime, including the distinct first hold.

In their printed order, the quotients are bounded by fixed constants times the following expressions:

(12.13)

term	upper comparison
$Z^\flat$ first; second	$\Pi^{3-\mathsf{d}} N^{\mathsf{h}-1}$; $\quad \Pi^{4-2\mathsf{d}} N^{2\mathsf{h}-2}$
$\{U, q\}$	$\Pi^{4-\mathsf{d}} N^{\mathsf{h}-1}$
$\partial_t Z^\flat$ first; second	$\Pi^{5-\mathsf{d}} N^{\mathsf{h}-1}$; $\quad \Pi^{6-2\mathsf{d}} N^{2\mathsf{h}-2}$
b gradient, fast; slow	$\Pi^{4-\mathsf{d}} N^{\mathsf{h}-1}$; $\quad \Pi^{5-2\mathsf{d}} N^{2\mathsf{h}-2}$
$ch \cdot \nabla_a V$	$\Pi^{4-\mathsf{d}} N^{\mathsf{h}-1}$
$2\varpi V h \cdot \nabla_a q$	$\Pi^{5-\mathsf{d}} N^{\mathsf{h}-1}$
fast; slow transport	$\Pi^{2-\mathsf{d}}$; $\quad \Pi^{3-2\mathsf{d}} N^{\mathsf{h}-1}$
fast; slow circulation square	$\Pi^{2-\mathsf{d}}$; $\quad \Pi^{4-2\mathsf{d}} N^{\mathsf{h}-1}$.

Here $\mathsf{h} = \bar{\beta}$. To check the last row directly, the fast term is $\mathcal{T}\sigma\varpi_* z_0 Y_*/\delta$; its N-powers cancel and its remaining power is $\Pi^{2-\mathsf{d}}$. The slow term contains its actual additional $h_0\lambda/N$ and δ^{-1}. The other rows follow by direct substitution of (12.12), so neither a radial weight nor a lower-elliptic piece has been dropped. Since $\mathsf{d} = 16$, all these bounds are at most one on one initial ray. Their constants are fixed size constants, not derivative-order constants.

Lemma 12.2 (One scalar threshold). *With the fixed choices above there is a finite y_*, independent of the number of layers and of the requested order, such that every $y \geq y_*$ realizes all scalar hypotheses of Proposition 11.1 for every finite initial sequence. The smaller-clock logarithmic-head and inherited-generation comparisons are used for $m \geq 2$; the native first-clock comparisons are used at $m = 1$. The admission sets are nested, nonempty and exhaustive.*

Proof. Lemma 12.1 gives nonemptiness, nesting and exhaustion of the admission sets.

For clarity, there is one threshold, not an implicit list of order-dependent thresholds. Form the following finite collection: the first-radius, first-history, first-pulse and first-clock tests; the positive scalar endpoint, denominator and geometric margins; the fixed power comparisons (12.7) and (12.13); and the fixed constants in the generation ray and clock-tail estimates. For a negative carrier power appearing in this collection, first retain half its exponent. The logarithm of its earlier-carrier factors is at most $C_o x/(q_0+1)$, already smaller than the reserved exponent. The remaining fixed polynomial is dominated on a common ray by Lemma 12.1. Infinite geometric sums are bounded there by their first term times a fixed constant. The generation factor uses the finite supremum in (12.6), including two copies where prescribed. The moment comparisons use the single propagating $x/\log x$ inequality proved above. The separate first tests are fixed polynomial or negative-power comparisons in y. Their finite maximum supplies y_*.

No fixed profile norm of arbitrarily high order enters this finite maximum: it enters $\mathcal{K}_{\mathrm{ho}}$ before admission instead. The allocation $r_* = k + 4J_m + 9$, the extra r_X, r_F, the primitive orders r_F+1, r_F+3, and the backwards old-core rectangles are exactly those in (11.1). The estimates above include all their four routes. Terminal sources run through $2J_m+4$ and composites through J_m+2. Thus the finite collection enumerated here covers the referenced scalar formulas as well as the additional formulas of Section 11. On all-trial older histories their prescribed upper integrals dominate truncations; on the youngest history they use only its selected completion. The threshold uses no unselected field or unknown future clock. □

12.3. **Physical force estimates for this sequence.** The distinction between small scalar quotients and large fixed derivative constants is useful here. The latter are the numbers in $\mathcal{K}_{\mathrm{ho}}(k_m)$; after its defining substitutions every one is at most x_m. This includes the actual profile norms before any coarsening, the finite level recursion, its terminal sums and the constants multiplying both means. It also includes fixed coordinate and compact-recovery costs. Products occurring in the following bounds have a fixed number of such factors, independently of m, k. The number of terms is at most a fixed power of J_m+1. It follows, for example, that their combined bound is

$$Cx_m^{16}C^{k+1}.$$

For the lower-elliptic vorticity mean the expanded product is $C_{\mathrm{ev}}H_{r_*}^2 a_u a_v$, followed by the fixed recovery factor. One of the two H_{r_*} factors is inside the lower-elliptic composite. These six scalar factors and at most two finite index sums are bounded by the displayed sixteenth power. Every terminal or activation term uses fewer factors. Since $k \le k_m \le Q_m/c_Q$ and $Q_m/x_m \to 0$ uniformly as $y\to\infty$, increasing the same threshold gives

$$Cx_m^{16}C^{k+1} \le N_m^{1/4} \qquad (k\le k_m). \tag{12.14}$$

This step does not modify the fixed order function or the sequence.

The rates from (12.12) satisfy

$$\Lambda_{\mathrm{ph}} \le CN\Pi^2, \qquad \Lambda_{\mathrm{slow}} \le C\Pi^3.$$

They are different because a mean has no phase derivative. Retain the amplitude sum $Y_*\sum a_n\delta^n$, the terminal sum $\sum_{g=J+1}^{2J+4} s_g\delta^g$, and the lower-elliptic sum in both mean formulas (10.44). Their finite constants have just been included in (12.14). The spatial size bracket in (10.46) is bounded by $C\Pi^3$, since

$$\frac{N\chi}{\sigma^2}(z_1^*+z_2^*)(1+\lambda/N) \le C\Pi^3, \qquad \frac{\chi\Lambda_K}{\sigma^2} + \varpi_* h_0 \le C\Pi.$$

Here $\Pi^4/N \le \Pi^3$; the first circulation mean is bounded by $C(1+\lambda/N)$. In particular neither the reciprocal-symbol derivative nor the circulation-square mean has been set to zero. After its outer divergence factor λ, a comfortable bound for the entire slow part is

$$Cx_m^{16}C^{k+1}\Pi_m^{3k+6}N_m^{-2\bar\beta}.$$

The exponent six is larger than the four obtained directly above.

Using the actual activation estimate (12.8), the physical factor $1+N/\sigma \le CN$, and $\mathcal{T}^{-1}\le\Pi$, the remaining two terms are bounded by

$$Cx_m^{16}C^{k+1}\left[N_m^{k+1-k_m-\mathsf{e}}\Pi_m^{2k+1} + N_m^{k+1-\bar\beta-\mathsf{h}(J_m+1)}\Pi_m^{2k+1+\mathsf{d}(J_m+1)}\right].$$

This uses $\mathcal{T} = \max(|I_m|^{\mathrm{bd}}, \Pi_m^{-1})$, not the duration alone. Zero duration and the beginning of activation therefore cause no division by zero.

Put $\mathsf{v}_m = \mathsf{q}/Q_m$, a dimensionless exponent for these estimates. Thus $\Pi_m \le N_m^{\mathsf{v}_m}$, $\mathsf{v}_m \le \mathsf{v}$, and $\mathsf{v}_m k_m \le \mathsf{u}$. The respective three exponents of N_m, before the harmless factor in (12.14), are at most

$$\begin{aligned} \text{activation}: &\quad 1 - \mathsf{e} + 2\mathsf{u} + \mathsf{v}, \\ \text{terminal}: &\quad -(2\mathsf{h}_* - 1)k_m + \beta - 9\mathsf{h}_* + 2\mathsf{u} + \mathsf{v}, \\ \text{means}: &\quad -2\bar{\beta} + 3\mathsf{u} + 6\mathsf{v}. \end{aligned} \tag{12.15}$$

With (12.2), the first two are less than -6, and the last is less than $-3/2$. The additional $1/4$ from (12.14) leaves more than enough room for

$$\mathfrak{F}_{m,k} \le N_m^{-1/2} \qquad (0 \le k \le k_m) \tag{12.16}$$

after a last fixed increase of y_*. This estimate applies to the full physical force, not to a phase-projected residual. Compatibility uses activation plus terminal remainder together and each complete mean separately, as in Proposition 6.2.

12.4. The infinite selected history and the extension.

Proposition 12.3 (Simultaneous choice and smooth force)**.** *Fix the data and the one geometric magnification as above. For any $y \ge y_*$, the sequence (12.3)–(12.4) has an infinite causal selected continuation. Its successive insertion times t_m^{in} increase to a finite $T_* > 0$. One fixed compact set away from the axis contains the base force and every physical increment throughout $[0, T_*)$. The orders k_m tend to infinity, and*

$$\|f_m\|_{C^k_{x,t}([0,T_*))} \le C_k \varepsilon_m, \quad \varepsilon_m = N_m^{-1/2}, \quad k \le k_m, \qquad \sum_m \varepsilon_m < \infty. \tag{12.17}$$

Every fixed increment and the base force have bounded mixed derivatives of every order on this interval. Their sum extends smoothly through T_* with the same support. The growth-duration and complete-gradient bounds of the finite construction hold on this same history.

Proof. Begin with the fixed smooth base preparation. Lemma 12.2 and Proposition 11.1 construct the first insertion and its selected return. Suppose the selected past has been fixed through $m-1$. It has its prescribed principal clock window. The same scalar threshold verifies the next finite problem, uniformly over the full current trial interval for its older fields. The nonempty compact return-zero set has its least member in $(0, c_p)$; choose that member and extend the selected history. This uses causality, not differentiability of the selector. All constructions agree on the already selected time interval. The flat endpoint profiles supply the joining derivatives. Induction therefore defines one smooth finite history on each finite collection of windows, with the scalar data already fixed.

Use the complete insertion window W_m and the growth interval $G_m = I_m^{\mathrm{gr}}$ of (9.8). For $m \ge 2$ the finite bounds give

$$c_T \frac{\Lambda_m}{\Gamma_m^{\mathrm{gr}}} \le |I_m^{\mathrm{gr}}| \le |W_m| \le C_T' \frac{\Lambda_m}{\Gamma_m^{\mathrm{gr}}} \le \frac{C_T''}{\sigma_{m-2}}. \tag{12.18}$$

The first window has the finite separate bound printed in (11.6). Every window has positive growth length. The geometric clock increase makes the sum of their lengths finite, so the strictly increasing insertion times have a finite positive limit. The geometric and segment estimates above give the same containing region at every finite stage. Compact recovery then gives the same off-axis support for every f_m, including the base, on the union of these intervals.

The selected complete gradient is not the principal clock. Denote the complete circulation at the current finite window by Γ^{cur}. The cone and complete correction estimate give separately

$$|\nabla\Gamma^{\mathrm{cur}}(Z(t),t)| \geq (\nu - \varepsilon_G)\left(A_0 + \sum_{i\leq m} A_i(t)\right) \geq c_G\sigma_m^2 \tag{12.19}$$

on the selected hold and the subsequent growth before the next return, with the small ramp included as in the finite theorem. At the selected endpoint $c_\sigma N_m^\beta \leq \sigma_m^2 \leq C_\sigma N_m^\beta$. These inequalities and (12.18) retain the later gradient and duration inputs; no conclusion about the limiting velocity, pressure or time integral is needed here.

At any finite horizon the input arrays yield (12.16). Their lifetime upper integrals include the entire future scalar tail, so the estimate is independent of that horizon. Taking its supremum proves the asserted lifetime bound. The series of $N_m^{-1/2}$ converges by $x_m \geq (q_0+1)^{m-1}y$. A fixed physical rescaling merely multiplies this bound by the fixed order constant C_k; it does not change the choice at any derivative order.

We verify separately that every early summand has all-order bounded derivatives. Fix a layer i and a mixed order k. Its insertion and any finite number of subsequent windows form a finite smooth beginning, so their norms are finite. For its tail, the clock moments of order required by k, including the finite raised and rotation shifts, are finite by (12.10). Every such order is eventually admitted. The fixed-layer conclusion of Proposition 10.2, applied with these actual moments, gives bounded derivatives on the full material support of this fixed layer. This is its full-support fixed-order conclusion, not an old-annulus estimate at all young orders. Its finite depth J_i requires only finitely many higher coefficient and profile orders to evaluate its complete residual and compact recovery. The exact residual formulas and the fixed inverse therefore give the asserted bounded $C^k_{x,t}$ norm of this fixed f_i. There is no required small bound for this finite beginning.

The base requires the rotation primitive as well as the ordinary base clock. Its first terms were retained above, and its higher moments are bounded by the same finite-initial-sum argument. Thus the rotation angle and all of its derivatives have finite norms. The base direction and angular velocity are smooth functions of this angle and its first derivative. Its compact profiles are fixed. Leibniz and the chain rule show that all derivatives of Γ^b, Ψ^b, and hence of v^b and $\xi^b = L\Psi^b$, are bounded. In particular their actual residuals

$$F_b^\Gamma = \partial_t\Gamma^b + v^b\cdot\nabla\Gamma^b, \qquad E_b^\xi = \partial_t\xi^b + v^b\cdot\nabla\xi^b - W\partial_1(\Gamma^b)^2$$

have bounded mixed derivatives of every order and their original compact support. The compact recovery and the fixed off-axis coordinate factors give the same conclusion for f^b. The finite initial preparation has bounded derivatives and its flat join preserves these bounds.

Set $f = f^b + \sum_i f_i$ on $\mathbb{R}^3\times[0,T_*)$. For each fixed mixed order k, there are only finitely many indices with $k_i < k+1$. The preceding fixed-layer and base estimates bound their derivatives through order $k+1$; for the remaining indices, (12.17) gives a summable derivative bound through that order. All summands have the common fixed off-axis support proved above. These verify the hypotheses of Lemma 13.2, proved in Section 13, which makes f smooth on $\mathbb{R}^3\times[0,T_*]$ with the same support. The smooth off-axis coordinate factors preserve the bounds for the cylindrical components and their axisymmetric lift. Only the force is extended. □

13. Construction of the solution and the smooth force

We use the single sequence and selected history of Proposition 12.3. No parameter is chosen again below. The construction is first expressed in the magnified unit-ring coordinates of Lemma 2.3. In estimates for the profiles we suppress the primes and write the spatial variable as $\boldsymbol{y}$, as in (9.1). We restore the physical volume coordinate $y = (z, r^2/2)$ before recovering velocity and pressure.

Let τ_m be the insertion times and $T_* = \lim_m \tau_m \in (0,\infty)$. On every interval $[0,\tau]$, $\tau < T_*$, only finitely many insertions have occurred. The selected continuations agree on the preceding intervals, and the new profiles vanish to every order at activation. Thus the formulas

$$\Gamma = \Gamma^b + \sum_m \gamma_m, \qquad \Psi = \Psi^b + \sum_m \psi_m, \qquad f = f^b + \sum_m f_m \tag{13.1}$$

are finite smooth sums on every compact preterminal interval. In this display f_m denotes the physical three-dimensional force; the corresponding Γ, Ψ sums are transported between the two charts by (2.10). In particular their derivatives and products there are the derivatives and products of the actual finite construction. The force increments are the full residual differences recovered by the same fixed linear compact inverse; they are not residuals of isolated waves.

13.1. **Sizes of the complete increments.** We estimate on the entire transported support, including its cutoff region. Write $A_m^{\mathrm{inv}} = X_m^{-1}$ for the inverse map, reserving $A_m = -N_m T_m(0,t)|\zeta_m(0,t)|$ for the positive central amplitude. The constants in the following bounds depend on the fixed profiles, geometry and comparison intervals, not on the insertion number or the last time to which the selected history has been continued.

Lemma 13.1 (Summable circulation and velocity sizes)**.** *For the chosen sequence the complete increments satisfy*

$$\sup_{t<T_*} \big(\|\gamma_m(t)\|_\infty + \|J\nabla_{\boldsymbol{y}}\psi_m(t)\|_\infty\big) \le C N_m^{-1/2}. \tag{13.2}$$

Consequently the circulation and the meridional velocity in (13.1) *are uniformly bounded before* T_**.*

Proof. The principal streamfunction is the one in (3.3). Differentiation before estimation gives the exact identity

$$v_{m,0} = w_m \left[N_m B_m F g_m J\zeta_m + PJH_m^T(g_m\nabla_a B_m + B_m \nabla_a g_m)\right]^\natural, \qquad B_m = \frac{\widehat{\Omega}_m}{N_m\sigma_{m-1}\kappa_m}. \tag{13.3}$$

In particular the differentiated symbol is retained:

$$\nabla_a B_m = \frac{\nabla_a \widehat{\Omega}_m}{N_m \sigma_{m-1}\kappa_m} - B_m \nabla_a \log \kappa_m.$$

The first-label amplitude and symbol estimates, the fixed profile bounds, and (12.12) give

$$\|v_{m,0}\|_\infty \le C\frac{Y_{*,m}}{\sigma_{m-1}}\left(1 + \frac{\Pi_m^2}{N_m}\right), \qquad \|\gamma_{m,0}\|_\infty \le C Y_{*,m}, \qquad Y_{*,m} = C_\Theta N_m^{-7/8}.$$

Here one may replace C by the corresponding calculated first-order profile constant. Such constants are included in the finite sum below. Indeed κ_m^{-1} and ζ_m are bounded, while $\|H_m\|$, ℓ_m^{-1}, and the normalized first-label rates are bounded by fixed multiples of Π_m. This estimates the formula on its whole support, without extending a core linear-velocity formula to the cutoff region.

For every correction level, the chain rule is

$$\nabla_{\boldsymbol{y}} Q^\natural = \left[N_m \zeta_m \partial_s Q + H_m^T \nabla_a Q\right]^\natural. \tag{13.4}$$

Apply (6.5) with $r = 1$ to every $1 \le n \le J_m$. Its allocation permits this, since $1 + 2J_m \le r_*$. Write $a_{m,n}$ for its scalar constant a_n at this insertion and $H_{*,m}$ for its constant H_{r_*}, distinct from the inverse Jacobian H_m. As $0 < \rho_m \le 1$, the definition of the weighted norm gives

$$\|\gamma_m\|_\infty \le Y_{*,m} \sum_{n=0}^{J_m} a_{m,n}\delta_m^n,$$
$$\sum_{n=1}^{J_m} \|J\nabla\psi_{m,n}\|_\infty \le C\frac{Y_{*,m}}{\sigma_{m-1}}\left(1+\frac{\Pi_m^2}{N_m}\right) H_{*,m}\sum_{n=1}^{J_m} a_{m,n}\delta_m^n.$$

All levels are present in these sums. Their constants, the first-order principal constants above, and the number of summands are among the finite products and sums estimated in (12.14). More explicitly, taking $\delta_m^n \le 1$, their sum is bounded by the product of $J_m + 1$, the maximum of the retained $a_{m,n}$, $1 + H_{*,m}$, and the fixed evaluation and profile factors. This uses fewer factors and index sums than the bound $C(\log N_m)^{16}C^{k+1}$ proved there. At order zero the same fixed choice therefore bounds this entire constant by $CN_m^{1/4}$.

The scalar schedule gives $\sigma_{m-1} \ge \sigma_0 > 0$ and $\Pi_m \le N_m^{\mathsf{v}}$, where $\mathsf{v} = 10/(q_0 + 1)$ and $2\mathsf{v} < 1$. It follows that the preceding sums are at most $CN_m^{-5/8}$, hence satisfy (13.2). The first layer uses its native Π_1 and its fixed σ_0; it is also covered by the same parameter estimate. Alternatively its finite fixed constant can be included in C. Since $\log N_m \ge (q_0 + 1)^{m-1} \log N_1$, the series of the right-hand sides in (13.2) converges.

The base circulation has the bound

$$\|\Gamma^b\|_\infty \le \Gamma_0'\|\chi_1\|_\infty + \frac{A_0}{N_0}\|F\|_\infty\|\chi_0\|_\infty, \qquad \Gamma_0' = (2\varepsilon_{\mathrm{geom}})^{-1}.$$

Its meridional velocity is $\alpha J\nabla H$. The fixed-order base estimate in Proposition 12.3 bounds $\sup_{t<T_*}|\alpha(t)|$, and H is one fixed compact smooth function. The base is therefore bounded too. Summing proves the assertion in magnified coordinates. The fixed changes $\Gamma = \varepsilon_{\mathrm{geom}}\Gamma'$ and $v = \varepsilon_{\mathrm{geom}}v'$, followed by $u^z = v_1$, $u^r = v_2/r$, preserve these bounds. $\square$

13.2. **Physical equations, pressure and endpoint force.** The fixed containing region of Proposition 12.3 contains the base, all complete increments and all recovered force supports. Lemma 2.3 places its physical image strictly inside $\mathcal{T}_R$, with $0 < R \le 1/4$, for the unit ring. Put $r_- = 1 - R > 0$ and $r_+ = 1 + R$. We now use the physical volume coordinate $y = (z, r^2/2)$ and the unmagnified coefficients L, A, W in (2.2).

Define u by (2.3). It is smooth, axisymmetric and divergence-free on each compact preterminal interval, and vanishes in a fixed neighborhood of the axis. Differentiation does not enlarge support, so Γ, $\omega = rL\Psi$ and u have support in $\overline{\mathcal{T}_R}$. For the complete fields put

$$F^\Gamma = \partial_t\Gamma + v\cdot\nabla_y\Gamma, \qquad E^\xi = \partial_t L\Psi + v\cdot\nabla_y L\Psi - W\partial_{y_1}(\Gamma^2).$$

All these expressions are finite smooth sums on the interval under consideration. In particular the full square and all cross interactions are included. Its compatibility integral is

$$\int E^\xi\,dy = 0, \qquad dy = r\,dz\,dr. \tag{13.5}$$

To see this, $L\Psi = \mathrm{div}_y(\mathsf{A}\nabla_y\Psi)$ has zero integral, $v\cdot\nabla_y L\Psi$ is a divergence, and W depends only on y_2, so its product with $\partial_{y_1}\Gamma^2$ integrates to zero. Compact support justifies all three integrations.

The exact residual-difference formulas telescope. Since the compact inverse $\mathcal{R}_y$ is fixed and linear, the very force in (13.1) obeys

$$f^\varphi = F^\Gamma/r, \qquad (f^z, f^r/r) = \mathcal{R}_y E^\xi, \qquad \partial_z f^r - \partial_r f^z = rE^\xi. \tag{13.6}$$

Both phase means and the full terminal remainder were included before this recovery. The alternative pair in (z, r) would have input rE^ξ, whose integral in $dz\,dr$ is zero by (13.5).

For completeness, the pressure can be recovered globally, without an integrability condition on it. Set $h = f - \partial_t u - (u\cdot\nabla)u$. The circulation equation gives $h^\varphi = 0$. The reduced-vorticity equation and (13.6) give $\partial_z h^r - \partial_r h^z = 0$. Hence all three components of $\operatorname{curl} h$ vanish off the axis. They also vanish near the axis, where the fields are zero. Thus h is a smooth curl-free field on $\mathbb{R}^3$, and

$$p(x,t) = \int_0^1 h(sx,t)\cdot x\,ds \tag{13.7}$$

is a smooth pressure, with $p(0,t) = 0$. Indeed, symmetry of $D_x h$ gives

$$\partial_{x_i} p = \int_0^1 \big[s x_j \partial_{x_j} h_i(sx,t) + h_i(sx,t)\big]\,ds = h_i(x,t).$$

This calculation includes points on the axis and all finite activation joins. It proves the momentum equation with the required sign.

Lemma 13.2 (Extension of the force)**.** *Suppose the base force and every fixed increment have bounded mixed derivatives of all orders and one common fixed off-axis compact support. If $k_m \to \infty$ and $\|f_m\|_{C^k_{x,t}} \le C_k N_m^{-1/2}$ for $k \le k_m$, then the force in (13.1) is smooth on $\mathbb{R}^3 \times [0,T_*]$, with the same support.*

Proof. For each fixed k, separate the finitely many indices with $k_m < k+1$. Their derivatives are bounded, and the remaining derivative series converges uniformly through order $k+1$. Every derivative of order at most k is therefore uniformly Cauchy as $t \uparrow T_*$, by its bounded time derivative. The spatial and temporal fundamental theorems of calculus identify the limits as the derivatives of the extended force. This argument for every k proves smoothness on the closed interval. Uniform limits preserve the fixed support. Equivalently, these are exactly the hypotheses of [1, Lemma 8.3] for each cylindrical component; the off-axis lift preserves the conclusion. Only this force series is evaluated at T_*. ☐

Proposition 13.3 (The preterminal solution)**.** *The selected history defines a smooth axisymmetric Euler solution (u,p) on $\mathbb{R}^3 \times [0,T_*)$, with force as in Lemma 13.2 and the initial datum (2.7). It satisfies (1.2) and, for every $\tau < T_*$,*

$$u \in L^\infty([0,\tau]; L^2(\mathbb{R}^3)), \qquad \nabla u \in L^\infty(\mathbb{R}^3 \times [0,\tau]).$$

The datum is divergence-free, smooth and compactly supported in $\mathcal{T}_R$, with nonzero swirl and zero meridional velocity.

Proof. The finite smooth sums and (13.7) prove the equations and preterminal smoothness. The boundedness and summability estimates of Proposition 12.3 supply every hypothesis of Lemma 13.2, so its conclusion applies to the actual force.

Lemma 13.1 proves the three uniform bounds in (1.2); the support was checked above. Since $r \ge r_-$ on the support, $\|u^\varphi\|_\infty \le r_-^{-1}\|\Gamma\|_\infty$. The velocity is bounded on a fixed compact set and zero elsewhere, so it has the asserted uniform energy bound. For a fixed $\tau < T_*$ it is a finite smooth sum on that same compact set, giving the uniform spatial-gradient bound. All increments are flat at their insertion times, so at zero the velocity is exactly the base datum. The initial angular rate is zero, and the initial circulation at the unit ring is $1/2$. The datum therefore has zero meridional velocity and nonzero swirl, and its support is strictly inside the prescribed containing torus. ☐

14. Blowup and preterminal uniqueness

The circulation gradient gives two components of the full vorticity. We first prove that its lower bound covers all late times, including transition and return. We then use a growth interval of positive length to prove the separate time-integral assertion.

14.1. **The complete gradient on consecutive windows.** We return to the magnified notation for the profile estimates and restore the physical variables in (14.5) below. Let t_m^a denote the growth hit, characterized by $-T_m(0,t_m^a) = N_m^{-\bar\beta}$, and let t_m^b be the selected return endpoint. The m-th insertion window runs from τ_m through growth, transition and return. Its selected hold starts at $t_m^b = \tau_{m+1}$ and continues through the next growth. The time t_m^a is not an insertion time.

At the full-flow point $Z(t)$, every active material profile is evaluated at its actual inverse label. For example, if $V_m = \sum_{n=0}^{J_m} V_{m,n}$ and $a_m^Z = A_m^{\mathrm{inv}}(Z(t),t)$, then

$$\nabla_{\boldsymbol{y}}\gamma_m(Z,t) = H_m(a_m^Z,t)^T\big(N_m p_m\partial_s V_m + \nabla_a V_m\big)(N_m p_m\cdot a_m^Z, a_m^Z, t). \tag{14.1}$$

The gradient estimate (9.49), with its inherited phase and label offsets, and the ramp estimate (9.50) give

$$\begin{gathered}\nabla_{\boldsymbol{y}}\Gamma(Z,t) = -A_0 e_b(t) - \sum_i w_i(t)A_i(t)e_i(t) + \mathfrak{g}(t),\\ |\mathfrak{g}(t)| \le \varepsilon_G\left(A_0 + \sum_i w_i(t)A_i(t)\right),\\ -e(t)\cdot\nabla_{\boldsymbol{y}}\Gamma(Z,t) \ge (\nu-\varepsilon_G)\left(A_0 + \sum_i w_i(t)A_i(t)\right).\end{gathered} \tag{14.2}$$

Here $e_i = \zeta_i(0,t)/|\zeta_i(0,t)|$, and the common unit direction $e(t)$ has projection at least $\nu > \varepsilon_G$ onto every direction in this sum. All sums are finite at a preterminal time. The estimate contains the complete correction series, the slow principal derivatives and the offsets of Z; it does not set them to zero at the central label. In particular an uncompleted activation does not multiply every correction by w_i. The separate ramp bound is what permits the displayed aggregate error estimate.

Lemma 14.1 (Lower bounds on the full time windows)**.** *There are constants $c_1, c_2 > 0$, independent of $m \ge 2$, such that*

$$\begin{aligned} |\nabla_{\boldsymbol{y}}\Gamma(Z(t),t)| &\ge c_1 N_{m-1}^{\beta} \quad (\tau_m \le t \le t_m^a),\\ |\nabla_{\boldsymbol{y}}\Gamma(Z(t),t)| &\ge c_2 N_m^{\beta} \qquad (t_m^a \le t \le t_m^b).\end{aligned} \tag{14.3}$$

Proof. For the first interval, (12.19) at layer $m-1$ applies on its selected hold and the next growth, including activation. Together with (11.3) it gives the first line, with $c_1 = c_G c_\sigma$.

For the second interval use the middle part of the central pair history. The scalar intervals in (9.46) and (9.47), realized by the chosen sequence, give a uniform finite number S_* such that

$$\int_{t_m^a}^{t_m^b} \|\mathcal{B}_m(0,t)\|\,dt \le S_m^{\mathrm{hist}} \le S_*.$$

Both return pulses occur in this integral. The component-ratio estimate used in deriving (9.49), restricted to this middle interval, gives a fixed $C_{\mathrm{mid}} < \infty$ with

$$T_m(0,t) < 0, \qquad |R_m(0,t)| \le C_{\mathrm{mid}}(-T_m(0,t)).$$

To specify this constant, let U_{gr} be the uniform hit-ratio upper bound from Lemma 9.2. On transition the two coefficients are nonnegative, so for $u = \widehat{\Omega}_m/T_m$ its equation gives $u' = \widehat{b}_m - \widehat{a}_m u^2 \le \widehat{b}_m$. The positive invariant interval and bounded quotient exclude a zero of T_m; hence one may take $U_{\mathrm{tr}} = U_{\mathrm{gr}} + S_*$. On return the generic specialization in Proposition 7.7 sets $c_0 = \mathfrak{z}_e^{-1}$ in (7.13), and (11.13) gives $\mathfrak{z}_e \le 1 + \delta_{\mathrm{ret}}$. Thus $|\widehat{\Omega}_m/T_m| = \mathfrak{z}_e|v| \le (1+\delta_{\mathrm{ret}})B$, where B is the fixed number in Lemma 7.5. One may take

$$C_{\mathrm{mid}} = \sqrt{1 + \max\{U_{\mathrm{tr}}, (1+\delta_{\mathrm{ret}})B\}^2}.$$

These are scalar coefficient and quotient intervals, not later derivative constants. Likewise (9.48), with the fixed C_R supplied by the parameter proof, permits $S_* = (\log C_R)/2$. The separate first return is unnecessary here, since $m \geq 2$. The covector bound on the same interval is $|\zeta_m(0,t)| \geq \zeta_- > 0$.

The pair equation, integrated backwards from any point of this interval to t_m^a, gives

$$|R_m(0,t)| \geq e^{-S_*}|R_m(0,t_m^a)| \geq e^{-S_*}N_m^{-\bar{\beta}}.$$

Consequently $A_m(t) \geq \zeta_- C_{\mathrm{mid}}^{-1} e^{-S_*} N_m^{\beta}$. Activation has finished by the hit, so $w_m = 1$ on this interval. Insert this positive summand into (14.2); all other principal summands have nonnegative projection. This gives the second line with $c_2 = (\nu - \varepsilon_G)\zeta_- C_{\mathrm{mid}}^{-1} e^{-S_*}$. Endpoint continuity covers the joins. □

Proposition 14.2 (The two limits and the vorticity integral)**.** *The solution of Proposition 13.3 satisfies* (1.3) *and* (1.4)*.*

Proof. The two intervals in (14.3) cover each complete insertion window. Since N_m increases, they imply, for a fixed $c_* > 0$,

$$|\nabla_{\boldsymbol{y}}\Gamma(Z(t),t)| \geq c_* N_n^{\beta} \qquad (\tau_{n+1} \leq t < T_*,\ n \geq 1). \tag{14.4}$$

For a time after τ_{n+1}, choose its current insertion window $m \geq n+1$ and use the corresponding line of (14.3). Thus this conclusion concerns every such time, not only the selected endpoints.

The magnification preserves the gradient exactly: $\nabla_y \Gamma = \nabla_{\boldsymbol{y}}\Gamma'$. At the physical point corresponding to $Z(t)$,

$$\begin{gathered}(\partial_z\Gamma, \partial_r\Gamma) = (\partial_{y_1}\Gamma, r\partial_{y_2}\Gamma),\\ \omega^r = -r^{-1}\partial_{y_1}\Gamma, \qquad \omega^z = \partial_{y_2}\Gamma, \qquad \boldsymbol{\omega} = \omega^r e_r + \omega e_\varphi + \omega^z e_z.\end{gathered} \tag{14.5}$$

Therefore, with constants depending only on $r_\pm$,

$$\|\nabla\Gamma(t)\|_\infty \geq \min(1, r_-)c_* N_n^{\beta}, \qquad \|\boldsymbol{\omega}(t)\|_\infty \geq \min(1, r_+^{-1})c_* N_n^{\beta} \quad (t \geq \tau_{n+1}).$$

Given any finite bound, choose n so that these lower bounds exceed it. Since $\tau_{n+1} < T_*$, both full limits follow.

For the time integral retain the whole growth interval I_{n+1}^{gr}, for $n \geq 2$. The first line of (14.3) gives a vorticity lower bound cN_n^{β} on this interval. Its duration satisfies

$$\begin{gathered}|I_{n+1}^{\mathrm{gr}}| \geq c_T \frac{\Lambda_{n+1}}{\Gamma_{n+1}^{\mathrm{gr}}} = \frac{c_T}{\sigma_{n-1}} \frac{s_{n+1}}{\sin s_{n+1}} \geq \frac{c_T}{\sigma_{n-1}},\\ \int_{I_{n+1}^{\mathrm{gr}}} \|\boldsymbol{\omega}(t)\|_\infty \, dt \geq cc_T C_\sigma^{-1/2} N_{n-1}^{\beta(Q_n - 1/2)} \longrightarrow \infty.\end{gathered} \tag{14.6}$$

Here $s_{n+1} = \Lambda_{n+1}\sigma_{n-1}/\sigma_n$, $\Gamma_{n+1}^{\mathrm{gr}} = \sigma_n \sin s_{n+1}$, and $\sigma_{n-1} \leq C_\sigma^{1/2} N_{n-1}^{\beta/2}$. All these indices are generic indices at least two, and $N_n = N_{n-1}^{Q_n}$ is used only in its defined range. The nonnegative integral over $[0, T_*)$ dominates each of the displayed integrals, proving (1.4) separately from the two norm limits. The same interval gives the remaining-time form of this estimate: with $G_n = c_* N_n^{\beta}$,

$$\begin{aligned}G_n(T_* - \tau_{n+1}) &\geq G_n|I_{n+1}^{\mathrm{gr}}|\\ &\geq c_* c_T C_\sigma^{-1/2} N_{n-1}^{\beta(Q_n - 1/2)} \longrightarrow \infty \qquad (n \geq 2).\end{aligned} \tag{14.7}$$

□

14.2. **Uniqueness in the full three-dimensional class.**

Proposition 14.3 (Preterminal uniqueness)**.** *The solution in Proposition 13.3 is unique in the class stated in Theorem 1.1. No symmetry of a competing solution or integrability of its pressure is required.*

Proof. Fix $\tau < T_*$ and let U be a competing solution on $\mathbb{R}^3 \times [0,\tau]$. It is locally space-time Lipschitz, has datum $U(\cdot,0) = u_0$, and obeys the asserted uniform L^2 and spatial-gradient bounds. Local continuity and Fatou's lemma extend the essential L^2 bound to every time section. The spatial Lipschitz bound extends to every section by approximation with times for which that bound holds. Testing divergence with compact spatial functions and using continuity in time gives $\operatorname{div} U(t) = 0$ for every $t \in [0,\tau]$.

For a spatially Lipschitz $h \in L^2(\mathbb{R}^3)$, with Lipschitz constant $L > 0$, its value of modulus M at a point gives $|h| \geq M/2$ on the ball of radius $M/(2L)$. Hence

$$\|h\|_\infty \leq (24/\pi)^{1/5} \|h\|_2^{2/5} L^{3/5}.$$

If $L = 0$, the finite-energy constant field is zero. Applying this at each time shows that U is uniformly bounded. The constructed solution u has these properties by Proposition 13.3.

Put $w = U - u$ and $Y = (U \cdot \nabla)w + (w \cdot \nabla)u$. The same force cancels, and for a distribution π,

$$\partial_t w + Y = -\nabla \pi \quad \text{in distributions.} \tag{14.8}$$

Here $Y \in L^\infty(0,\tau; L^2(\mathbb{R}^3))$, since

$$\|Y(t)\|_2 \leq \|U(t)\|_2 \|\nabla w(t)\|_\infty + \|w(t)\|_2 \|\nabla u(t)\|_\infty.$$

Write M_Y for a finite upper bound of this expression. For every $0 \leq s \leq t \leq \tau$ define the L^2 field

$$G_{s,t} = w(t) - w(s) + \int_s^t Y(r)\, dr. \tag{14.9}$$

The integral is a Bochner integral in the separable space L^2. Taking the curl of (14.8) and pairing with a fixed compact smooth spatial test gives a scalar distributional time derivative with an integrable right-hand side. Its left-hand pairing is continuous up to both endpoints by local continuity of w. The scalar fundamental theorem of calculus therefore proves $\operatorname{curl} G_{s,t} = 0$ for every s, t, including $s = 0$.

A curl-free L^2 field is orthogonal to every divergence-free L^2 field: their Fourier transforms at almost every nonzero frequency are parallel and perpendicular to that frequency, respectively, and Plancherel's identity gives the assertion. The frequency zero has no L^2 mass. Pairing (14.9) with the divergence-free field $w(t) - w(s)$ gives

$$\|w(t) - w(s)\|_2^2 = -\left\langle \int_s^t Y(r)\, dr, w(t) - w(s) \right\rangle, \qquad \|w(t) - w(s)\|_2 \leq M_Y |t - s|.$$

Thus w is Lipschitz as an L^2-valued function, and in particular its strong initial trace is the given zero field. This trace has been derived from the stated class, not added as a hypothesis.

Pairing instead with $w(t) + w(s)$, then summing over a partition of $[s,t]$, yields

$$\|w(t)\|_2^2 - \|w(s)\|_2^2 = -2 \int_s^t \langle Y(r), w(r) \rangle\, dr. \tag{14.10}$$

Indeed the replacement of the two endpoint values by $2w(r)$ on a subinterval of length at most h costs at most $2M_Y^2 h$ times that length; this tends to zero after summation. This proves the energy identity without pairing a distributional pressure with w.

The transport term in it is zero. Choose a smooth cutoff equal to one on B_R, zero outside B_{2R}, with derivative at most C/R. Integration by parts, using $\operatorname{div} U = 0$, bounds the cutoff transport integral by $CR^{-1}\|U\|_\infty\|w\|_2^2$. The uncut integrand is integrable, since $\int |U||\nabla w||w| \le \|\nabla w\|_\infty \|U\|_2\|w\|_2$. Letting $R \to \infty$ proves $\langle (U\cdot\nabla)w, w\rangle = 0$. Consequently (14.10) implies, almost everywhere,

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 = -\langle (w\cdot\nabla)u, w\rangle \le \|\nabla u\|_{L^\infty(\mathbb{R}^3\times[0,\tau])}\|w(t)\|_2^2.$$

Gronwall's inequality and $w(0) = 0$ give $w = 0$ on $[0,\tau]$. Local continuity upgrades equality almost everywhere to equality at every point. Since $\tau < T_*$ was arbitrary, uniqueness holds on the full preterminal interval. □

Proof of Theorem 1.1. For the unit ring, Proposition 12.3 supplies one selected history and a fixed containing region. Apply Lemma 2.3 with the already fixed $\varepsilon_{\mathrm{geom}} > 0$ to place its physical fields and force strictly inside a torus of radius $R < 1/2$. Proposition 13.3 supplies the datum, solution, pressure, support and size assertions, and Lemma 13.2 supplies smoothness of the force through T_*. Proposition 14.2 proves both full limits and the separate divergent time integral. Proposition 14.3 proves uniqueness in the stated class.

For any prescribed $r_0 > 0, z_0 \in \mathbb{R}$, use $\widehat{x} = (x - z_0 e_z)/r_0$ and $u = r_0\widehat{u}$, $p = r_0^2\widehat{p}$, $f = r_0\widehat{f}$. Lemma 2.2 and (2.9) verify the equations, compact torus, initial datum, force smoothness and solution class under this exact time-preserving change. In particular the full vorticity and its time integral are unchanged, while the circulation gradient is multiplied by the fixed positive factor r_0. The inverse change takes every competitor in the claimed class to a unit-ring competitor, so uniqueness also transfers. This proves every assertion of the theorem. □

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